APPROXIMATE SOLUTION OF BOUNDARY VALUE
PROBLEM WITH BERNSTEIN POLYNOMIAL
LAPLACE DECOMPOSITION METHOD

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Abstract: In this paper, a new approach enabling Laplace Adomian decomposition method in combination with Bernstein polynomials for numerical solution of boundary value problem is introduced. The accuracy of the method is dependent on the size of the set of Bernstein polynomials. Numerical results with comparisons are given to establish the reliability of the proposed method for solving singular and nonsingular boundary value problem. Convergence analysis of the presented technique is also discussed.

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Key Words: Laplace Adomian decomposition method, Bernstein polynomials, singular and nonsingular boundary value problem

1. Introduction

Nonlinear boundary value problems (BVPs) eminently emerge in the modelling of real life problems. A huge research has been conducted to find the numerical solutions of these problems by considering their applications in many branches of science and engineering for example, fluid mechanics, quantum mechanics, aero-dynamics, reaction-diffusion process, chemical reactor, geophysics [10], [21], astrophysics, boundary layer theory, the study of stellar interiors, op-

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timal control theory and flow networks in biology [1]. In recent times such kinds of problems have been solved analytically and numerically by some methods like Galerkin and collocation method, Sinc-Galerkin method, homotopy perturbation method, homotopy analysis method, homotopy asymptotic method, He-Laplace method, Adomian decomposition method, Laplace differential transform method and shooting method, etc.

The aim of the present research is motivated to achieve the numerical solutions by using a new reliable modification in Laplace Adomian decomposition method. Due to the effectiveness of the technique, Laplace decomposition method is adopted by many researchers to solve various functional equations [15], [17], [2], [19], [7]. Mainly the method is based on representing the series solution of the problem and decomposing the nonlinear term by Adomian polynomials. Various authors investigated and modified this technique [4], [8] to find the solutions of different nonlinear functional equations. In [5] Khuri extended the method by combining Laplace transform with Adomian decomposition method for solving nonlinear differential equations. Diversified modifications and improvements have been investigated further to improve the standard Laplace Adomian decomposition method [18], [11], [16].

In this paper, a novel and simple technique is introduced based on Bernstein polynomials. For finding the approximate solutions of BVPs, the technique expands the source function in terms of a set of continuous polynomials over a closed interval, i.e. Bernstein polynomials and then the standard Laplace decomposition method is adopted which provides tremendous results.

2. Basic Concepts of Bernstein Polynomials

Polynomials are the mathematical tools as these can be defined, calculated, differentiated and integrated easily. The Bernstein basis polynomials are exercised to approximate the functions and curves. Bernstein polynomials are the better approximation to a function with a few terms. These polynomials are used in the fields of applied mathematics, physics and computer aided-geometric designs and are also combined with other methods like Galerkin and collocation method to solve some differential and integral equations [3], [20].

Following are some basic definitions and results [13]:

Definition 1 (Bernstein basis polynomials). The Bernstein basis polynomials of degree m form a complete basis over the interval $[0, 1]$ and are defined

by

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}$$

where the binomial coefficient is

$$\binom{m}{i} = \frac{m!}{i!(m-i)!}$$

Definition 2 (Bernstein polynomials). A linear combination of Bernstein basis polynomials

$$B_m(x) = \sum_{i=0}^m B_{i,m}(x) \beta_i \quad (1)$$

is called the Bernstein polynomials of degree m , where β_i are the Bernstein coefficients.

Definition 3. With f a real valued function defined and bounded on $[0, 1]$, let $B_m(f)$ be the polynomial on $[0, 1]$, that assigns to x the value

$$B_m(f) = \sum_{i=0}^m \binom{m}{i} x^i (1-x)^{m-i} f\left(\frac{i}{m}\right) \quad (2)$$

where $B_m(f)$ is the m th Bernstein polynomials for $f(x)$ [13].

With the help of these polynomials Weierstrass approximation theorem is proved.

Theorem 4. For all functions f in $C[0, 1]$, the sequence of $B_m(f)$ converges uniformly to f , where $B_m(f)$ is defined by (2).

3. Description of Modified Technique

Let us consider the nonlinear singular boundary value problem

$$y^{(n+1)} + \frac{m}{x} y^{(n)} + Ny = g(x) \quad (3)$$

subject to the conditions

$$y(0) = a_0, y(0) = a_1, \dots, y^{r-1}(0) = a_{r-1}, y(b) = c_0, y(b) = c_1 \dots y(n-r) = c_{n-r} \quad (4)$$

where Ny is the nonlinear operator, $g(x)$ is the given source function and $a_0, a_1, \dots, a_{r-1}, c_0, c_1 \dots c_{n-r}, b$ are the known constants with $m \leq r \leq n, r \geq 1$.

Initially multiplying with x , then taking Laplace transform on both side of (3) and using the derivative property of Laplace transform, we get

$$(-1) \frac{d}{ds} [s^{n+1}L(y) - s^n y(0) - s^{n-1}y(0) \dots - y^n(0)] + \\ m[s^n L(y) - s^{n-1}y(0) - s^{n-1}y(0) \dots - y^{n-1}(0)] \\ + L[xNy] = L[xg(x)] \quad (5)$$

By substituting the conditions given in (4) and solving (5), gives

$$L(y) = -\frac{1}{s^{n+1}} [(n+1-m)s^n L(y) - (n-m)a_0 s^{n-1} \dots - (1-m)a_{n-1}] \\ - \frac{1}{s^{n+1}} L[xg(x)] + \frac{1}{s^{n+1}} L[xNy] \quad (6)$$

Now according to standard Laplace Adomian decomposition method, we write the approximate solution as:

$$y(x) = \sum_{k=0} y_k(x) \quad (7)$$

and the nonlinear term is decomposed as follows:

$$N[y(x)] = \sum_{k=0} A_k \quad (8)$$

where A_k 's are the Adomian polynomials

$$A_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[f \sum_{i=0}^k \lambda^i (u_i) \right]_{\lambda=0} \quad (9)$$

Substituting these values in (6) and comparing the terms yields the iterative algorithm:

$$L(y_0) = -\frac{1}{s^{n+1}} [(n+1-m)s^n L(y) - (n-m)a_0 s^{n-1} \dots - (1-m)a_{n-1}] \\ - \frac{1}{s^{n+1}} L[xg(x)] \quad (10)$$

$$L(y_1) = \frac{1}{s^{n+1}} L[xA_0] \quad (11)$$

$$L(y_2) = \frac{1}{s^{n+1}} L[xA_1] \quad (12)$$

In general,

$$L(y_{k+1}) = \frac{1}{s^{n+1}} L[xA_k]$$

After this integrating and employing Laplace inverse transform on (10), we obtain

$$y_0 = L^{-1} \left\{ \int \left[-\frac{1}{s^{n+1}} [(n+1-m)s^n L(y) - (n-m)a_0 s^{n-1} \dots - (1-m)a_{n-1}] ds \right] \right\} - L^{-1} \left\{ \int \frac{1}{s^{n+1}} L[xg(x)] ds \right\} \quad (13)$$

The new modification in LADM is introduced here by using the Bernstein polynomial approximation to the given function $g(x)$ in (13). After that the value of y_0 is used to compute A_0 and applying the same process to (11) and (12), gives

$$y_1 = L^{-1} \left[\int \frac{1}{s^{n+1}} L[xA_0] ds \right] \quad (14)$$

$$y_2 = L^{-1} \left[\int \frac{1}{s^{n+1}} L[xA_1] ds \right] \quad (15)$$

Thus we obtain the components $y_0, y_1, y_2 \dots$ successively. Hence the k th approximate solution of nonlinear boundary value problem is given by

$$y_k(x) = \sum_{i=0}^k y_i \quad (16)$$

Now we consider the nonsingular boundary value problem of the form

$$y^{(n+1)} + Ny = g(x), 0 < x < b \quad (17)$$

subject to the conditions

$$y(0) = a_0, y(0) = a_1, \dots, y^{r-1}(0) = a_{r-1}, y(b) = c_0 \quad (18)$$

where Ny is the nonlinear operator, $g(x)$ is the given source function and $a_0, a_1, \dots, a_{r-1}, c_0$ are given constants.

Here we are adapting the same procedure as stated above. In nonsingular BVPs, we obtain the solution components $y_0, y_1, y_2 \dots$ successively as:

$$y_0 = L^{-1} \left\{ \left[-\frac{1}{s^{n+1}} [(n+1-m)s^n L(y) - (n-m)a_0 s^{n-1} \dots - (1-m)a_{n-1}] ds \right] - L^{-1} \left\{ \frac{1}{s^{n+1}} L[g(x)] ds \right\} \right\} \quad (19)$$

$$y_1 = L^{-1} \left[\frac{1}{s^{n+1}} L[A_0] ds \right] \quad (20)$$

$$y_2 = L^{-1} \left[\frac{1}{s^{n+1}} L[A_1] ds \right] \quad (21)$$

Therefore, we attain the k th approximate solution of nonsingular boundary value problem given by (16).

4. Convergence Analysis

In this section, we study the convergence of above technique using the analysis in [14], [9]. Let us consider that $(C[I], \|\cdot\|)$ be the Banach space of all continuous function on I . Here we first convert the boundary value problem into the integral equation and discuss the convergence analysis. Then for the integral equation we also assume that the kernel given by the converted equation; i.e. $|k(x, t)| \leq S$ and the nonlinear function $F(y)$ satisfy the Lipschitz condition such that $|F(y) - F(z)| \leq T|y - z|$. For this we write the nonlinear boundary value problem in operator form

$$L(y) + q(x)F(y(x)) = g(x)$$

where $L(y)$ is the differential part, $F(y)$ is the nonlinear term, $q(x)$ be any function of x and $g(x)$ is the source term.

For instance, here we develop the following theorem for second order nonlinear boundary value problem.

Theorem 5. *The approximate solution y_n corresponding to second order nonlinear boundary value problem*

$$y + F(y(x)) = g(x), 0 < x < \pi \quad (22)$$

where we suppose $y(0) = 0$, $y(\pi) = 0$ converges to the exact solution y if $0 < \alpha < 1$, where $\alpha = \left(\frac{xT\pi}{2} + STx\right)$.

Proof. Now reformulating (22) into integral equation, for this integrate two times w.r.to x and taking limits from 0 to x and use the given boundary conditions, we have

$$y(x) - xy(0) + \int_0^x (x-t)F(y(t))dt = \int_0^x (x-t)g(t)dt \quad (23)$$

Here using the boundary condition $y(\pi) = 0$ in (23), gives the value of $y(0)$. Hence putting the value of $y(0)$ in (23), we attain the integral equation

$$y(x) = \frac{x}{\pi} \int_0^\pi (\pi-t)[F(y(t)) + g(t)]dt + \int_0^x (x-t)[F(y(t)) + g(t)]dt \quad (24)$$

Analyzing the convergence, we suppose here that y_n be the n th approximate solution of (22), therefore

$$\begin{aligned} ||y(x) - y_n(x)|| &= \max_{x \in I} \left| \frac{x}{\pi} \int_0^\pi (\pi-t)[F(y(t)) + g(t)]dt + \int_0^x (x-t)[F(y(t)) + g(t)]dt \right. \\ &\quad \left. + \frac{x}{\pi} \int_0^\pi (\pi-t)[F(y_n(t)) + B_m(g(t))]dt - \int_0^x (x-t)[F(y_n(t)) + B_m(g(t))]dt \right| \end{aligned} \quad (25)$$

where $B_m(g(t))$ is the Bernstein polynomial of function $g(t)$.

$$\begin{aligned} ||y(x) - y_n(x)|| &\leq \max_{x \in I} \frac{x}{\pi} \int_0^\pi |\pi-t| |F(y_n(t)) - F(y(t)) + B_m(g(t)) - g(t)|dt + \\ &\quad \int_0^x |x-t| |F(y(t)) + g(t) - F(y_n(t)) + B_m(g(t))|dt \end{aligned} \quad (26)$$

Adopting the convergence of Bernstein polynomial and above said conditions for kernel and nonlinear function, we have

$$\begin{aligned} ||y(x) - y_n(x)|| &\leq \max_{x \in I} \frac{x}{\pi} \int_0^\pi |\pi-t| |T| |y_n(t) - y(t)| + \epsilon dt + \\ &\quad S \int_0^x T |y(t) - y_n(t)| + \epsilon dt \end{aligned} \quad (27)$$

$$\begin{aligned} ||y(x) - y_n(x)|| &\leq \frac{xT\pi}{2} \max_{x \in I} |y - y_n| + STx \max_{x \in I} |y - y_n| \\ ||y(x) - y_n(x)|| &\leq \left(\frac{xT\pi}{2} + STx \right) \max_{x \in I} |y - y_n| \end{aligned} \quad (28)$$

Now by choosing $0 < \alpha = \left(\frac{xT\pi}{2} + STx \right) < 1$, when $n \rightarrow \infty$, then $||y(x) - y_n(x)|| \rightarrow 0$, therefore the approximate solution converges to exact one. $\square$

5. Test Problems

We are demonstrating the efficiency of our method for nonlinear singular and nonsingular boundary value problem.

Example 5.1. Consider the following nonlinear boundary value problem [6]

$$y + \frac{3}{x}y - y^3 = g(x) \quad (29)$$

$$y(0) = 0, y(1) = 0, y(1) = e \text{ where } g(x) = 24e^x + 36xe^x + 12x^2e^x + x^3e^x - x^9e^{3x}.$$

The standard LADM which provides the series solutions given by (7) and the nonlinear terms is decomposed as in (8). In this problem the nonlinear term is $Ny = y^3$, henceforth using (9) the Adomian polynomials are given by:

$$\begin{aligned} A_0 &= y_0^3 \\ A_1 &= 3y_0^2y_1 \\ A_2 &= 3y_0^2y_2 + 3y_0y_1^2 \\ A_3 &= 3y_0^2y_3 + 6y_0y_1y_2 + y_1^3 \end{aligned}$$

Using the above modified technique, we expand $g(x)$ in the terms of Bernstein polynomials of order 10,

$$\begin{aligned} g(x) \approx & -0.2461796x^{10} - 1.8849352x^9 - 5.3246642x^8 - 6.9170682x^7 - 4.0861802x^6 \\ & + 0.5894802x^5 + 9.3947682x^4 + 32.0678222x^3 + 64.3915632x^2 \\ & + 66.3644302x + 24 \end{aligned} \quad (30)$$

In Figure 1. the numerical approximate solution obtained from presented technique is compared with exact solution that exhibits the effectiveness of method.

Example 5.2. Consider the nonlinear boundary value problem [12]

$$y - y^3 = -\sin x(1 + \sin^2 x) \quad (31)$$

where $y(0) = 0, y(\pi) = 0$ with exact solution $y(x) = \sin x$.

Now using the above modified technique, we expand $g(x)$ in the terms of Bernstein polynomials of order 5,

$$\begin{aligned} g(x) \approx & 0.000788114x^5 + 0.136755981x^4 - 0.1877757x^3 \\ & - 0.354509063x^2 - 1.032553553x \end{aligned} \quad (32)$$

Figure 1: Comparison of Approximate solution with Exact solution

Figure 2. shows that expanding the source term in the Bernstein polynomials, the solution is very much close to the exact solution which proves that our method is well suited to this problem.

6. Discussion

Bernstein polynomials are exclusively used to modify the standard Laplace decomposition method. The proposed new technique is analyzed and applied to the nonlinear singular and nonsingular boundary value problems. The results shown in the figures exclude that the numerical approximations are in good agreement with the exact one. The convergence analysis exhibits the applicability of technique.

References

- [1] M. El-Gamel, Sinc-Galerkin method for solving nonlinear boundary value problems, *Computers and Mathematics with Applications*, **48** (2004), 1285-1298.
- [2] D. J. Evans and K. R. Raslan, The Adomian decomposition method for solving delay

Figure 2: Comparison of Approximate solution with Exact solution

differential equation, *International Journal of Computer Mathematics*, **82** (2005), 49-54, **doi:** 10.1080/00207160412331286815.

- [3] R. T. Farouki, The Bernstein polynomial basis: A centennial retrospective, *Computer Aided Geometric Design*, **29** (2012), 379-419, **doi:** 10.1016/j.cagd.2012.03.001.
- [4] M. M. Hosseini, Adomian decomposition method for solution of nonlinear differential algebraic equations, *Applied Mathematics and Computation*, **181** (2006), 1737-1744, **doi:** 10.1016/j.amc.2006.03.027.
- [5] S. A. Khuri, A Laplace decomposition algorithm applied to a class of nonlinear differential equation, *Journal of Applied Mathematics*, **1** (2001), 141-155, **doi:** 10.1155/S1110757X01000183.
- [6] W. Kim and C. Chun, A modified Adomian decomposition method for solving higher-order singular boundary value problems, *Verlag der Zeitschrift fur Naturforschung*, **65a** (2010), 1093-1100, **doi:** 10.1515/zna-2010-1213.
- [7] A. Kumar and R. D. Pankaj, Laplace decomposition method to study solitary wave solutions of coupled nonlinear partial differential equation, *ISRN Computational Mathematics*, **2012** (2012), 1-5, **doi:** 10.5402/2012/423469.
- [8] Y. Liu, Adomian decomposition method with orthogonal polynomials: Legendre polynomial, *Mathematical and Computer Modelling*, **49** (2009), 1268-1273, **doi:** 10.1016/j.mcm.2008.06.020.
- [9] F. Mirzaee and A. A. Hoseini, Numerical solution of nonlinear volterra fredholm integral equations using hybrid block pulse functions and taylor series, *Alexandria Engineering Journal*, **52**, (2013), 551-555, **doi:** 10.1016/j.aej.2013.02.004.

- [10] H. K. Mishra, He-Laplace method for the solution of two-point boundary value problems, *American Journal of Mathematical Analysis*, **2** (2014), 45-49.
- [11] M. A. Mohamed and M. S. Torky, Numerical solution of nonlinear system of partial differential equations by the Laplace decomposition method and the pade approximation, *American Journal of Computational Mathematics*, **3** (2013), 175-184, **doi:** 10.4236/ajcm.2013.33026.
- [12] A. Mohsena and M. El-Gamelb, On the Galerkin and collocation methods for two-point boundary value problems using sinc bases, *Computers and Mathematics with Applications*, **56** (2008), 930-941, **doi:** 10.1016/j.camwa.2008.01.023.
- [13] W. Quain, M. D. Riedel and I. Rosenberg, Uniform approximation and bernstein polynomial with coefficients in the unit interval, *European Journal of Combinatorics*, **32** (2011), 448-463.
- [14] K. Sayevand, M. Fardi, E. Moradi and F. H. Boroujeni, Convergence analysis of homotopy perturbation method for Volterra integro-differential equations of fractional order, *Alexandria Engineering Journal*, **52** (2013), 807-812.
- [15] A. M. Wazwaz, The combined Laplace transform-Adomian decomposition method for handling nonlinear volterra integro-differential equations, *Applied Mathematics and Computation*, **216** (2010), 1304-1309.
- [16] S. Widatalla and M. Z. Liu, New iterative method based on Laplace decomposition algorithm, *Journal of Applied Mathematics*, **2013** (2013), 1-7, **doi:** 10.1155/2013/286529.
- [17] C. Yang and J. Hou, An approximate solution of nonlinear fractional differential equation by Laplace transform and Adomian polynomials, *Journal of Information and Computational Science*, **10** (2013), 213-222.
- [18] F. K. Yin, W. Y. Han and J. Q. Song, Modified Laplace decomposition method for Lane-Emden type differential equations, *International Journal of Applied Physics and Mathematics*, **3** (2013), 98-102, **doi:** 10.7763/IJAPM.2013.V3.184.
- [19] J. B. Yindoula, P. Yousseuf, G. Bissanga, F. Bassono and B. Some, Application of the Adomian decomposition method and Laplace transform method to solving the convection diffusion-dissipation equation, *International Journal of Applied Mathematical Research*, **3** (2014), 30-35, **doi:** 10.14419/ijamr.v3i1.1596.
- [20] S. A. Yousefi and M. Behroozifar, Operational matrices of Bernstein polynomials and their applications, *International Journal of Systems Science*, **41** (2010), 709-716, **doi:** 10.1080/00207720903154783.
- [21] S. Zuhra, S. Islam, M. Idress, R. Nawaz, I. A. Shah and H. Ullah, Solving singular boundary value problems by optimal homotopy asymptotic method, *International Journal of Differential Equations*, **2014** (2014), 1-10, **doi:** 10.1155/2014/287480.