ON $T_i^{\widetilde{\alpha^m}}$ $i=0,1,2$ -SPACES
IN TOPOLOGICAL SPACES

K. Sivakumar¹ §, R. Parimelazhagan²

¹Department of Mathematics

Sathyabama University

Chennai - 119, Tamilnadu, INDIA

²Department of Mathematics

RVS Technical Campus

Coimbatore - 641 402, Tamilnadu, INDIA

Abstract: In this paper, we introduce the spaces called $T_0^{\widetilde{\alpha^m}}$, $T_1^{\widetilde{\alpha^m}}$ and $T_2^{\widetilde{\alpha^m}}$ by using α^m -open sets in topological spaces and study the characterizations and their properties.

AMS Subject Classification: 57N40

Key Words: $T_0^{\widetilde{\alpha^m}}$, $T_1^{\widetilde{\alpha^m}}$ and $T_2^{\widetilde{\alpha^m}}$

1. Introduction

In 1967, A. Wilansky [1] has introduced the concept of US spaces. In 1968, C. E. Aull [2] studied some separation axioms between the T_1 and T_2 spaces, namely, S_1 and S_2 . Next, in 1982, S. P. Arya et al.[3] have introduced and studied the concept of semi- US spaces and also they made study of s -convergence, sequentially semi-closed sets, sequentially s -compact notions. G. B. Navlagi studied P -Normal Almost- P -Normal and Mildly- P -Normal spaces. Closedness are basic concept for the study and investigation in general topological spaces. This concept has been generalized and studied by many authors from different points of views.

Received: February 10, 2017

Revised: April 19, 2017

Published: June 7, 2017

© 2017 Academic Publications, Ltd.

url: www.acadpubl.eu

§Correspondence author

O. Njastad [4] introduced and defined an α -open and α -closed set. After the works of O. Njastad on α -open sets, various mathematicians turned their attention to the generalizations of various concepts in topology by considering semi-open, α -open sets. The concept of g -closed [5], s -open [6] and α -open sets has a significant role in the generalization of continuity in topological spaces. The modified form of these sets and generalized continuity were further developed by many mathematicians ([7], [8], [9], [10], [11]).

In 1970, Levine generalized the concept of closed sets to generalized closed sets. After that there is a vast progress occurred in the field of generalized open sets (complement of respective closed sets). In topological spaces, it is well known that normality is preserved under closed continuous surjections. Many authors have tried to weaken the condition “closed” in this theorem. In 1978, Long and Herrington [12] used almost closedness due to Singal [13].

Quite recently, we have introduced the notion of α^m -closed sets which are strictly weaker than α -closed sets. We use α^m -open sets to define a spaces called $T_i\widehat{\alpha^m}_{i=0,1,2}$ -Spaces. Also we derive the properties of α^m -interior in topological spaces.

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned.

For a subset A of a space (X, τ) , $cl(A)$, $int(A)$ and A^c denote the closure of A , the interior of A and the complement of A in X , respectively.

Definition 2.1. A subset A of a topological space (X, τ) is called:

- (a) a preopen set [14] if $A \subseteq int(cl(A))$ and pre-closed set if $cl(int(A)) \subseteq A$.
- (b) a semiopen set [6] if $A \subseteq cl(int(A))$ and semi closed set if $int(cl(A)) \subseteq A$.
- (c) an α -open set [4] if $A \subseteq int(cl(int(A)))$ and an α -closed set if $cl(int(cl(A))) \subseteq A$.
- (d) a semi-preopen set [15] (β -open set) if $A \subseteq cl(int(cl(A)))$ and semi-preclosed set if $int(cl(int(A))) \subseteq A$.
- (e) an α^m -closed set [16] if $int(cl(A)) \subseteq U$, whenever $A \subseteq U$ and U is α -open. The complement of α^m -closed set is called an α^m -open set.

Definition 2.2. A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is called

- (a) an α^m -continuous [17] if $f^{-1}(V)$ is α^m -closed in (X, τ) for every closed set V of (Y, σ) .
- (b) an α^m -irresolute [17] if $f^{-1}(V)$ is α^m -closed in (X, τ) for every α^m -closed set V of (Y, σ) .

Definition 2.3. [18] (a) Let X be a topological space and let $x \in X$. A subset N of X is said to be α^m -nbhd of x if there exists an α^m -open set G such that $x \in G \subset N$.

The collection of all α^m -nbhd of $x \in X$ is called an α^m -nbhd system at x and shall be denoted by $n^{\#_{\alpha^m}}(x)$.

(b) Let X be a topological space and A be a subset of X , A subset N of X is said to be α^m -nbhd of A if there exists an α^m -open set G such that $A \subseteq G \subseteq N$.

(c) Let A be a subset of X . A point $x \in A$ is said to be an α^m -interior point of A , if A is an $n^{\#_{\alpha^m}}(x)$. The set of all α^m -interior points of A is called an α^m -interior of A and is denoted by $I_{\alpha^m}(A)$.

$$I_{\alpha^m}(A) = \bigcup \{G : G \text{ is } \alpha^m\text{-open, } G \subset A\}.$$

(d) Let A be a subset of X . A point $x \in A$ is said to be an α^m -closure of A . Then

$$C_{\alpha^m}(A) = \bigcap \{F : A \subset F \in \alpha^m C(X, \tau_X)\}.$$

3. On $T_i \widetilde{\alpha^m}_{i=0,1,2}$ -Spaces

Definition 3.1. A topological space (X, τ) is said to be

(a) $T_0 \widetilde{\alpha^m}$ if for each pair of distinct points x, y in X , there exists a α^m -open set U such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.

(b) $T_1 \widetilde{\alpha^m}$ if for each pair of distinct points x, y in X , there exist two α^m -open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

(c) $T_2 \widetilde{\alpha^m}$ if for each distinct points x, y in X , there exist two disjoint α^m -open sets U and V containing x and y respectively.

Proposition 3.2. A topological space (X, τ) is $T_0 \widetilde{\alpha^m}$ if and only if for each pair of distinct points x, y of X , $C_{\alpha^m}(\{x\}) \neq C_{\alpha^m}(\{y\})$.

Proof. Necessity. Let (X, τ) be a $T_0 \widetilde{\alpha^m}$ -space and x, y be any two distinct points of X . There exists a α^m -open set U containing x or y , say x but not y . Then X/U is a α^m -closed set which does not contain x but contains y . Since $C_{\alpha^m}(\{y\})$ is the smallest α^m -closed set containing y , $C_{\alpha^m}(\{y\}) \subseteq X/U$ and therefore $x \notin C_{\alpha^m}(\{y\})$. Consequently $C_{\alpha^m}(\{x\}) \neq C_{\alpha^m}(\{y\})$.

Sufficiency. Suppose that $x, y \in X$, $x \neq y$ and $C_{\alpha^m}(\{x\}) \neq C_{\alpha^m}(\{y\})$. Let z be a point of X such that $z \in C_{\alpha^m}(\{x\})$ but $z \notin C_{\alpha^m}(\{y\})$. We claim that $x \notin C_{\alpha^m}(\{y\})$. For, if $x \in C_{\alpha^m}(\{y\})$ then $C_{\alpha^m}(\{x\}) \subseteq C_{\alpha^m}(\{y\})$. This contradicts the fact that $z \notin C_{\alpha^m}(\{y\})$. Consequently x belongs to the α^m -open set $X/C_{\alpha^m}(\{y\})$ to which y does not belong. $\square$

Proposition 3.3. A topological space (X, τ) is $T_1^{\widetilde{\alpha^m}}$ if and only if the singletons are α^m -closed sets.

Proof. Let (X, τ) be $T_1^{\widetilde{\alpha^m}}$ and x any point of X . Suppose $y \in X/\{x\}$, then $x \neq y$ and so there exists a α^m -open set U such that $y \in U$ but $x \notin U$. Consequently $y \in U \subseteq X/\{x\}$, that is $X/\{x\} = \cup\{U : y \in X/\{x\}\}$ which is α^m -open.

Conversely, suppose $\{p\}$ is α^m -closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X/\{x\}$. Hence $X/\{x\}$ is a α^m -open set contains y but not x . Similarly $X/\{y\}$ is a α^m -open set contains x but not y . Accordingly X is a $T_1^{\widetilde{\alpha^m}}$ -space. $\square$

Theorem 3.4. Let A and B be subsets of X . Then

- (a) $I_{\alpha^m}(X) = X$ and $I_{\alpha^m}(\phi) = \phi$,
- (b) $I_{\alpha^m}(A) \subset A$,
- (c) If B is any α^m -open set contained in A , then $B \subset I_{\alpha^m}(A)$,
- (d) If $A \subset B$, then $I_{\alpha^m}(A) \subset I_{\alpha^m}(B)$.

Proof. (a) Since X and ϕ are α^m -open sets, $I_{\alpha^m}(X) = \bigcup\{G : G \text{ is } \alpha^m\text{-open, } G \subset X\} = X \bigcup \{\text{all } \alpha^m\text{-open sets}\} = X$. That is $I_{\alpha^m}(X) = X$. Since ϕ is the only α^m -open set contained in ϕ , $I_{\alpha^m}(\phi) = \phi$.

(b) Let $x \in I_{\alpha^m}(A) \Rightarrow x$ is an α^m -interior point of A .

$\Rightarrow A$ is an $n^{\#_{\alpha^m}}(x)$.

$\Rightarrow x \in A$.

Thus $x \in I_{\alpha^m}(A) \Rightarrow x \in A$. Hence $I_{\alpha^m}(A) \subset A$.

(c) Let B be any α^m -open set such that $B \subset A$. Let $x \in B$, since B is an α^m -open set contained in A , then x is an α^m -interior point of A . That is $x \in I_{\alpha^m}(A)$. Hence $B \subset I_{\alpha^m}(A)$.

(d) Let A and B be subsets of X such that $A \subset B$. Let $x \in I_{\alpha^m}(A)$. Then x is an α^m -interior point of A and so A is an $n^{\#_{\alpha^m}}(x)$. Since $B \supset A$, B is also an $n^{\#_{\alpha^m}}(x)$. This implies that $x \in I_{\alpha^m}(B)$. Thus we have shown that $x \in I_{\alpha^m}(A) \Rightarrow x \in I_{\alpha^m}(B)$. Hence $I_{\alpha^m}(A) \subset I_{\alpha^m}(B)$. $\square$

Proposition 3.5. The following statements are equivalent for a topological space (X, τ) :

(a) X is $T_2^{\widetilde{\alpha^m}}$.

(b) Let $x \in X$. For each $y \neq x$, there exists a α^m -open set U containing x such that $y \notin C_{\alpha^m}(U)$.

(c) For each $x \in X$, $\cap\{C_{\alpha^m}(U) : U \in \alpha^m O(X) \text{ and } x \in U\} = \{x\}$.

Proof. (a) $\implies$ (b) Since X is $T_2 \widetilde{\alpha^m}$, there exist disjoint α^m -open sets U and V containing x and y respectively. So, $U \subseteq X/V$. Therefore, $C_{\alpha^m}(U) \subseteq X/V$. So $y \notin C_{\alpha^m}(U)$.

(b) $\implies$ (c) If possible for some $y \neq x$, we have $y \in C_{\alpha^m}(U)$ for every α^m -open set U containing x , which then contradicts (b).

(c) $\implies$ (a) Let $x, y \in X$ and $x \neq y$. Then there exists a α^m -open set U containing x such that $y \notin C_{\alpha^m}(U)$. Let $V = X/C_{\alpha^m}(U)$, then $y \in V$ and $x \in U$ and also $U \cap V = \phi$. $\square$

Theorem 3.6. If A and B are subsets of X , then $I_{\alpha^m}(A) \cup I_{\alpha^m}(B) \subset I_{\alpha^m}(A \cup B)$.

Proof. We know that $A \subset A \cup B$ and $B \subset A \cup B$, $I_{\alpha^m}(A) \subset I_{\alpha^m}(A \cup B)$ and $I_{\alpha^m}(B) \subset I_{\alpha^m}(A \cup B)$. This implies that $I_{\alpha^m}(A) \cup I_{\alpha^m}(B) \subset I_{\alpha^m}(A \cup B)$. $\square$

Proposition 3.7. Let (X, τ) be a topological space, then every $T_2 \widetilde{\alpha^m}$ space is $T_1 \widetilde{\alpha^m}$.

Proof. The proof is straightforward from the definitions. $\square$

Theorem 3.8. If A and B are subsets of a space X , then $I_{\alpha^m}(A \cap B) = I_{\alpha^m}(A) \cap I_{\alpha^m}(B)$.

Proof. We know that $A \cap B \subset A$ and $A \cap B \subset B$, $I_{\alpha^m}(A \cap B) \subset I_{\alpha^m}(A)$ and $I_{\alpha^m}(A \cap B) \subset I_{\alpha^m}(B)$. This implies that $I_{\alpha^m}(A \cap B) \subset I_{\alpha^m}(A) \cap I_{\alpha^m}(B) \longrightarrow (i)$.

Again, let $x \in I_{\alpha^m}(A) \cap I_{\alpha^m}(B)$, then $x \in I_{\alpha^m}(A)$ and $x \in I_{\alpha^m}(B)$. Hence x is an α^m -interior point of each sets A and B . It follows that A and B are $n^{\# \alpha^m}(x)$, so that their intersection $A \cap B$ is also an $n^{\# \alpha^m}(x)$. Hence $x \in I_{\alpha^m}(A \cap B)$. Thus $x \in I_{\alpha^m}(A) \cap I_{\alpha^m}(B)$ implies that $x \in I_{\alpha^m}(A \cap B)$.

Therefore $I_{\alpha^m}(A) \cap I_{\alpha^m}(B) \subset I_{\alpha^m}(A \cap B) \longrightarrow (ii)$.

From (i) and (ii), we get $I_{\alpha^m}(A \cap B) = I_{\alpha^m}(A) \cap I_{\alpha^m}(B)$. $\square$

References

- [1] A. Wilansky, Between T_1 and T_2 , Amer.Math. Monthly. 74 (1967), 261-266.
- [2] C.E. Aull, Sequences in topological spaces, Comm. Math. (1968), 329-36.
- [3] S.P. Arya and M.P. Bhamini, A note on semi-US spaces, Ranchi Uni. Math. J. Vol. 13 (1982), 60-68.
- [4] Njastad. O, On some classes of nearly open sets, Pacific J. Math., , 15 (1965), 961-970.
- [5] Levine. N, Generalized closed sets in topology, Rend. Circ. Math. Palermo, 19 (1970), 89-96.

- [6] Levine. N, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.
- [7] K. Balachandran, P. Sundran and H. Maki, On Generalized Continuous Maps In Topological Spaces. Mem. Fac.Sci. Kochi Univ., ser. A. math.,12, 5-13(1991).
- [8] R. Devi, H. Maki and K. Balachandran, Semi-Generalised Closed Maps And Generalized Semi-Closed Maps, Mem. Fac. Sci. Kochi Univ. Ser. A. math., 14, 41-54(1993).
- [9] I. Arockiarani, Studies on Generalizations of Generalized Closed Sets and Maps in Topological Spaces, Ph. D Thesis, Bharathiar University, Coimbatore, (1997).
- [10] A. S.Mashhour, I.A.Hasanein, S.N.El-Deeb: α -continuous and α -open mappings. Acta-Math. Hung. 41 (1983), 213-218.
- [11] Mashhour. A. S, Abd EI-Monsef. M. E. and EI-Deeb. S. N., On Precontinuous and weak pre-continuous mapping, Proc. Math., Phys. Soc. Egypt, 53 (1982), 47-53.
- [12] P. E. Long and L. L. Herington, Basic Properties of Regular Closed Functions, Rend. Cir. Mat. Palermo, 27(1978), 20-28.
- [13] M.K. Singal, A.R. Singal: Almost-continuous mappings. Yokohama Math. J. 16 (1968), 63-73.
- [14] S.R.Malghan, Generalized closed maps, J. Karnataka Univ. Sci., 27(1982), 82-88.
- [15] Andrijevic. D, Semi-preopen sets, Mat. Vesnik, 38 (1986), 24-32.
- [16] Milby Mathew and R. Parimelazhagan, α^m -Closed Sets in Topological Spaces, International Journal of Mathematical Analysis Vol. 8, 2014, no. 47, 2325-2329.
- [17] Milby Mathew and R. Parimelazhagan, α^m -Continuous Maps in Topological Spaces. (submitted)
- [18] R. Parimelazhagan and Milby Mathew, On $I_{\alpha^m}(A)$, $C_{\alpha^m}(A)$ and α^m -open and closed Functions (submitted).