A COMMON FIXED POINT THEOREM UNDER AN AUXILIARY FUNCTION

T. Phaneendra^{1 §}, Swatmaram²

¹Department of Mathematics

School of Advanced Sciences,

VIT University

Vellore, 632014, Tamil Nadu, INDIA

²Department of Mathematics

Chaitanya Bharati Institute of Technology

Ranga Reddy District-500 075, Telangana, INDIA

Abstract: A generalization of a result of Badshah and Singh [1] was proved in [5] for a pair of compatible maps and dropping the continuity of one of the self-maps. A generalization of the result of [5] is obtained in this paper, by employing an auxiliary function.

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1. Introduction

Badshah and Singh [1] proved the following result for commuting self-maps:

Theorem 1.1. *Let f and g be self-maps on a complete metric space X satisfying the inclusion*

$$f(X) \subset g(X) \tag{1}$$

and the inequality

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[§]Correspondence author

$$\begin{aligned}
[d(fx, fy)]^2 \leq & \alpha [d(fx, gx)d(fy, gy) + d(fy, gx)d(fx, gy)] \\
& + \beta [d(fx, gx)d(fx, gy) + d(fy, gx)d(fy, gy)] \\
& \text{for all } x, y \in X,
\end{aligned} \tag{2}$$

where

- (a) α and β are nonnegative constants with $\alpha + 2\beta < 1$,
- (b) (f, g) is a commuting pair,
- (c) f and g are continuous.

Then f and g have a unique common fixed point.

A generalization of Theorem 1.1 was obtained in [5], by dropping the continuity of f and using a compatible pair¹ (f, g) in (b) with the choice:

$$\lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0 \tag{3}$$

whenever $\langle x_n \rangle_{n=0}^{\infty}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t \tag{4}$$

for some $t \in X$.

It is easy to observe that every commuting pair of self-maps is necessarily compatible. Converse is not true. For instance, see [2], [3] and [4].

The generalization proved in [5] is the following:

Theorem 1.2. *Let f and g be self-maps on a complete metric space X satisfying the inclusion (1) and the inequality (2), where α and β are nonnegative constants with $\alpha + 2\beta < 1$. If g is continuous, and (f, g) is a compatible pair, then f and g have a unique common fixed point.*

We prove a generalization of Theorem 1.2 by replacing (2) with a general inequality involving an auxiliary function.

¹Compatible maps was introduced by Gerald Jungck [2] as a generalization of commuting maps

2. Preliminary Notations

Several fixed point theorems in metric space setting have been proved through contraction conditions involving different types of auxiliary functions. Given a positive integer α , a generalized class Φ_α of auxiliary functions was introduced in [6] as follows:

$$\Phi_\alpha = \{\phi : [0, \infty) \rightarrow [0, \infty) | \phi(0) = 0, \phi(\alpha t) < t \text{ for } t > 0\}. \quad (5)$$

It is obvious that, for $\alpha = 1$, Φ_α reduces to the class Ψ of all contractive moduli ψ [7] such that $\psi(0) = 0$ and $\psi(t) < t$ for $t > 0$.

Definition 2.1. A mapping $\phi \in \Phi_\alpha$ is said to be upper semicontinuous at $t_0 \geq 0$ if $\limsup_{n \rightarrow \infty} \phi(t_n) \leq \phi(t_0)$ whenever $\{t_n\}_{n=1}^\infty$ is such that $\lim_{n \rightarrow \infty} t_n = t_0$, and ϕ is u.s.c if it is u.s.c. at every $t \geq 0$.

Our main result is

Theorem 2.1. Let f and g be self-maps on a complete metric space X satisfying the inclusion (1), and the inequality

$$\begin{aligned} [d(fx, fy)]^2 &\leq \phi(\max\{d(fx, gx)d(fy, gy) + d(fy, gx)d(fx, gy), \\ &\quad d(fx, gx)d(fx, gy) + d(fy, gx)d(fy, gy)\}) \\ &\quad \text{for all } x, y \in X, \end{aligned} \quad (6)$$

where $\phi \in \Phi_2$ is nondecreasing and upper semicontinuous. If g is continuous, and (f, g) is a compatible pair, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary.

In view of (1), we can choose points $x_1, x_2, \dots, x_n, \dots$ in X inductively such that

$$fx_{n-1} = gx_n = y_n \quad \text{for all } n \geq 1. \quad (7)$$

Writing $x = x_{n-1}$ and $y = x_n$ in (6) and using (7), we get

$$\begin{aligned} [d(y_n, y_{n+1})]^2 &= [d(fx_{n-1}, fx_n)]^2 \\ &\leq \phi(\max\{d(fx_{n-1}, gx_{n-1})d(fx_n, gx_n) + d(fx_n, gx_{n-1})d(fx_{n-1}, gx_n), \\ &\quad d(fx_{n-1}, gx_{n-1})d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})d(fx_n, gx_n)\}) \\ &= \phi(\max\{d(y_n, y_{n-1})d(y_{n+1}, y_n), d(y_{n+1}, y_{n-1})d(y_{n+1}, y_n)\}) \\ &\leq \phi(d(y_n, y_{n+1})[d(y_n, y_{n-1}) + d(y_{n+1}, y_n)]) \end{aligned} \quad (8)$$

We now prove that

$$d(y_n, y_{n-1}) \geq d(y_{n+1}, y_n) \text{ for } n \geq 2. \quad (9)$$

If possible, suppose that $d(y_m, y_{m-1}) < d(y_{m+1}, y_m)$ for some $m \geq 2$. Then $d(y_{m+1}, y_m) > 0$. Since ϕ is nondecreasing, from (8) it follows that

$$0 < [d(y_{m+1}, y_m)]^2 \leq \phi(2[d(y_m, y_{m+1})]^2) < [d(y_{m+1}, y_m)]^2,$$

which is a contradiction. This proves (9). In other words, $\langle d(y_{n+1}, y_n) \rangle_{n=1}^\infty$ is a decreasing sequence of nonnegative real numbers and hence converges to some $t \geq 0$. Now using (9) in (8), we get

$$d(y_{n+1}, y_n) \leq \phi(d(y_{n+1}, y_n) + d(y_{n+2}, y_{n+1})) \leq \phi(2d(y_{n+1}, y_n)) \text{ for } n \geq 1.$$

Taking the limit superior as $n \rightarrow \infty$ in this and then using the upper semicontinuity of ϕ , we obtain that

$$t \leq \phi(2t). \quad (10)$$

If $t > 0$ in (10), then the choice of ϕ implies that $t \leq \phi(2t) < t$, which is a contradiction. Thus

$$t = \lim_{n \rightarrow \infty} d(y_{n+1}, y_n) = \lim_{n \rightarrow \infty} d(y_{n+1}, y_n) = 0. \quad (11)$$

We now prove that $\langle y_n \rangle_{n=1}^\infty$ is a Cauchy sequence in X .

If possible we suppose that $\langle y_n \rangle_{n=1}^\infty$ is not Cauchy. Then for some $\epsilon > 0$, we choose sequences $\langle y_{m_k} \rangle_{k=1}^\infty$ and $\langle y_{n_k} \rangle_{k=1}^\infty$ of positive integers such that $m_k > n_k > k$ and

$$d(y_{m_k}, y_{n_k}) \geq \epsilon \text{ for } k = 1, 2, 3, \dots \quad (12)$$

Suppose that m_k is the smallest integer exceeding n_k which satisfies (12). That is

$$d(y_{m_k-1}, y_{n_k}) < \epsilon. \quad (13)$$

Now by triangle inequality of d , we see that

$$\begin{aligned} \epsilon &\leq d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{m_k-1}) + d(y_{m_k-1}, y_{n_k}) \\ &< d(y_{m_k}, y_{m_k-1}) + \epsilon \end{aligned} \quad (14)$$

and from (11), we see that

$$\lim_{k \rightarrow \infty} d(y_{m_k-1}, y_{m_k}) = 0 \quad (15)$$

and

$$\lim_{k \rightarrow \infty} d(y_{n_k-1}, y_{n_k}) = 0 \quad (16)$$

Using (15) in (14), we get

$$\lim_{k \rightarrow \infty} d(y_{m_k}, y_{n_k}) = \epsilon. \quad (17)$$

Again, by the triangle inequality of d , we get

$$d(y_{n_k-1}, y_{m_k}) \leq d(y_{n_k-1}, y_{n_k}) + d(y_{n_k}, y_{m_k}).$$

As $k \rightarrow \infty$ this in view of (16) and (17), gives

$$\lim_{k \rightarrow \infty} d(y_{n_k-1}, y_{m_k}) = \epsilon. \quad (18)$$

On the other hand, writing $x = x_{m_k-1}$, $y = x_{n_k-1}$ in (6), we have

$$\begin{aligned} [d(fx_{m_k-1}, fx_{n_k-1})]^2 &\leq \phi(\max\{d(fx_{m_k-1}, gx_{m_k-1})d(fx_{n_k-1}, gx_{n_k-1}) \\ &\quad + d(fx_{n_k-1}, gx_{m_k-1})d(fx_{m_k-1}, gx_{n_k-1}), \\ &\quad d(fx_{m_k-1}, gx_{m_k-1})d(fx_{m_k-1}, gx_{n_k-1}) \\ &\quad + d(fx_{n_k-1}, gx_{m_k-1})d(fx_{n_k-1}, gx_{n_k-1})\}) \end{aligned}$$

or

$$\begin{aligned} \epsilon^2 &\leq [d(y_{m_k}, y_{n_k})]^2 \\ &\leq \phi(\max\{d(y_{m_k}, y_{m_k-1}))d(y_{n_k}, y_{n_k-1}) + d(y_{n_k}, y_{m_k-1})d(y_{m_k}, y_{n_k-1}), \\ &\quad d(y_{m_k}, y_{m_k-1})d(y_{m_k}, y_{n_k-1}) + d(y_{n_k}, y_{m_k-1})d(y_{n_k}, y_{n_k-1})\}) \end{aligned} \quad (19)$$

Since ϕ is nondecreasing, proceeding the limit as $n \rightarrow \infty$ in this, and then using upper semicontinuity of ϕ , (13), (15), (16), (17) and (18) we get

$$0 < \epsilon^2 \leq \phi(\max\{0 + \epsilon^2, 0\}) = \phi(\epsilon^2) \leq \phi(2\epsilon^2) < \epsilon^2,$$

which is a contradiction. Hence $\langle y_n \rangle_{n=1}^\infty$ must be a G -Cauchy sequence in X .

Since (X, G) is G -Complete, there exists a point $p \in X$ such that $\langle y_n \rangle_{n=1}^\infty$ is G -convergent to p . That is

$$\lim_{n \rightarrow \infty} y_{n-1} = \lim_{n \rightarrow \infty} y_n = p. \quad (20)$$

Now the compatibility of f and g , and (20) imply that

$$\lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0, \quad (21)$$

while the sequential property of the continuity of g and (20) give

$$\lim_{n \rightarrow \infty} gfx_n = \lim_{n \rightarrow \infty} g^2x_n = gz. \quad (22)$$

Hence it follows from (21) and (22), that

$$\lim_{n \rightarrow \infty} d(fgx_n, gz) = 0 \quad \text{or} \quad \lim_{n \rightarrow \infty} gfx_n = gz. \quad (23)$$

But the use of (6) yields

$$\begin{aligned} [d(fgx_n, fz)]^2 &\leq \phi(\max\{d(fgx_n, g^2x_n)d(fz, gz) + d(fz, g^2x_n)d(fgx_n, gz), \\ &\quad d(fgx_n, g^2x_n)d(fgx_n, gz) + d(fz, g^2x_n)d(fz, gz)\}). \end{aligned}$$

Applying the limit as $n \rightarrow \infty$ in this, and using (22) and (23), we obtain that

$$\begin{aligned} [d(gz, fz)]^2 &\leq \phi(\max\{d(gz, gz)d(fz, gz) + d(fz, gz)d(gz, gz), \\ &\quad d(gz, gz)d(gz, gz) + d(fz, gz)d(fz, gz)\}). \end{aligned}$$

or

$$[d(gz, fz)]^2 \leq \phi([d(fz, gz)]^2).$$

If $fz \neq gz$, then the nondecreasing nature of ϕ would lead to a contradiction that

$$0 < [d(gz, fz)]^2 \leq \phi([d(fz, gz)]^2) \leq \phi(2[d(fz, gz)]^2) < [d(fz, gz)]^2.$$

Hence we must have

$$gz = fz. \quad (24)$$

Finally from (6), we see that

$$[d(fx_n, fz)]^2 \leq \phi(\max\{d(fx_n, gx_n)d(fz, gz) + d(fz, gx_n)d(fx_n, gz),$$

$$d(fx_n, gx_n)d(fx_n, gz) + d(fz, gx_n)d(fz, gz).$$

The limiting case of this as $n \rightarrow \infty$, (20), and (22) would imply that

$$[d(z, fz)]^2 \leq \phi([d(fz, z)]^2),$$

which with a similar argument as above yields that $d(z, fz) = 0$ or $fz = z$. Thus z is a common fixed point of f and g .

The uniqueness of the common fixed point follows easily from (6). $\square$

Remark 2.1. Theorem 2.1 does not require the continuity of f .

Since every commuting pair is compatible, writing $\phi(t) = qt$ for all $t \geq 0$, where $q < 1/2$, we obtain

Corollary 2.1. *Let f and g be self-maps on a complete metric space X satisfying the inclusion (1), and the inequality*

$$\begin{aligned} [d(fx, fy)]^2 &\leq q \max\{d(fx, gx)d(fy, gy) + d(fy, gx)d(fx, gy), \\ &\quad d(fx, gx)d(fx, gy) + d(fy, gx)d(fy, gy)\} \\ &\text{for all } x, y \in X, \end{aligned} \quad (25)$$

If g is continuous, and (f, g) is a commuting, then f and g have a unique common fixed point.

Choosing α and β such that $\alpha + 2\beta < 1/2$, then it is easily seen that the right hand side of (2) is less than or equal to the right hand side of (25), where $r = \alpha + 2\beta$. Thus Theorem 1.2 will become a particular case of Corollary 2.1.

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