FILTERS IN SEMITRELLISES

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Abstract: In this paper we have introduced the concept of semitrellises as a generalization of semilattices. Some characterizations of filters in semitrellises are obtained. Also we have proved that a dense subset $\mathbf{D}$ of a semitrllis $\mathbf{S}$ is an upset if $\mathbf{D}$ is meet associative.

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1. Introduction

A reflexive and antisymmetric binary relation $\trianglelefteq$ on a set A is called a *pseudo-order* on A and $\langle A, \trianglelefteq \rangle$ is called a *pseudo-ordered set* or a *psoet*. For $a, b \in A$ if $a \trianglelefteq b$ and $a \neq b$, then we write $a \triangleleft b$. For a subset B of A , the notions of a lower bound, an upper bound, the greatest lower bound (meet, denoted by $\wedge B$), the least upper bound (join, denoted by $\vee B$) are defined analogous to the corresponding notions in a poset (refer[2]).

It is shown in [5] that any psoet can be regarded as a digraph(possibly infinite) in which for any pair of distinct elements u and v there is no directed line between u and v or if there is a directed line from u to v , there is no directed line from v to u .

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A poset $\langle A, \leq \rangle$ in which every pair of elements has a least upper bound and a greatest lower bound is called a *trellis*. In other words, a trellis is an algebra $\langle L, \wedge, \vee \rangle$ where the binary operations $\wedge$ and $\vee$ satisfy the following properties:

- (i) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ for all $a, b \in L$;
- (ii) $a \vee (b \wedge a) = a = a \wedge (b \vee a)$ for all $a, b \in L$;
- (iii) $a \vee ((a \wedge b) \vee (a \wedge c)) = a = a \wedge ((a \vee b) \wedge (a \vee c))$ for all $a, b, c \in L$.

The notion of a trellis as a non-associative generalisation of a lattice is due to E.Fried [1] and H.L.Skala [6]. A poset S is said to be a meet semitrellis if for all $a, b \in S, a \wedge b \in S$. Dually we can define a join semitrellis. In this paper by a semitrellis we mean a meet semitrellis.

2. definitions and Results

Definition 2.1. A meet semitrellis is called *directed above* if for all $a, b \in S$, there exists an element $c \in S$ such that $c \geq a$ and $c \geq b$.

Meet semitrellis S directed above will be denoted by S^Δ for all future references.

Definition 2.2. A nonempty subset I of S^Δ is called a *downset* if for $x \in I$ and $y \in S, y \leq x$ imply $y \in I$. Downset is also called an *ideal* if for $x, y \in I, x \vee y \in I$.

Definition 2.3. A nonempty set F of S^Δ is called an *upset* if for $x \in F$ and $y \in S, y \geq x$ imply $y \in F$. An upset F of S is also called a *filter* if for all $x, y \in F, x \wedge y \in F$.

Remark 2.4. Every ideal is a downset as well as an upset but not the converse

Definition 2.5. An ideal (downset) P of S^Δ is called a prime ideal (downset) if $a \wedge b \in P$ imply either $a \in P$ or $b \in P$.

Definition 2.6. A filter F of S^Δ is called prime if $S - F$ is a prime ideal.

Definition 2.7. A filter F of S^Δ is called a maximal filter if $F \neq S$ and F is not contained in any other proper filter of S^Δ .

Definition 2.8. A prime downset is called a minimal prime downset if it does not contain any other prime downset of S .

Definition 2.9. In S^Δ , an ideal J is called semiprime ideal if for $x, y, z \in S, x \wedge y \in J$ and $x \wedge z \in J$ imply that $x \wedge (y \vee z) \in J$.

Theorem 2.10. *Let F be a nonempty proper subset of a semitrellis S . then F is a filter if and only if $S - F$ is a prime downset.*

Proof. Let F be a filter of a semitrellis S . Let $x \in S - F$ and $y \trianglelefteq x$. Then clearly $y \notin F$. Therefore $y \in S - F$ and $S - F$ becomes a proper downset.

Claim: $S - F$ is a prime downset.

Let $a, b \in S$ such that $a \wedge b \in S - F$. Then $a \wedge b \notin F$ and hence either $a \notin F$ or $b \notin F$ as F is a filter. Therefore either $a \in S - F$ or $b \in S - F$ and hence $S - F$ is a prime downset.

Conversely, let $S - F$ be a prime downset. For $x, y \in F$ we have $x \wedge y \notin S - F$. Therefore $x \wedge y \in F$. Suppose $x \in F$ and $y \trianglerighteq x$. Then $x \notin S - F$ and hence $y \notin S - F$. This implies $y \in F$, proving that F is a filter. $\square$

Theorem 2.11. *Let F be a nonempty subset of a semitrellis S . Then F is a maximal filter if and only if $S - F$ is a minimal prime downset.*

Proof. Let F be a maximal filter of S . If $S - F$ is not a minimal prime downset, then there exists a prime downset $I \subseteq S - F$. This implies $F \subseteq S - I$, contradicting the maximality of F .

Conversely, let $S - F$ be a minimal prime downset. If F is not a maximal filter, then there exists a proper filter G of S such that $F \subseteq G$. Then $S - G \subseteq S - F$, contradicting the minimality of $S - F$. $\square$

Definition 2.12. For a subset A of a poset P the set

$$A^u = \{x \in P : x \trianglerighteq a \ \forall a \in A\}$$

is called the upper cone of A .

Lower cone A^l of A is the dual notion of A^u .

Definition 2.13. An element d of a trellis L is said to be:

- (i) right transitive if $d \trianglelefteq x \trianglelefteq y$ imply $d \trianglelefteq y$;
- (ii) left transitive if $x \trianglelefteq y \trianglelefteq d$ imply $x \trianglelefteq d$;
- (iii) middle transitive if $x \trianglelefteq d \trianglelefteq y$ imply $x \trianglelefteq y$.

Definition 2.14. A subset I of a poset P is called an ideal if $a, b \in I$ implies $(a, b)^u \subseteq I$.

A proper ideal I is called prime if $(a, b)^l \subseteq I$ imply either $a \in I$ or $b \in I$.

Filter and prime filter can be defined dually.

Given a left transitive element a of a poset P , the subset $\{x \in P : x \trianglelefteq a\}$ is an ideal, called principal ideal generated by a and is denoted by $(a]$. Dually we have the definition of principal filter.

Definition 2.15. An ideal I of a poset P is called semiprime if $(a, b)^l \subseteq I$ and $(a, c)^l \subseteq I$ together imply that $\{a, (b, c)^u\}^l \subseteq I$

Lemma 2.16. Let I be an ideal of a poset P . If I is prime, then I is semiprime.

Proof. Let I be a prime ideal of a poset P and for $a, b, c \in P$ let $(a, b)^l \subseteq I$ and $(a, c)^l \subseteq I$. Since I is prime, we have:

Case (i): If $a \in I$, then $\{a, (b, c)^u\}^l \subseteq a^l \subseteq I$.

Case (ii): If $a \notin I$, then $b, c \in I$ and hence $(b, c)^u \subseteq I$. Therefore $\{a, (b, c)^u\}^l \subseteq (b, c)^u \subseteq I$. Thus I is semiprime. $\square$

Remark 2.17. Converse of lemma 2.17 need not be true. In the poset of Figure 1, $I = \{c, d\}$ is semiprime ideal which is not prime.

Figure 1

Definition 2.18. An element x of a poset is meet irreducible if x can not be obtained as a meet of two elements different from x .

Theorem 2.19. Every prime ideal of a poset P is a meet irreducible element of $Id(P)$

Proof. Let I be a prime ideal of P such that $I = J \cap K$ for $J, K \in Id(P)$. We need to prove that either $I = J$ or $I = K$. Clearly $I \subseteq J$ and $I \subseteq K$. Suppose $I \neq J$ and $I \neq K$, then there exists elements $x, y \in P$ such that $x \in J - I$ and $y \in K - I$. But as J and K are ideals, we must have $(x, y)^l \subseteq J \cap K = I$ and I is a prime ideal, so either $x \in I$ or $y \in I$, a contradiction to the fact that $x, y \notin I$. $\square$

3. Weakly Distributive Semitrellises and * Semitrellises

Definition 3.1. A semitrellis S with least element 0 is called weakly-distributive semitrellis if for any $a, b, c \in S$ such that $a \wedge b = 0 = a \wedge c$ implies $a \wedge d = 0$ for some $d \geq b, c$.

Figure 2

The semitrellis in Figure 2 is an example for a weakly distributive semitrellis.

Lemma 3.2. Let S be a directed above semitrellis with 0 . If S is not weakly distributive, then the set of all middle transitive elements $F = \{x \in S \mid x \geq a \wedge y \neq 0 \ \forall \ y \geq b, c\}$ where $a, b, c \in S$ such that $a \wedge b = a \wedge c = 0$ is a proper filter.

Proof. If S is not weakly distributive, there are elements $p, q, r \in S$ such that $p \wedge q = p \wedge r = 0$ and $p \wedge d \neq 0$ for all $d \geq q, r$. As $p \geq p \wedge d$, we have $p \in F$. Hence F is non empty. Clearly $0 \notin F$. We prove that F is a filter. Let $x \in F$ and $z \geq x$. Then $x \geq a \wedge y$ for all $y \geq b, c$ so that $a \wedge y \leq x \leq z$. Since x is middle transitive $z \geq a \wedge y$. Hence $z \in F$.

Let $x, z \in F$. Then $x \geq a \wedge y$ and $z \geq a \wedge y$ for all $y \geq b, c$. Then $x \wedge z \geq a \wedge y$ for all $y \geq b, c$ so that $x \wedge z \in F$. Thus F is a filter. $\square$

The following definition of weakly distributive semitrellis is equivalent to definition 3.1.

Definition 3.3. A semitrellis S with 0 is said to be a weakly distributive

semitrellis if (a) is an ideal of S for any $a \in S$, where

$$(a) = \{x \in S / x \wedge a = 0, a \in S\}.$$

Definition 3.4. If S is a semitrellis with 0, then for any subset A of S , we define $A = \{x \in S / x \wedge a = 0, \forall a \in A\}$

Remark 3.5. If $A = \{a\}$ then $A = (a)$

Definition 3.6. A semitrellis S with 0 is said to be a $*$ semitrellis if and only if for any $a \in S$ there exists $a^1 \in S$ such that $(a) = (a^1)$ where $a \wedge a^1 = 0$

Figure 3

Semitrellis of Figure 3 is weakly distributive as well as $*$ semitrellis.

Definition 3.7. An element $a \in S$ is called a *dense* element of S if and only if $(a) = \{0\}$

Definition 3.8. A subset D of a semitrellis S is called *dense* if all the elements in D are dense elements.

Definition 3.9. An element x of a semitrellis S is called meet associative if $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ for all $y, z \in S$

Definition 3.10. A subset D of a semitrellis S is said to be meet associative if every element of D are meet associative.

Proposition 3.11. A dense subset D of a semitrellis S is an upset if D is meet associative.

Proof. Let D be a meet associative dense subset of S . If $x \in D$ and $a \in S$ such that $a \geq x$. If $a \notin D$, then there exists an element $y \in S$ such that $a \wedge y = 0$. Consider $x \wedge y = (x \wedge a) \wedge y = x \wedge (a \wedge y) = 0$ which is a contradiction to the fact that x is a dense element. Therefore $a \in D$ and hence D is an upset. $\square$

Remark 3.12. Meet associativity in proposition 3.11 is only a sufficient condition but not necessary. Consider the semitrellis in the in Figure 4. Here if $D = \{a, b, c\}$, thrn D is an upset but D is not meet associative.

Figure 4

Theorem 3.13. In a weakly distributive semitrellis S , the following are equivalent.

- (i) S is a $*$ semitrellis;
- (ii) For any $x \in S$, there exists $x^1 \in S$ such that $x \wedge x^1 = 0$ and $[x] \cap [x^1] \subseteq D$ where D is a dense filter.

Proof. Let S be a weakly distributive semitrellis, then for every $a \in S$, $(a) = \{x \in S / x \wedge a = 0\}$ is an ideal.

(i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) Clearly $(x^1) \subseteq (x)$ because if $s \in (x^1)$ then $s \wedge r = 0$ for all $r \in (x^1)$. Therefore $s \wedge x = 0$ as $x \in (x^1)$ and hence $s \in (x)$.

Also let $t \in s$ be such that $t \wedge x = 0$ and $r \in S$ be such that $r \wedge x^1 = 0$. Then as x is meet associative $x \wedge (r \wedge t) = (r \wedge t) \wedge x = r \wedge (t \wedge x) = r \wedge 0 = 0$ which imply $x \in (r \wedge t)$. Similarly we can prove that $x^1 \in (r \wedge t)$. Since for $t, r \in S$, we have $t \wedge r \in S$ and $t \wedge r$ is an ideal of S it follows that there exists an element $z \in S$ such that $z \supseteq x, x^1$ and $z \in (t \wedge r)$ which imply $z \wedge (t \wedge r) = 0$. Since $z \supseteq x, x^1$, we have $z \in [x]$ and $z \in [x^1]$ so that $z \in [x] \cap [x^1] \subseteq D$. Therefore $z \in D$ and hence $t \wedge r = 0$. Thus $t \in (r)$. But $r \in (x^1)$ and $t \wedge r = 0$

implies that $t \in (x^1)$ so that $(x) \subseteq (x^1)$. Therefore $(x) = (x^1)$ proving that S is a $*$ semitrellis. $\square$

4. Nodal Filters in Semitrellises

In [7], the concept of nodal elements is introduced in trellises and its characterizations are obtained. In this section we obtain properties of nodal filters in semitrellises.

Let $F(S)$ denotes the set of all filters of an updirected semitrellis S . When ordered by set inclusion $F(S)$ is a lattice in which for every $F, G \in F(S)$, $F \wedge G = F \cap G$ and $F \vee G$ is the filter generated by $F \cup G$. That is $F \vee G = [F \cup G]$.

Definition 4.1. A node in a poset P is an element a which is comparable with every element of P

Definition 4.2. A node in a pset P is an element a which is middle transitive and comparable with every element of P .

Theorem 4.3. For a filter F of a semitrellis S , the following conditions are equivalent:

- (i) the relation $x \triangleright y$ is satisfied for right transitive elements $x \in F$ and $y \in S - F$;
- (ii) any filter G of S is comparable with F ;
- (iii) F is a node of $F(S)$ where $F(S)$ is set of all filters of F .

Proof. (i) $\Rightarrow$ (ii): Let us assume that there exists a filter G which is incomparable with F . Then there are elements x and y such that $x \in F - G$ and $y \in G - F$ and $x \not\triangleright y$, a contradiction.

(ii) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (i): If F is a node in $F(S)$, then for every right transitive element $x \in F$ and for $y \notin F$, we have $[y] \supset F$ hence $[y] \supset F \subset [x] \Rightarrow x \triangleright y$. $\square$

A filter satisfying one of the conditions in Theorem 4.3 is called a *nodal filter*. Obviously semitrellis S is an improper nodal filter.

Theorem 4.4. For a right transitive element x , $[x]$ is nodal filter if and only if x is a node in S .

Proof. Follows easily. $\square$

Let $N(S)$ be the set of all nodal filters of a semitrellis S , ordered by set inclusion. Then $N(S)$ is a chain with the greatest element S . If $1 \in S$ then $N(S)$ has the least element $\{1\}$.

Definition 4.5. Two elements x and y of a semitrellis S are said to be connected (ie $(x, y) \in R$) if there is no nodal filter separating x and y .

Note that $(x, y) \notin R \Rightarrow$ either $x \triangleright y$ or $x \triangleleft y$

Theorem 4.6. In any semitrellis S , the relation R satisfies the following properties.

- 1) R is a congruence relation of $\langle S, \wedge \rangle$;
- 2) Any R -class contains at most one right transitive node.

Proof. (1) Clearly $(x, x) \in R$ for all $x \in S$. Also if $(x, y) \in R$ then $(y, x) \in R$ for all $x, y \in S$. Suppose (x, y) and (y, z) belong to R . If possible let $(x, z) \notin R$. Then there exists a nodal filter F such that $x \in F$ and $z \notin F$. Then $y \in F$ and $(y, z) \notin R$ a contradiction. Therefore $(x, z) \in R$ and R is an equivalence relation. Now if $(x, y) \in R$ then $(x \wedge s, y \wedge s) \in R$ for every $s \in S$. Otherwise, $x \wedge s \in F$ and $y \wedge s \notin F$ for some nodal filter F . Hence $x \in F, s \in F$ and $y \notin F$ which is a contradiction as $(x, y) \in R$. Therefore R is a congruence relation in $\langle S, \wedge \rangle$.

(2) Let a and b be two right transitive connected nodes of S . Then we have either $a \triangleleft b$ or $b \triangleleft a$. If $a \triangleleft b$ then a and b are separated by a nodal filter $[b)$ which imply $(a, b) \notin R$, a contradiction. $\square$

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