***L*-FUZZY (K, E) -SOFT TOPOLOGIES AND
L-FUZZY (K, E) -SOFT CLOSURE OPERATORS
INDUCED BY FUZZY SOFT MAPS**

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Abstract: In this paper, we study *L*-fuzzy (K, E) -soft topologies and *L*-fuzzy (K, E) -soft closure spaces induced by fuzzy soft maps in complete residuated lattices. We give some examples.

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1. Introduction

In 1999, Molodtsov [13] initiated the theory of soft sets as a new mathematical tool to deal with uncertainties while modeling problems in engineering physics, computer science, economics, social sciences and medical sciences. In [14], Molodtsov applied successfully in directions such as, smoothness of functions, game theory, operations research, Riemann-integration, Perron integra-

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tion, probability and theory of measurement. Maji et al. [10,11] gave the first practical application of soft sets in decision making problems. In 2003, Maji et al. [11] defined and studied several basic notions of soft set theory. Many researchers have contributed towards the algebraic structure of soft set theory [1-5,7]. In 2011, Shabir and Naz [19] initiated the study of soft topological spaces. They defined soft topology on the collection of soft sets over X and established their several properties. Aygünoglu et.al [2] introduced the concept of (K, E) -soft topology in the sense of Šostak [9]. Cetkin et.al [3] studied (K, E) -soft proximities and discuss their properties.

Hájek [8] introduced a complete residuated lattice which is an algebraic structure for many valued logic and decision rules in complete residuated lattices. Höhle [9] introduced L -fuzzy topologies with algebraic structure L (cqm, quantales, MV -algebra). It has developed in many directions [16,17]. Kim et al. [10] defined L -fuzzy (K, E) -soft topologies and L -fuzzy (K, E) -soft closure spaces in complete residuated lattices.

In this paper, we study L -fuzzy (K, E) -soft topologies and L -fuzzy (K, E) -soft closure spaces induced by fuzzy soft maps in complete residuated lattices. We give their examples.

2. Preliminaries

Definition 1. [8,9] An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we always assume that $(L, \leq, \odot, \rightarrow, \oplus, *)$ is a complete residuated lattice with an order reversing involution $*$ which is defined by

$$x \oplus y = (x^* \odot y^*)^*, \quad x^* = x \rightarrow 0$$

unless otherwise specified.

Remark 2. Every completely distributive lattice $(L, \leq, \odot = \wedge, \oplus = \vee, *)$ with order reversing involution $*$ is a complete residuated lattice.

Lemma 3. [8,9] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \rightarrow x = x, 0 \odot x = 0,$

- (2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$,
- (3) $x \leq y$ iff $x \rightarrow y = 1$.
- (4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,
- (5) $x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i)$,
- (6) $(\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)$,
- (7) $x \rightarrow (\bigvee_i y_i) \geq \bigvee_i (x \rightarrow y_i)$,
- (8) $(\bigwedge_i x_i) \rightarrow y \geq \bigvee_i (x_i \rightarrow y)$,
- (9) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
- (10) $x \odot y = (x \rightarrow y^*)^*$ and $x \oplus y = x^* \rightarrow y$,
- (11) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$,
- (12) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,
- (13) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \oplus z) \rightarrow (y \oplus w)$.
- (14) $x \rightarrow y = y^* \rightarrow x^*$.
- (15) $(x \vee y) \odot (z \vee w) \leq (x \vee z) \vee (y \odot w) \leq (x \oplus z) \vee (y \odot w)$.
- (16) $\bigvee_{i \in \Gamma} x_i \rightarrow \bigvee_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)$ and $\bigwedge_{i \in \Gamma} x_i \rightarrow \bigwedge_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)$,
- (17) $(x \odot y) \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)$.

Throughout this paper, X refers to an initial universe, E and K are the sets of all parameters for X , and L^X is the set of all L -fuzzy sets on X .

Definition 4. [4] A map f is called an L -fuzzy soft set on X , where f is a mapping from E into L^X , i.e., $f_e := f(e)$ is an L -fuzzy set on X , for each $e \in E$. The family of all L -fuzzy soft sets on X is denoted by $(L^X)^E$. Let f and g be two L -fuzzy soft sets on X .

(1) f is an L -fuzzy soft subset of g and we write $f \sqsubseteq g$ if $f_e \leq g_e$, for each $e \in E$. f and g are equal if $f \sqsubseteq g$ and $g \sqsubseteq f$.

(2) The intersection of f and g is an L -fuzzy soft set $h = f \sqcap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

(3) The union of f and g is an L -fuzzy soft set $h = f \sqcup g$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(4) An L -fuzzy soft set $h = f \odot g$ is defined as $h_e = f_e \odot g_e$, for each $e \in E$.

(5) An L -fuzzy soft set $h = f \oplus g$ is defined as $h_e = f_e \oplus g_e$, for each $e \in E$.

(6) The complement of an L -fuzzy soft sets on X is denoted by f^* , where $f^* : E \rightarrow L^X$ is a mapping given by $f_e^* = (f_e)^*$, for each $e \in E$.

(7) f is called a null L -fuzzy soft set and is denoted by 0_X , if $f_e(x) = 0$, for each $e \in E$, $x \in X$.

(8) f is called an absolute L -fuzzy soft set and is denoted by 1_X , if $f_e(x) = 1$, for each $e \in E$, $x \in X$ and $(1_X)_e(x) = 1$.

Definition 5. [2,4] Let $\varphi : X \rightarrow Y$ and $\psi : E_1 \rightarrow E_2$ be two mappings, where E_1 and E_2 are parameters sets for the crisp sets X and Y , respectively. Then $\varphi_\psi : (L^X)^{E_1} \rightarrow (L^Y)^{E_2}$ is called a fuzzy soft mapping.

(1) For $f \in (L^X)^{E_1}$, the image of f under the fuzzy soft mapping φ_ψ defined by, $\forall k \in K, \forall y \in Y$,

$$\varphi(f)_{e_2}(y) = \begin{cases} \bigvee_{\varphi(x)=y} (\bigvee_{\psi(e_1)=e_2} f_{e_1}(x)), & \text{if } x \in \varphi^{-1}(y), e_1 \in \psi^{-1}(e_2) \\ 0, & \text{otherwise.} \end{cases}$$

(2) For $f \in (L^X)^{E_1}$, the pre-image of g defined by

$$\varphi_\psi^{-1}(g)_e(x) = g_{\psi(e)}(\varphi(x)), \forall e \in E, \forall x \in X.$$

(3) The fuzzy soft mapping $\varphi_\psi : (L^X)^{E_1} \rightarrow (L^Y)^{E_2}$ is called injective (resp. surjective, bijective) if f and ϕ are both injective (resp. surjective, bijective).

Definition 6. [2, 10] A mapping $\mathcal{T} : K \rightarrow L^{(L^X)^E}$ (where $\mathcal{T}_k := \mathcal{T}(k) : (L^X)^E \rightarrow L$ is a mapping for each $k \in K$) is called an L -fuzzy (K, E) -soft topology on X if it satisfies the following conditions for each $k \in K$.

- (O1) $\mathcal{T}_k(0_X) = \mathcal{T}_k(1_X) = 1$,
- (O2) $\mathcal{T}_k(f \odot g) \geq \mathcal{T}_k(f) \odot \mathcal{T}_k(g) \quad \forall f, g \in (L^X)^E$,
- (O3) $\mathcal{T}_k(\bigsqcup_i f_i) \geq \bigwedge_{i \in I} \mathcal{T}_k(f_i) \quad \forall f_i \in (L^X)^E, i \in I$.

The pair $(X, \mathcal{T})$ is called an L -fuzzy (K, E) -soft topological space. Let $\mathcal{T}^1$ and $\mathcal{T}^2$ be L -fuzzy (K, E) -soft topologies on X . Then $\mathcal{T}^1$ is coarser than $\mathcal{T}^2$ if $\mathcal{T}_k^1(f) \leq \mathcal{T}_k^2(f)$, for all $f \in (L^X)^E, k \in K$.

Let $(X, \mathcal{T}^1)$ be an L -fuzzy (K_1, E_1) -soft topological space and $(Y, \mathcal{T}^2)$ be an L -fuzzy (K_2, E_2) -soft topological space. Let $\varphi : X \rightarrow Y$, $\psi : E_1 \rightarrow E_2$ and $\eta : K_1 \rightarrow K_2$ be mappings. Then $\varphi_{\psi, \eta}$ from $(X, \mathcal{T}^1)$ into $(Y, \mathcal{T}^2)$ is called L -fuzzy soft continuous if

$$\mathcal{T}_{\eta(k)}^2(f) \leq \mathcal{T}_k^1(\varphi_\psi^{-1}(f)) \quad \forall f \in (L^Y)^{E_2}, k \in K_1.$$

Definition 7. [10] A map $\mathcal{C} : K \times (L^X)^E \times L_0 \rightarrow (L^X)^E$ is called an L -fuzzy (K, E) -soft closure operator if it satisfies the following conditions;

- (C1) $\mathcal{C}(k, 0_X, r) = 0_X$,
- (C2) $\mathcal{C}(k, f, r) \supseteq f$,
- (C3) If $f_1 \sqsubseteq f_2$, then $\mathcal{C}(k, f_1, r) \sqsubseteq \mathcal{C}(k, f_2, r)$,
- (C4) If $r_1 \leq r_2$, then $\mathcal{C}(k, f, r_1) \sqsubseteq \mathcal{C}(k, f, r_2)$,
- (C5) $\mathcal{C}(k, f_1 \oplus f_2, r \odot s) \sqsubseteq \mathcal{C}(k, f_1, r) \oplus \mathcal{C}(k, f_2, s)$.

The pair $(X, \mathcal{C})$ is called an L -fuzzy (K, E) -soft closure space.

An L -fuzzy (K, E) -soft closure operator is called topological if

(T) $\mathcal{C}(k, \mathcal{C}(k, f, r), r) \subseteq \mathcal{C}(k, f, r)$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be L -fuzzy (K, E) -soft closure operators on X . Then $\mathcal{C}_1$ is finer than $\mathcal{C}_2$ if $\mathcal{C}_1(k, f, r) \subseteq \mathcal{C}_2(k, f, r)$, for all $f \in (L^X)^E, r \in L_0$.

Let $(X, \mathcal{C}_X)$ be L -fuzzy (K_1, E_1) -soft closure spaces and $(Y, \mathcal{C}_Y)$ be L -fuzzy (K_2, E_2) -soft closure spaces. Let $\varphi : X \rightarrow Y$, $\psi : E_1 \rightarrow E_2$ and $\eta : K_1 \rightarrow K_2$ be maps. Then $\varphi_{\psi, \eta}$ is called an L -fuzzy soft closed map if, for each $k \in K_1, f \in (L^X)^{E_1}, r \in L_0$,

$$\varphi_{\psi, \eta}(\mathcal{C}_X(k, f, r)) \subseteq \mathcal{C}_Y(\eta(k), \varphi_{\psi}(f), r).$$

Lemma 8. [10] Let $\varphi_{\psi} : (L^X)^{E_1} \rightarrow (L^Y)^{E_2}$ be a fuzzy soft mapping. Then we have the following properties. For $f, f_i \in (L^X)^{E_1}$ and $g, g_i \in (L^Y)^{E_2}$,

- (1) $g \supseteq \varphi_{\psi}(\varphi_{\psi}^{-1}(g))$ with equality if φ_{ψ} is surjective,
- (2) $f \subseteq \varphi_{\psi}^{-1}(\varphi_{\psi}(f))$ with equality if φ_{ψ} is injective,
- (3) if φ_{ψ} is injective,

$$\varphi(f)_{e_2}(y) = \begin{cases} f_{e_1}(x), & \text{if } x \in \varphi^{-1}(\{y\}), e_1 \in \psi^{-1}(\{e_2\}) \\ 0, & \text{otherwise,} \end{cases}$$

- (4) $\varphi_{\psi}^{-1}(g^*) = (\varphi_{\psi}^{-1}(g))^*$,
- (5) $\varphi_{\psi}^{-1}(\bigvee_{i \in I} g_i) = \bigvee_{i \in I} \varphi_{\psi}^{-1}(g_i)$,
- (6) $\varphi_{\psi}^{-1}(\bigwedge_{i \in I} g_i) = \bigwedge_{i \in I} \varphi_{\psi}^{-1}(g_i)$,
- (7) $\varphi_{\psi}(\bigvee_{i \in I} f_i) = \bigvee_{i \in I} \varphi_{\psi}(f_i)$,
- (8) $\varphi_{\psi}(\bigwedge_{i \in I} f_i) \subseteq \bigwedge_{i \in I} \varphi_{\psi}(f_i)$ with equality if φ_{ψ} is injective,
- (9) $\varphi_{\psi}^{-1}(g_1 \odot g_2) = \varphi_{\psi}^{-1}(g_1) \odot \varphi_{\psi}^{-1}(g_2)$,
- (10) $\varphi_{\psi}^{-1}(g_1 \oplus g_2) = \varphi_{\psi}^{-1}(g_1) \oplus \varphi_{\psi}^{-1}(g_2)$,
- (11) $\varphi_{\psi}(f_1 \odot f_2) \subseteq \varphi_{\psi}(f_1) \odot \varphi_{\psi}(f_2)$ with equality if φ_{ψ} is injective,
- (12) $\varphi_{\psi}(f_1 \oplus f_2) \subseteq \varphi_{\psi}(f_1) \oplus \varphi_{\psi}(f_2)$ with equality if φ_{ψ} is injective.

Theorem 9. [10] (1) Let $(X, \mathcal{T})$ be an L -fuzzy (K, E) -soft topological space. Define $\mathcal{C}_{\mathcal{T}} : K \times (L^X)^E \times L_0 \rightarrow (L^X)^E$ as

$$\mathcal{C}_{\mathcal{T}}(k, f, r) = \bigwedge \{g \in (L^X)^E \mid g \supseteq f, \mathcal{T}_k(g^*) \geq r\}.$$

Then (1) $\mathcal{C}_{\mathcal{T}}$ is a topological L -fuzzy (K, E) -soft closure operator.

(2) Let $(X, \mathcal{C})$ be an L -fuzzy (K, E) -soft closure space. Define $\mathcal{T}_{\mathcal{C}} : (L^X)^E \rightarrow L$ as

$$(\mathcal{T}_{\mathcal{C}})_k(f) = \bigvee \{r \in L \mid \mathcal{C}(k, f^*, r) \subseteq f^*\}.$$

Then $\mathcal{T}_{\mathcal{C}}$ is an L -fuzzy (K, E) -soft topology on X .

(3) Let $(X, \mathcal{T})$ be an L -fuzzy (K, E) -soft topological space. Then $\mathcal{T} = \mathcal{T}_{\mathcal{C}_{\mathcal{T}}}$.

Theorem 10. [10] Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be L -fuzzy (K_1, E_1) -soft and L -fuzzy (K_2, E_2) -soft topological spaces, respectively. Let $\varphi_\psi : (L^X)^{E_1} \rightarrow (L^Y)^{E_2}$ be a soft map. If $\varphi_{\psi, \eta} : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ is an L -fuzzy soft continuous map, then $\varphi_{\psi, \eta} : (X, \mathcal{C}_{\mathcal{T}_X}) \rightarrow (Y, \mathcal{C}_{\mathcal{T}_Y})$ is an L -fuzzy soft closure map.

Theorem 11. [10] Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L -fuzzy (K_1, E_1) -soft and L -fuzzy (K_2, E_2) -soft closure spaces, respectively. Let $\varphi_\psi : (L^X)^{E_1} \rightarrow (L^Y)^{E_2}$ be a fuzzy soft map. Then $\varphi_{\psi, \eta} : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ is an L -fuzzy soft closure map iff $\mathcal{C}_X(k, \varphi_\psi^{-1}(g), r) \sqsubseteq \varphi_\psi^{-1}(\mathcal{C}_Y(\eta(k), g, r))$.

Theorem 12. [10] Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be L -fuzzy (K_1, E_1) -soft and L -fuzzy (K_2, E_2) -soft closure spaces, respectively. Let $\varphi_\psi : (L^X)^E \rightarrow (L^Y)^{E_2}$ be a fuzzy soft map. Then the following properties;

- (1) If $\varphi_{\psi, \eta} : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ is an L -fuzzy soft closure map, then $\varphi_{\psi, \eta} : (X, \mathcal{T}_{\mathcal{C}_X}) \rightarrow (Y, \mathcal{T}_{\mathcal{C}_Y})$ is an L -fuzzy soft continuous map.
- (2) $\varphi_{\psi, \eta} : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ is an L -fuzzy soft continuous map iff $\varphi_{\psi, \eta} : (X, \mathcal{C}_{\mathcal{T}_X}) \rightarrow (Y, \mathcal{C}_{\mathcal{T}_Y})$ is an L -fuzzy soft closure map.

3. L -fuzzy (K, E) -soft topologies and L -fuzzy (K, E) -soft closure operators induced by fuzzy soft maps

Theorem 13. Let $(Y, \mathcal{T}_Y)$ be L -fuzzy (K, E) -soft topological spaces and $\varphi_\psi : (L^X)^E \rightarrow (L^Y)^{E_2}$ be a soft map. Define $\mathcal{T}_X : K \rightarrow L^{(L^X)^E}$ as

$$(\mathcal{T}_X)_k(f) = \bigvee \{(\mathcal{T}_Y)_{\eta(k)}(g) \mid f = \varphi_\psi^{-1}(g)\}$$

Then $\mathcal{T}_X$ is the coarsest L -fuzzy (K, E) -soft topology for which $\varphi_{\psi, \eta} : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ is an L -fuzzy soft continuous map.

Proof. (O1) is easily proved from the definition of $\mathcal{T}_X$.

(O2)

$$\begin{aligned} & (\mathcal{T}_X)_k(f_1) \odot (\mathcal{T}_X)_k(f_2) \\ &= (\bigvee \{(\mathcal{T}_Y)_{\eta(k)}(g_1) \mid f_1 = \varphi_\psi^{-1}(g_1)\}) \odot \bigvee \{(\mathcal{T}_Y)_{\eta(k)}(g_2) \mid f_2 = \varphi_\psi^{-1}(g_2)\} \\ &= \bigvee \{(\mathcal{T}_Y)_{\eta(k)}(g_1) \odot (\mathcal{T}_Y)_{\eta(k)}(g_2) \mid f_1 \odot f_2 = \varphi_\psi^{-1}(g_1) \odot \varphi_\psi^{-1}(g_2)\} \\ &\leq \bigvee \{(\mathcal{T}_Y)_{\eta(k)}(g_1 \odot g_2) \mid f_1 \odot f_2 = \varphi_\psi^{-1}(g_1 \odot g_2)\} \\ &= (\mathcal{T}_X)_k(f_1 \odot f_2). \end{aligned}$$

(O3)

$$\begin{aligned}
\bigwedge_{i \in \Gamma} (\mathcal{T}_X)_k(f_i) &= \bigwedge_{i \in \Gamma} (\bigvee \{(\mathcal{T}_Y)_{\eta(k)}(g_i) \mid f_i = \varphi_\psi^{-1}(g_i)\}) \\
&= \bigvee \{ \bigwedge_{i \in \Gamma} (\mathcal{T}_Y)_{\eta(k)}(g_i) \mid \bigvee_{i \in \Gamma} f_i = \varphi_\psi^{-1}(\bigvee_{i \in \Gamma} g_i) \} \\
&= (\mathcal{T}_X)_k(\bigvee_{i \in \Gamma} f_i).
\end{aligned}$$

Let $\varphi_{\psi, \eta} : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}_Y)$ be an L -fuzzy soft continuous map. Then $\mathcal{T}_k(\varphi_\psi^{-1}(g)) \geq (\mathcal{T}_Y)_{\eta(k)}(g)$. Hence

$$\mathcal{T}_k(\varphi_\psi^{-1}(g)) \geq (\mathcal{T}_X)_k(\varphi_\psi^{-1}(g)).$$

Thus $\mathcal{T}_k \geq (\mathcal{T}_X)_k$.

Theorem 14. Let $(Y, \mathcal{C}_Y)$ be a (resp. topological) L -fuzzy (K_2, E_2) -soft closure space, $\varphi_\psi : (L^X)^{E_1} \rightarrow (L^Y)^{E_2}$ be a soft map and $\eta : E_1 \rightarrow E_2$. Define $\mathcal{C}_X : K_1 \times (L^X)^{E_1} \times L_0 \rightarrow (L^X)^{E_1}$ as

$$\mathcal{C}_X(k, f, r) = \varphi_{\psi, \eta}^{-1}(\mathcal{C}_Y(\eta(k), \varphi_\psi(f), r))$$

Then $\mathcal{C}_X$ is the coarsest (resp. topological) L -fuzzy (K_1, E_1) -soft closure operator for which $\varphi_{\psi, \eta} : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ is an L -fuzzy soft closure map.

Proof. (C1)-(C3) and (C4) are easily proved from the definition of $\mathcal{C}_X$.
(C5)

$$\begin{aligned}
&\mathcal{C}_X(k, f_1, r) \oplus \mathcal{C}_X(k, f_2, s) \\
&= \varphi_{\psi, \eta}^{-1}(\mathcal{C}_Y(\eta(k), \varphi_\psi(f_1), r)) \oplus \varphi_{\psi, \eta}^{-1}(\mathcal{C}_Y(\eta(k), \varphi_\psi(f_2), s)) \\
&= \varphi_{\psi, \eta}^{-1}(\mathcal{C}_Y(\eta(k), \varphi_\psi(f_1), r) \oplus \mathcal{C}_Y(\eta(k), \varphi_\psi(f_2), s)) \\
&\sqsupseteq \varphi_{\psi, \eta}^{-1}(\mathcal{C}_Y(\eta(k), \varphi_\psi(f_1) \oplus \varphi_\psi(f_2), r \odot s)) \\
&\sqsupseteq \varphi_{\psi, \eta}^{-1}(\mathcal{C}_Y(\eta(k), \varphi_\psi(f_1 \oplus f_2), r \odot s)) \\
&= \mathcal{C}_X(k, f_1 \oplus f_2, r \odot s).
\end{aligned}$$

From the definition of $\mathcal{C}_X$,

$$\begin{aligned}
\varphi_{\psi, \eta}(\mathcal{C}_X(k, f, r)) &= \varphi_{\psi, \eta}(\varphi_{\psi, \eta}^{-1}(\mathcal{C}_Y(\eta(k), \varphi_\psi(f), r))) \\
&\sqsubseteq \mathcal{C}_Y(\eta(k), \varphi_\psi(f), r).
\end{aligned}$$

Hence $\varphi_{\psi, \eta} : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ is an L -fuzzy soft closure map.

If $\varphi_{\psi, \eta} : (X, \mathcal{C}) \rightarrow (Y, \mathcal{C}_Y)$ is an L -fuzzy soft closure map, then $\varphi_{\psi, \eta}(\mathcal{C}(k, f, r)) \sqsubseteq \mathcal{C}_Y(\eta(k), \varphi_\psi(f), r)$. Thus $\mathcal{C}(k, f, r) \sqsubseteq \varphi_\psi^{-1}(\mathcal{C}_Y(\eta(k), \varphi_\psi(f), r)) = \mathcal{C}_X(k, f, r)$.

If $\mathcal{C}_Y$ is topological, then

$$\begin{aligned} & \mathcal{C}_X(k, \mathcal{C}_X(k, f, r), r) \\ &= \varphi_{\psi, \eta}^{-1}(\mathcal{C}_Y(\eta(k), \varphi_{\psi}(\mathcal{C}_X(k, f, r)), r)) \\ &\sqsubseteq \varphi_{\psi}^{-1}(\mathcal{C}_Y(\eta(k), \mathcal{C}_Y(\eta(k), \varphi_{\psi}(f), r)), r) \\ &= \varphi_{\psi}^{-1}(\mathcal{C}_Y(\eta(k), \varphi_{\psi}(f), r)) = \mathcal{C}_X(k, f, r). \end{aligned}$$

Hence $\mathcal{C}_X$ is topological. $\square$

Example 15. Let $X = \{h_i \mid i = \{1, \dots, 5\}\}$ with h_i =house and $E_X = \{e, b, w, c\}$ with e =expensive, b = beautiful, w =wooden, c = creative. Define a binary operation $\wedge$ on $[0, 1]$ by

$$x \wedge y = \min\{x, y\}, x^* = 1 - x, x \vee y = (x^* \wedge y^*)^*$$

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$$

Then $([0, 1], \wedge, \rightarrow, 0, 1)$ is a complete residuated lattice (ref.[8,9]). Let $E_1 = \{e, w, b\} \subset E_X$ and $f_1, f_2 \in ([0, 1]^X)^{E_1}$ as follows:

$$\begin{aligned} (f_1)_e &= (0.8, 0.1, 0.5, 0.9, 0.6), & (f_1)_b &= (0.7, 0.9, 0.4, 0.5, 0.7) \\ (f_1)_w &= (0.4, 0.7, 0.5, 0.6, 0.5), & (f_2)_e &= (0.5, 0.9, 0.4, 0.8, 0.4) \\ (f_2)_b &= (0.3, 1.0, 0.2, 0.4, 0.5), & (f_2)_w &= (0.5, 0.4, 0.8, 0.5, 0.1) \end{aligned}$$

Then we obtain:

$$\begin{aligned} (f_1 \vee f_2)_e &= (0.8, 0.9, 0.5, 0.9, 0.6), & (f_1 \vee f_2)_b &= (0.7, 1.0, 0.4, 0.5, 0.7) \\ (f_1 \vee f_2)_w &= (0.5, 0.7, 0.8, 0.6, 0.5), & (f_1 \wedge f_2)_e &= (0.5, 0.1, 0.4, 0.8, 0.4) \\ (f_1 \wedge f_2)_b &= (0.3, 0.9, 0.2, 0.4, 0.5), & (f_1 \wedge f_2)_w &= (0.4, 0.4, 0.5, 0.5, 0.1) \end{aligned}$$

For $K = \{k_1, k_2\}$, we define a $[0, 1]$ -fuzzy (K, E_1) -soft topology $\mathcal{T}_X : K \rightarrow [0, 1]^{([0, 1]^X)^{E_1}}$ as follows:

$$\begin{aligned} (\mathcal{T}_X)_{k_1}(f) &= \begin{cases} 1, & \text{if } f = 0_X \text{ or } 1_X, \\ 0.7, & \text{if } f = f_1 \\ 0.4, & \text{if } f = f_2 \\ 0.5, & \text{if } f = f_1 \vee f_2 \\ 0.6, & \text{if } f = f_1 \wedge f_2 \\ 0, & \text{otherwise.} \end{cases} \\ (\mathcal{T}_X)_{k_2}(f) &= \begin{cases} 1, & \text{if } f = 0_X \text{ or } 1_X, \\ 0.5, & \text{if } f = f_1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

From Theorem 14(1), we obtain a $[0, 1]$ -fuzzy (K, E_1) -soft closure operator $\mathcal{C}_{\mathcal{T}_X} : K \times ([0, 1]^X)^{E_1} \times (0, 1] \rightarrow ([0, 1]^X)^{E_1}$ as follows:

$$\mathcal{C}_{\mathcal{T}_X}(k_1, f, r) = \begin{cases} 1_X, & \text{if } f = 1_X, r \in (0, 1], \\ f_1^*, & \text{if } f \sqsubseteq f_1^*, f \not\sqsubseteq f_1^* \wedge f_2^*, r \leq 0.7, \\ f_2^*, & \text{if } f \sqsubseteq f_2^*, f \not\sqsubseteq f_1^* \wedge f_2^*, r \leq 0.4, \\ f_1^* \wedge f_2^* & \text{if } f \sqsubseteq f_1^* \wedge f_2^*, r \leq 0.5 \\ f_1^* \vee f_2^* & \text{if } f \sqsubseteq f_1^* \vee f_2^*, f \not\sqsubseteq f_1^*, f \not\sqsubseteq f_2^*, r \leq 0.6, \\ 0_X, & \text{otherwise.} \end{cases}$$

$$\mathcal{C}_{\mathcal{T}_X}(k_2, f, r) = \begin{cases} 1_X, & \text{if } f = 1_X, r \in (0, 1], \\ f_1^*, & \text{if } f \sqsubseteq f_1^*, r \leq 0.5, \\ 0_X, & \text{otherwise.} \end{cases}$$

(2) Let $Z = \{z_1, z_2, z_3, z_4\}$ and $E_3 = \{p, q, r, s\}$ be given. Define $\chi_\phi : (L^X)^{E_3} \rightarrow (L^X)^{E_1}$ as follows

$$\chi(z_1) = \chi(z_2) = h_1, \chi(z_3) = h_4, \chi(z_4) = h_5$$

$$\phi(p) = \phi(q) = e, \phi(r) = b, \phi(s) = w.$$

Since $\chi_\phi^{-1}(f_1)(p)(z_1) = f_1(\phi(p))(\chi(z_1)) = 0.8$, we have

$$\begin{aligned} \chi_\phi^{-1}(f_1)(p) &= (0.8, 0.8, 0.9, 0.6), & \chi_\phi^{-1}(f_1)(q) &= (0.8, 0.8, 0.9, 0.6) \\ \chi_\phi^{-1}(f_1)(r) &= (0.7, 0.7, 0.5, 0.7), & \chi_\phi^{-1}(f_1)(s) &= (0.4, 0.4, 0.6, 0.5) \\ \chi_\phi^{-1}(f_2)(p) &= (0.5, 0.5, 0.8, 0.4), & \chi_\phi^{-1}(f_2)(q) &= (0.5, 0.5, 0.8, 0.4) \\ \chi_\phi^{-1}(f_2)(r) &= (0.3, 0.3, 0.4, 0.5), & \chi_\phi^{-1}(f_2)(s) &= (0.5, 0.5, 0.5, 0.1). \end{aligned}$$

From Theorem 13, we obtain a $[0, 1]$ -fuzzy (K, E_3) -soft topology $\mathcal{T}_Z : K \rightarrow [0, 1]^{([0, 1]^Z)^{E_3}}$ as follows:

$$(\mathcal{T}_Z)_{k_1}(h) = \begin{cases} 1, & \text{if } h = 0_Z \text{ or } 1_Z, \\ 0.7, & \text{if } h = \chi_\phi^{-1}(f_1) \\ 0.4, & \text{if } h = \chi_\phi^{-1}(f_2) \\ 0.5, & \text{if } h = \chi_\phi^{-1}(f_1) \vee \chi_\phi^{-1}(f_2) \\ 0.6, & \text{if } h = \chi_\phi^{-1}(f_1) \wedge \chi_\phi^{-1}(f_2) \\ 0, & \text{otherwise.} \end{cases}$$

$$(\mathcal{T}_Z)_{k_2}(h) = \begin{cases} 1, & \text{if } h = 0_Z \text{ or } 1_Z, \\ 0.5, & \text{if } h = \chi_\phi^{-1}(f_1) \\ 0, & \text{otherwise.} \end{cases}$$

From Theorem 14, we obtain a $[0, 1]$ -fuzzy (K, E_3) -soft closure operator $\mathcal{C}_{\mathcal{T}_Z} : K \times ([0, 1]^Z)^{E_3} \times (0, 1] \rightarrow ([0, 1]^Z)^{E_3}$ as follows:

$$\mathcal{C}_{\mathcal{T}_Z}(h, r) = \begin{cases} 1_X, & \text{if } h = 1_X, r \in (0, 1], \\ \chi_\phi^{-1}(f_1^*), & \text{if } h \sqsubseteq \chi_\phi^{-1}(f_1^*), h \not\sqsupseteq \chi_\phi^{-1}(f_1^*) \wedge \chi_\phi^{-1}(f_2^*), r \leq 0.7 \\ \chi_\phi^{-1}(f_2^*), & \text{if } h \sqsubseteq \chi_\phi^{-1}(f_2^*), h \not\sqsupseteq \chi_\phi^{-1}(f_1^*) \wedge \chi_\phi^{-1}(f_2^*), r \leq 0.4 \\ \chi_\phi^{-1}(f_1^*) \wedge \chi_\phi^{-1}(f_2^*), & \text{if } h \sqsubseteq \chi_\phi^{-1}(f_1^*) \wedge \chi_\phi^{-1}(f_2^*), r \leq 0.5 \\ \chi_\phi^{-1}(f_1^*) \vee \chi_\phi^{-1}(f_2^*), & \text{if } h \sqsubseteq \chi_\phi^{-1}(f_1^*) \vee \chi_\phi^{-1}(f_2^*) \\ & h \not\sqsupseteq \chi_\phi^{-1}(f_1^*), h \not\sqsupseteq \chi_\phi^{-1}(f_2^*), r \leq 0.6 \\ 0_X, & \text{otherwise.} \end{cases}$$

□

Example 16. Let X and E_X be given as Example 3.3. Define a binary operation $\odot$ on $[0, 1]$ by

$$x \odot y = \max\{0, x + y - 1\}, \quad x \rightarrow y = \min\{1 - x + y, 1\}$$

$$x \oplus y = \min\{1, x + y\}, \quad x^* = 1 - x.$$

Then $([0, 1], \odot, \oplus, \rightarrow, *, 0, 1)$ be a complete residuated lattice. Let $E = \{b, w, c\} \subset E_X$, $f \in ([0, 1]^X)^E$ be a fuzzy soft set as follow:

$$\begin{aligned} f_b &= (0.5, 0.3, 0.5, 0.6, 0.2) & f_c &= (0.1, 0.2, 0.6, 0.5, 0.5) \\ f_w &= (0.4, 0.4, 0.5, 0.6, 0.6) \end{aligned}$$

$$\begin{aligned} (f \odot f)_b &= (0.0, 0.0, 0.0, 0.2, 0.0), & (f \odot f)_c &= (0.0, 0.0, 0.2, 0.0, 0.0) \\ (f \odot f)_w &= (0.0, 0.0, 0.0, 0.2, 0.2). \end{aligned}$$

(1) Let $K = \{k_1, k_2\}$ be given. Define a $[0, 1]$ -fuzzy (K, E) -soft topology $\mathcal{T}_X : K \rightarrow [0, 1]^{([0, 1]^X)^E}$ as follows:

$$\begin{aligned} (\mathcal{T}_X)_{k_1}(h) &= \begin{cases} 1, & \text{if } h = 0_X \text{ or } h = 1_X, \\ 0.6, & \text{if } h = f \\ 0.3, & \text{if } h = f \odot f \\ 0, & \text{otherwise.} \end{cases} \\ (\mathcal{T}_X)_{k_2}(h) &= \begin{cases} 1, & \text{if } h = 0_X \text{ or } h = 1_X, \\ 0.4, & \text{if } h = f \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We obtain a $[0, 1]$ -fuzzy (K, E) -soft closure operator $\mathcal{C}_{\mathcal{T}_X} : K \times ([0, 1]^X)^E \times (0, 1] \rightarrow ([0, 1]^X)^E$ as follows:

$$\mathcal{C}_{\mathcal{T}_X}(k_1, h, r) = \begin{cases} 1_X, & \text{if } h = 1_X, r \in (0, 1], \\ f^*, & \text{if } h \sqsubseteq f^*, r \leq 0.6 \\ f^* \oplus f^* & \text{if } h \sqsubseteq f^* \oplus f^*, h \not\sqsubseteq f^*, r \leq 0.3 \\ 0_X, & \text{otherwise.} \end{cases}$$

$$\mathcal{C}_{\mathcal{T}_X}(k_2, h, r) = \begin{cases} 1_X, & \text{if } h = 1_X, r \in (0, 1], \\ f^*, & \text{if } h \sqsubseteq f^*, r \leq 0.4 \\ 0_X, & \text{otherwise.} \end{cases}$$

(2) Let $Z = \{z_1, z_2, z_3, z_4\}$ and $E_4 = \{p, q, r\}$ be given. Define $\rho_\psi : ([0, 1]^Z)^{E_4} \rightarrow ([0, 1]^Z)^E$ as follows

$$\rho(z_1) = h_2, \rho(z_2) = h_1, \rho(z_3) = h_5, \rho(z_4) = h_3$$

$$\psi(p) = c, \psi(q) = w, \psi(r) = b.$$

$\rho_\psi^{-1}(f)$	z_1	z_2	z_3	z_4
p	0.2	0.1	0.5	0.6
q	0.4	0.4	0.6	0.5
r	0.3	0.5	0.2	0.5

$\rho_\psi^{-1}(f \odot f)$	z_1	z_2	z_3	z_4
p	0.0	0.0	0.0	0.2
q	0.0	0.0	0.2	0.0
r	0.0	0.0	0.0	0.0

$\rho_\psi^{-1}(f^* \oplus f^*)$	z_1	z_2	z_3	z_4
p	1.0	1.0	1.0	0.8
q	1.0	1.0	0.8	1.0
r	1.0	1.0	1.0	1.0

From Theorem 13, we obtain a $[0, 1]$ -fuzzy (K, E_4) -soft topology $\mathcal{T}_Z : K \rightarrow [0, 1]^{([0, 1]^Z)^{E_4}}$ as follows:

$$(\mathcal{T}_Z)_{k_1}(h) = \begin{cases} 1, & \text{if } h = 0_Z \text{ or } 1_Z, \\ 0.6, & \text{if } h = \varphi_\psi^{-1}(f) \\ 0.3, & \text{if } h = \varphi_\psi^{-1}(f) \odot \varphi_\psi^{-1}(f) \\ 0, & \text{otherwise.} \end{cases}$$

$$(\mathcal{T}_Z)_{k_2}(h) = \begin{cases} 1, & \text{if } h = 0_Z \text{ or } 1_Z, \\ 0.4, & \text{if } h = \varphi_\psi^{-1}(f) \\ 0, & \text{otherwise.} \end{cases}$$

From Theorem 14, we obtain a $[0, 1]$ -fuzzy (K, E_4) -soft closure operator $\mathcal{C}_{\mathcal{T}_Z} : K \times ([0, 1]^Z)^{E_4} \times (0, 1] \rightarrow ([0, 1]^Z)^{E_4}$ as follows:

$$\mathcal{C}_{\mathcal{T}_Z}(k_1, h, r) = \begin{cases} 1_Z, & \text{if } h = 1_Z, r \in (0, 1], \\ \rho_\psi^{-1}(f^*), & \text{if } h \sqsubseteq \rho_\psi^{-1}(f^*), ; r \leq 0.6 \\ \rho_\psi^{-1}(f^*) \oplus \rho_\psi^{-1}(f^*) & \text{if } h \sqsubseteq \rho_\psi^{-1}(f^* \oplus f^*) \\ h \not\sqsubseteq \rho_\psi^{-1}(f^*), r \leq 0.3, \\ 0_Z, & \text{otherwise.} \end{cases}$$

$$\mathcal{C}_{\mathcal{T}_Z}(k_2, h, r) = \begin{cases} 1_Z, & \text{if } h = 1_Z, r \in (0, 1], \\ \rho_\psi^{-1}(f^*), & \text{if } h \sqsubseteq \rho_\psi^{-1}(f^*), r \leq 0.4 \\ 0_Z, & \text{otherwise.} \end{cases}$$

□

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