A COMMON FIXED POINT THEOREM
UNDER AN AUXILIARY FUNCTION

T. Phaneendra$^1$, Swatmaram$^2$

$^1$Department of Mathematics
School of Advanced Sciences,
VIT University
Vellore, 632014, Tamil Nadu, INDIA

$^2$Department of Mathematics
Chaitanya Bharati Institute of Technology
Ranga Reddy District-500 075, Telangana, INDIA

Abstract: A generalization of a result of Badshah and Singh [1] was proved in [5] for a pair of compatible maps and dropping the continuity of one of the self-maps. A generalization of the result of [5] is obtained in this paper, by employing an auxiliary function.

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1. Introduction

Badshah and Singh [1] proved the following result for commuting self-maps:

**Theorem 1.1.** Let $f$ and $g$ be self-maps on a complete metric space $X$ satisfying the inclusion

$$f(X) \subseteq g(X)$$

and the inequality

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$^\S$Correspondence author
\[ [d(fx, fy)]^2 \leq \alpha [d(fx, gx)d(fy, gy) + d(fy, gx)d(fx, fy)] + \beta [d(fx, gx)d(fx, gy) + d(fy, gx)d(fy, gy)] \]

\[ \text{for all } x, y \in X, \] (2)

where

(a) \( \alpha \) and \( \beta \) are nonnegative constants with \( \alpha + 2\beta < 1 \),

(b) \((f, g)\) is a commuting pair,

(c) \( f \) and \( g \) are continuous.

Then \( f \) and \( g \) have a unique common fixed point.

A generalization of Theorem 1.1 was obtained in [5], by dropping the continuity of \( f \) and using a compatible pair\(^1\) \((f, g)\) in (b) with the choice:

\[ \lim_{n \to \infty} d(fgx_n, gfx_n) = 0 \] (3)

whenever \( \langle x_n \rangle \) is a sequence in \( X \) such that

\[ \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \] (4)

for some \( t \in X \).

It is easy to observe that every commuting pair of self-maps is necessarily compatible. Converse is not true. For instance, see [2], [3] and [4].

The generalization proved in [5] is the following:

**Theorem 1.2.** Let \( f \) and \( g \) be self-maps on a complete metric space \( X \) satisfying the inclusion (1) and the inequality (2), where \( \alpha \) and \( \beta \) are nonnegative constants with \( \alpha + 2\beta < 1 \). If \( g \) is continuous, and \((f, g)\) is a compatible pair, then \( f \) and \( g \) have a unique common fixed point.

We prove a generalization of Theorem 1.2 by replacing (2) with a general inequality involving an auxiliary function.

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\(^1\)Compatible maps was introduced by Gerald Jungck [2] as a generalization of commuting maps.
2. Preliminary Notations

Several fixed point theorems in metric space setting have been proved through contraction conditions involving different types of auxiliary functions. Given a positive integer $\alpha$, a generalized class $\Phi_\alpha$ of auxiliary functions was introduced in [6] as follows:

$$\Phi_\alpha = \{ \phi : [0, \infty) \to [0, \infty) | \phi(0) = 0, \phi(\alpha t) < t \text{ for } t > 0 \}. \quad (5)$$

It is obvious that, for $\alpha = 1$, $\Phi_\alpha$ reduces to the class $\Psi$ of all contractive moduli $\psi$ [7] such that $\psi(0) = 0$ and $\psi(t) < t$ for $t > 0$.

**Definition 2.1.** A mapping $\phi \in \Phi_\alpha$ is said to be upper semicontinuous at $t_0 \geq 0$ if $\limsup_{n \to \infty} \phi(t_n) \leq \phi(t_0)$ whenever $\langle t_n \rangle_{n=1}^\infty$ is such that $\lim_{n \to \infty} t_n = t_0$, and $\phi$ is u.s.c if it is u.s.c at every $t \geq 0$.

Our main result is

**Theorem 2.1.** Let $f$ and $g$ be self-maps on a complete metric space $X$ satisfying the inclusion (1), and the inequality

$$[d(fx, fy)]^2 \leq \phi(\max\{d(fx, gx)d(fy, gy) + d(fy, gx)d(fx, gy), d(fx, gx)d(fx, gy) + d(fy, gx)d(fy, gy)\})$$

for all $x, y \in X,$

where $\phi \in \Phi_2$ is nondecreasing and upper semicontinuous. If $g$ is continuous, and $(f, g)$ is a compatible pair, then $f$ and $g$ have a unique common fixed point.

**Proof.** Let $x_0 \in X$ be arbitrary.

In view of (1), we can choose points $x_1, x_2, \ldots, x_n, \ldots$ in $X$ inductively such that

$$fx_{n-1} = gx_n = y_n \quad \text{for all } n \geq 1. \quad (7)$$

Writing $x = x_{n-1}$ and $y = x_n$ in (6) and using (7), we get

$$[d(y_n, y_{n+1})]^2 = [d(fx_{n-1}, fx_n)]^2$$

$$\leq \phi(\max\{d(fx_{n-1}, gx_{n-1})d(fy_n, gy_n) + d(fy_n, gx_{n-1})d(fx_{n-1}, gx_n),$$

$$d(fx_{n-1}, gx_{n-1})d(fx_{n-1}, gx_n) + d(fx_n, gx_{n-1})d(fx_n, gx_n)\})$$

$$= \phi(\max\{d(y_n, y_{n-1})d(y_{n+1}, y_n), d(y_{n+1}, y_{n-1})d(y_{n+1}, y_n)\})$$

$$\leq \phi(d(y_n, y_{n+1})[d(y_n, y_{n-1}) + d(y_{n+1}, y_n)]) \quad (8)$$
We now prove that
\[ d(y_n, y_{n-1}) \geq d(y_{n+1}, y_n) \quad \text{for } n \geq 2. \] (9)

If possible, suppose that \( d(y_m, y_{m-1}) < d(y_{m+1}, y_m) \) for some \( m \geq 2 \). Then \( d(y_{m+1}, y_m) > 0 \). Since \( \phi \) is nondecreasing, from (8) it follows that
\[ 0 < \left[ d(y_{m+1}, y_m) \right]^2 \leq \phi(2\left[ d(y_m, y_{m+1}) \right]^2) < \left[ d(y_{m+1}, y_m) \right]^2, \]
which is a contradiction. This proves (9). In other words, \( \langle d(y_{n+1}, y_n) \rangle_{n=1}^\infty \) is a decreasing sequence of nonnegative real numbers and hence converges to some \( t \geq 0 \). Now using (9) in (8), we get
\[ d(y_{n+1}, y_n) \leq \phi(d(y_{n+1}, y_n) + d(y_{n+2}, y_{n+1})) \leq \phi(2d(y_{n+1}, y_n)) \quad \text{for } n \geq 1. \]

Taking the limit superior as \( n \to \infty \) in this and then using the upper semicontinuity of \( \phi \), we obtain that
\[ t \leq \phi(2t). \] (10)

If \( t > 0 \) in (10), then the choice of \( \phi \) implies that \( t \leq \phi(2t) < t \), which is a contradiction. Thus
\[ t = \lim_{n \to \infty} d(y_{n+1}, y_n) = \lim_{n \to \infty} d(y_{n+1}, y_n) = 0. \] (11)

We now prove that \( \langle y_n \rangle_{n=1}^\infty \) is a Cauchy sequence in \( X \).

If possible we suppose that \( \langle y_n \rangle_{n=1}^\infty \) is not Cauchy. Then for some \( \epsilon > 0 \), we choose sequences \( \langle y_{m_k} \rangle_{k=1}^\infty \) and \( \langle y_{n_k} \rangle_{k=1}^\infty \) of positive integers such that \( m_k > n_k > k \) and
\[ d(y_{m_k}, y_{n_k}) \geq \epsilon \quad \text{for } k = 1, 2, 3, \ldots. \] (12)

Suppose that \( m_k \) is the smallest integer exceeding \( n_k \) which satisfies (12). That is
\[ d(y_{m_k-1}, y_{n_k}) < \epsilon. \] (13)

Now by triangle inequality of \( d \), we see that
\[
\epsilon \leq d(y_{m_k}, y_{n_k}) \leq d(y_{m_k}, y_{m_k-1}) + d(y_{m_k-1}, y_{n_k}) < d(y_{m_k}, y_{m_k-1}) + \epsilon \] (14)
and from (11), we see that
\[ \lim_{k \to \infty} d(y_{mk-1}, y_{mk}) = 0 \] (15)
and
\[ \lim_{k \to \infty} d(y_{nk-1}, y_{nk}) = 0 \] (16)
Using (15) in (14), we get
\[ \lim_{k \to \infty} d(y_{mk}, y_{nk}) = \epsilon. \] (17)

Again, by the triangle inequality of \( d \), we get
\[ d(y_{nk-1}, y_{mk}) \leq d(y_{nk-1}, y_{mk}) + d(y_{nk-1}, y_{nk}). \]
As \( k \to \infty \) this in view of (16) and (17), gives
\[ \lim_{k \to \infty} d(y_{nk-1}, y_{mk}) = \epsilon. \] (18)
On the other hand, writing \( x = x_{mk-1}, y = x_{nk-1} \) in (6), we have
\[
[d(f x_{mk-1}, f x_{nk-1})]^2 \leq \phi(\max\{d(f x_{mk-1}, g x_{mk-1})d(f x_{nk-1}, g x_{nk-1})
+ d(f x_{nk-1}, g x_{mk-1})d(f x_{mk-1}, g x_{nk-1}),
\quad d(f x_{mk-1}, g x_{mk-1})d(f x_{nk-1}, g x_{nk-1})
+ d(f x_{nk-1}, g x_{mk-1})d(f x_{nk-1}, g x_{nk-1})\})
\]
or
\[
\epsilon^2 \leq [d(y_{mk}, y_{nk})]^2
\leq \phi(\max\{d(y_{mk}, y_{mk-1})d(y_{nk}, y_{nk-1}) + d(y_{nk}, y_{mk-1})d(y_{mk}, y_{nk-1}),
\quad d(y_{mk}, y_{mk-1})d(y_{mk}, y_{nk-1}) + d(y_{nk}, y_{mk-1})d(y_{mk}, y_{nk-1})\})
\] (19)
Since \( \phi \) is nondecreasing, proceeding the limit as \( n \to \infty \) in this, and then using upper semicontinuity of \( \phi \), (13), (15), (16),(17) and (18) we get
\[
0 < \epsilon^2 \leq \phi(\max\{0 + \epsilon^2, 0\}) = \phi(\epsilon^2) \leq \phi(2\epsilon^2) < \epsilon^2,
\]
which is a contradiction. Hence \( \langle y_n \rangle_{n=1}^\infty \) must be a \( G \)-Cauchy sequence in \( X \).
Since \((X, G)\) is \(G\)-Complete, there exists a point \(p \in X\) such that \(\langle y_n \rangle_{n=1}^\infty\) is \(G\)-convergent to \(p\). That is
\[
\lim_{n \to \infty} y_{n-1} = \lim_{n \to \infty} y_n = p. \tag{20}
\]
Now the compatibility of \(f\) and \(g\), and (20) imply that
\[
\lim_{n \to \infty} d(fgx_n, g^2x_n) = 0, \tag{21}
\]
while the sequential property of the contiuity of \(g\) and (20) give
\[
\lim_{n \to \infty} gfx_n = \lim_{n \to \infty} g^2x_n = gz. \tag{22}
\]
Hence it follows from (21) and (22), that
\[
\lim_{n \to \infty} d(fgx_n, gz) = 0 \text{ or } \lim_{n \to \infty} fgx_n = gz. \tag{23}
\]
But the use of (6) yields
\[
[d(fgx_n, fz)]^2 \leq \phi(\max\{d(fgx_n, g^2x_n)d(fz, gz) + d(fz, g^2x_n)d(fgx_n, gz),
\]
\[d(fgx_n, g^2x_n)d(fgx_n, gz) + d(fz, g^2x_n)d(fz, gz)\}).
\]
Applying the limit as \(n \to \infty\) in this, and using (22) and (23), we obtain that
\[
[d(gz, fz)]^2 \leq \phi(\max\{d(gz, g^2z)d(fz, gz) + d(fz, g^2z)d(gz, gz),
\]
\[d(gz, g^2z)d(gz, gz) + d(fz, g^2z)d(fz, gz)\}).
\]
or
\[
[d(gz, fz)]^2 \leq \phi([d(fz, gz)]^2).
\]
If \(fz \neq gz\), then the nondecreasing nature of \(\phi\) would lead to a contradiction that
\[
0 < [d(gz, fz)]^2 \leq \phi([d(fz, gz)]^2) \leq \phi(2[d(fz, gz)]^2) < [d(fz, gz)]^2.
\]
Hence we must have
\[
gz = fz. \tag{24}
\]
Finally from (6), we see that
\[
[d(fx_n, fz)]^2 \leq \phi(\max\{d(fx_n, gx_n)d(fz, gz) + d(fz, gx_n)d(fx_n, gz),
\]
...
\[ d(fx_n, gx_n) d(fx_n, gz) + d(fz, gx_n) d(fz, gz). \]

The limiting case of this as \( n \to \infty \), (20), and (22) would imply that

\[ |d(z, fz)|^2 \leq \phi(|d(fz, z)|^2), \]

which with a similar argument as above yields that \( d(z, fz) = 0 \) or \( fz = z \). Thus \( z \) is a common fixed point of \( f \) and \( g \).

The uniqueness of the common fixed point follows easily from (6).

**Remark 2.1.** Theorem 2.1 does not require the continuity of \( f \).

Since every commuting pair is compatible, writing \( \phi(t) = qt \) for all \( t \geq 0 \), where \( q < 1/2 \), we obtain

**Corollary 2.1.** Let \( f \) and \( g \) be self-maps on a complete metric space \( X \) satisfying the inclusion (1), and the inequality

\[
|d(f x, f y)|^2 \leq q \max\{d(f x, g x)d(f y, g y) + d(f y, g x)d(f x, g y), \\
\quad d(f x, g x)d(f x, g y) + d(f y, g x)d(f y, g y)\}
\]

for all \( x, y \in X \), (25)

If \( g \) is continuous, and \( (f, g) \) is a commuting, then \( f \) and \( g \) have a unique common fixed point.

Choosing \( \alpha \) and \( \beta \) such that \( \alpha + 2\beta < 1/2 \), then it is easily seen that the right hand side of (2) is less than or equal to the right hand side of (25), where \( r = \alpha + 2\beta \). Thus Theorem 1.2 will become a particular case of Corollary 2.1.

**References**


