A CLASSIFICATION OF RAMANUJAN COMPLEMENTS OF UNITARY CAYLEY GRAPHS

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Abstract: The unitary Cayley graph on n vertices, X_n , has vertex set $\mathbb{Z}_n$, where two vertices a and b are connected by an edge if and only if they differ by a multiplicative unit modulo n , i.e. $\gcd(ab, n) = 1$. A k -regular graph X is Ramanujan if and only if $\lambda(X) \leq 2\sqrt{k-1}$ where $\lambda(X)$ is the second largest absolute value of the eigenvalues of the adjacency matrix of X . We obtain a complete characterization of the cases in which the complements of unitary Cayley graph $\bar{X}_n$ is a Ramanujan graph.

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1. Introduction

The eigenvalues of a graph are eigenvalues of its adjacency matrix, and the spectrum of a graph is the collection of its eigenvalues together with multiplicities. If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct eigenvalues of a graph X and $m_1, m_2, \dots, m_k$ the corresponding multiplicities, then we denote the spectrum of X by

$$\text{Spec}(X) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}.$$

Given a finite group G and a symmetric subset S of G , we define the Cayley

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graph $X = \text{Cay}(G, S)$ to be the graph whose vertex set is G , in which two vertices v and u in G are connected by an edge if and only if vu^{-1} is in S . A Cayley graph of the form $\text{Cay}(G, S)$ with $G = \mathbb{Z}_n$ is called a circulant graph. It is easy to see that $\text{Cay}(G, S)$ is a simple, $|S|$ -regular graph.

The unitary Cayley graph on n vertices, X_n , is defined to be the undirected graph whose vertex set is $\mathbb{Z}_n$, in which two vertices a and b are connected by an edge if and only if $\gcd(a - b, n) = 1$. This can also be stated as $X_n = \text{Cay}(\mathbb{Z}_n, \mathbb{U}_n)$, where $\mathbb{Z}_n$ is the additive group of integers modulo n and $\mathbb{U}_n = \mathbb{Z}_n^*$ is the set of multiplicative units modulo n . X_n is a simple, $\varphi(n)$ -regular graph, where φ is the Euler totient function. When discussing X_n , we always assume $n > 3$.

Lemma 1. *The eigenvalues of any adjacency matrix of X_n are*

$$\lambda_m(n) = \mu\left(\frac{n}{(n, m)}\right) \frac{\varphi(n)}{\varphi\left(\frac{n}{(n, m)}\right)}, \quad (1)$$

where μ is the Mobius function.

Proof. see [4, Klotz, W. and Sander, T. (2007)] □

When $\frac{n}{(n, m)}$ is square-free,

$$|\lambda_m(n)| = \frac{\varphi(n)}{\varphi\left(\frac{n}{(n, m)}\right)}. \quad (2)$$

Corollary 2.

$$\text{Spec}(X_{p^\alpha}) = \begin{pmatrix} p^\alpha - p^{\alpha-1} & -p^{\alpha-1} & 0 \\ 1 & p-1 & p^\alpha-1 \end{pmatrix}. \quad (3)$$

The complement $\bar{X}$ of a graph X is the graph with the same vertex set as X such that two vertices are adjacent in $\bar{X}$ if and only if they are not adjacent in X .

Lemma 3 ([2]). *Let X be an k -regular graph with n vertices. Then X and $\bar{X}$ have the same eigenvectors, and their largest eigenvalues are k , and $n - k - 1$ respectively. Moreover, if the eigenvalues of X are $k, \lambda_2, \dots, \lambda_n$, then the eigenvalues of $\bar{X}$ are $n - k - 1, 1 - \lambda_2, \dots, 1 - \lambda_n$.*

Lemma 1 and Corollary 2 together imply the following result.

Corollary 4.

$$\text{Spec}(\bar{X}_{p^\alpha}) = \begin{pmatrix} p^{\alpha-1} - 1 & -1 \\ p & p^\alpha - p \end{pmatrix}. \quad (4)$$

2. Ramanujan Unitary Cayley Graphs

Recall that the adjacency matrix of any k -regular graph X has eigenvalues between k and $-k$, where k is an eigenvalue with multiplicity precisely equal to the number of connected components of X . Furthermore, if $\lambda(X)$ denotes the largest absolute value of the eigenvalues of the adjacency matrix of X , smaller than k , then the graph X is called Ramanujan if and only if

$$\lambda(X) \leq 2\sqrt{k-1} \quad (5)$$

Note that $\lambda(X)$ is only defined for regular graphs X with 3 or more vertices. Writing n in the form $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ for some distinct primes $p_1 < p_2 < \cdots < p_k$, and positive integers $\alpha_1, \dots, \alpha_k$, we can determine $\lambda(X_n)$ as follows. Since X_n is $\varphi(n)$ -regular, we find the maximum absolute value of an eigenvalue $\lambda_m(n)$ of the adjacency matrix of X_n , smaller than $\varphi(n)$. This can be accomplished by looking at 1. Indeed, we see that if $n = 2^\alpha$; then the eigenvalues have absolute value of either 0 or $\varphi(n)$ (since, the only values of $m, 0 \leq m \leq n-1$, which make $\frac{n}{(n,m)}$ square-free are $m = 0$ and $m = 2^{\alpha-1}$), resulting in eigenvalues $\varphi(n)$ and $-\varphi(n)$. Thus $\lambda(X_{2^\alpha}) = 0$ and so X_{2^α} satisfies 2 and thus is Ramanujan. It is known from [1] that the graph X_n is Ramanujan if and only if n satisfies one of the following conditions for some distinct odd primes $p < q$ and natural α .

1. $n = 2^\alpha, \alpha \geq 2$.
2. $n = p$.
3. $n = 2^\alpha p$ with $\alpha \geq 1, p > 2^{\alpha-3} + 1$.
4. $n = p^2, 2p^2, 4p^2$.
5. $n = p, 2pq$ with $p < q \leq 4p - 5$.
6. $n = 4pq$ with $p < q \leq 2p - 3$.

3. Ramanujan Complements of Unitary Cayley Graphs

Lemma 5 ([3]). Let $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, be the canonical factorization of an integer n into prime powers, where $p_1 < p_2 < \cdots < p_k$ are primes and each $\alpha_i \geq 1$. If $k \geq 3$ or $k = 2$ and $p_1 > 2$, then $2^{k-1} \varphi(n) > n$.

Theorem 6. Let $n \geq 2$ be a intager. Then $\bar{X}_n$ is Ramanujan if and only if n is one of the following forms:

1. $n = p^a$ with p a prime and $a \geq 1$
2. $n = p^a q^b$ with $p = 2, q = 3$ and $1 \leq a \leq 3, b = 1$, or $a = 1, 1 \leq b \leq 2$.
3. $n = pq$ with p and q are primes and $3 \leq p \leq 5, 5 \leq q \leq 7$.
4. $n = 2 \cdot 5, 2 \cdot 3 \cdot 5$

Proof. By (4), $\bar{X}_{p^a}$ is Ramanujan.

Case 1: $p_1 = 2$ and $k \geq 2$. In this case, $\lambda(\bar{X}_n) = \varphi(n) - 1$ and $\bar{X}_n$ is Ramanujan if and only if

$$\varphi(n) - 1 \leq 2\sqrt{n - \varphi(n) - 2} \quad (6)$$

This condition is satisfied if only $\varphi(n)^2 < 4n$. In particular, if $k \geq 4$, then by Lemma 5, $\varphi(n)^2 > 4n$ and so $\bar{X}_n$ is not Ramanujan. Assume $k \leq 3$.

Case 1.1: $k = 3$ and $n = 2^a p^b q^c$. Since $(p-1)^2 = p(p-2) + 1 > p(p-2)$, If $a \geq 3$, then

$$\frac{\varphi(n)^2}{n} = 2^{a-2} p^{b-1} q^{c-1} \frac{(p-1)^2}{p} \frac{(q-1)^2}{q} > 2(p-2)(q-2) > 4$$

and so $\bar{X}_n$ is not Ramanujan. It is easy to see that if $b \geq 2$ or $c \geq 2$, then $\varphi(n)^2 > 4n$, and so $\bar{X}_n$ is not Ramanujan. It remains to consider the case where $n = 2pq$ or $4pq$. If $n = 2pq$, then

$$\frac{\varphi(n)^2}{n} = \frac{(p-1)^2(q-1)^2}{2pq} > \frac{(p-2)(q-2)}{2}$$

If $p \neq 3$ or $p = 3$ and $q \geq 7$, then $\varphi(n)^2 > 4n$, and so $\bar{X}_n$ is not Ramanujan. It is easy to see that $n = 2 \cdot 3 \cdot 5$, then $\bar{X}_n$ is Ramanujan, whilst if $n = 2 \cdot 3 \cdot 7$, $\bar{X}_n$ is not Ramanujan. If $n = 4pq$, similar to earlier state if $p \neq 3$, or $p = 3$ and $q \neq 5$, then $\bar{X}_n$ is not Ramanujan. For $n = 4 \cdot 3 \cdot 5$, the condition (6) is not established and so $\bar{X}_n$ is not Ramanujan.

Case 1.2: $k = 2$ and $n = 2^a p^b$. In this case

$$\frac{\varphi(n)^2}{n} = \frac{2^{a-1} p^{b-1}}{2p} (p-1)$$

As case 1.1, If $a \geq 4$, $p \geq 7$ or $b \geq 3$, then $\varphi(n)^2 \geq 4n$ and so $\bar{X}_n$ is not Ramanujan. It is easy to see that $n = 2 \cdot 3, 2 \cdot 3^2, 2 \cdot 5, 2^2 \cdot 3, 2^3 \cdot 3$; then, $\bar{X}_n$ is Ramanujan.

Case 2: $p_1 \geq 3$. In this case $\lambda(\bar{X}_n) = \frac{\varphi(n)}{p_1-1} - 1$ and so $\bar{X}_n$ is Ramanujan if and only if

$$\frac{\varphi(n)}{p_1-1} - 1 \leq 2\sqrt{n - \varphi(n) - 2} \quad (7)$$

this inequality is equivalent to

$$\frac{\varphi(n)}{p_1-1} \leq -(2p-3) + \sqrt{4n-9+(2p-3)^2} \quad (8)$$

and this condition is not satisfied unless

$$\varphi(n)^2/n < 4(p-1)^2 \quad (9)$$

If $k \geq 4$, then by Lemma 5, $\varphi(n)^2/n > \varphi(n)/2^{k-1} > 4(p-1)$ and so $\bar{X}_n$ is not Ramanujan. Assume $k \leq 3$ and $n = p^a q^b r^c$.

Case 2.1: $k = 3$. In this case, we have

$$\frac{\varphi(n)^2}{n} = p^{a-1} q^{b-1} r^{c-1} \frac{(p-1)^2 (q-1)^2 (r-1)^2}{pqr} \geq \frac{p^{a-1}}{p} (p-1)^2 (q-2)(r-2)$$

Therefore, if $a \geq 3$, then $\varphi(n)^2/n \geq 4(p-1)^2$ and so $\bar{X}_n$ is not Ramanujan. Similarly, if $b \geq 2, c \geq 2$ or $p \geq 7$, or $n = 9pq, 25pq$, then $\bar{X}_n$ is not Ramanujan. It is easy to see that $n = 3 \cdot 5 \cdot 7$; then, (8) is violated and again $\bar{X}_n$ is not Ramanujan. Moreover, If $n = 3pq$, with $p \neq 3$ and $q \neq 7$, then

$$\frac{\varphi(n)^2}{n} = \frac{4(p-1)^2 (q-1)^2}{3pq} > (p-2)(q-2) \geq 16 = 4(3-1)^2$$

and so $\bar{X}_n$ is not Ramanujan.

Case 2.2: $k = 2$ and $n = p^a q^b$. In this case by Lemma 5,

$$\frac{\varphi(n)}{n} > \frac{1}{2} \Rightarrow \frac{\varphi(n)^2}{n} > \frac{\varphi(n)}{2} = \frac{p^{a-1} q^{b-1} (p-1)(q-1)}{2} > \frac{p^{a-1} q^{b-1} (p-1)^2}{2}$$

and so if $a \geq 3, b \geq 2$ or $a = b = 2$, then (9) is violated and $\bar{X}_n$ is not Ramanujan. If $n = pq$, then from (8), we have $(q - 2)^2 \leq 4p$ and so if $q > 7$, then $\bar{X}_n$ is not Ramanujan. It is easy to see that $n = 3 \cdot 5, 3 \cdot 7, 5 \cdot 7$, then $\bar{X}_n$ is not Ramanujan. If $n = pq^2$, then

$$\frac{\varphi(n)^2}{n} = \frac{(p-1)^2(q-1)^2}{p} > 4(p-1)^2$$

and so $\bar{X}_n$ is not Ramanujan. As in previous cases, if $n = p^2q$, then $\bar{X}_n$ is not Ramanujan. $\square$

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