EVASION DIFFERENTIAL GAME OF TWO PURSUERS AND ONE EVADER WITH COORDINATE-WISE INTEGRAL CONSTRAINTS

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Abstract: An evasion differential game of two pursuers and one evader in $\mathbb{R}^2$ is studied. Unlike the traditional integral constraints, in the present work, integral constraints are imposed on each component of control functions of the players. By definition, evasion is said to be possible if the state of a pursuer does not coincide with that of the evader for all $t \geq 0$. Sufficient conditions of evasion are obtained and then strategies for the evader are constructed.

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1. Introduction

A number of studies were carried out for differential games (see, e.g., [7]–[9]) and differential games with integral constraints were studied in many papers (see, e.g., [1], [3], [10]) but there are only a few papers for evasion differential games from many pursuers with integral constraints.

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In paper [4], a simple motion differential game of countably many pursuers and one evader in Hilbert space l_2 is studied. The problems were described by an infinite system of differential equations with integral constraints imposed on controls of players. It was shown that if the total energy of countably many pursuers is less than that of the evader, then evasion is possible.

Ibragimov et al [5], studied a two dimensional evasion differential game with several pursuers and one evader with integral constraints imposed on control functions of players. The researchers have solved the game by presenting an explicit strategy for the evader which guarantees evasion assuming that the total resource of the pursuers does not exceed that of the evader. The explicit strategy for the evader is given as a function of the positions and control parameters of pursuers. The distances between the evader and pursuers were estimated as well.

An evasion differential game of many pursuers and one evader with integral constraints in the plane is considered in another paper of Ibragimov et al [6]. The game is described by simple differential equations where each component of the control functions of players is subjected to integral constraint. Researchers have shown that evasion is possible where a sufficient condition of evasion have been obtained.

Recently, Alias et al [2] considered a simple motion evasion differential game of infinitely many pursuers and evaders in l_2 . Control functions of all players are subjected to integral constraints. An interesting result was obtained where evasion is possible if

$$\sum_{j=1} \sigma_j^2 > \sum_{i=1} \rho_i^2,$$

where σ_j^2 is the total resource of j th evader and ρ_i^2 is that of i th pursuer by letting the series $\sum_{j=1} \sigma_j^2$ be convergent.

Previously, pursuit differential game of two pursuers and one evader in $\mathbb{R}^2$ was studied for the problem in Section 3 below. In the present paper, we consider an evasion differential game for the same problem. We find sufficient conditions and construct strategies for the evader.

2. Statement of the Problem

In differential game of two pursuers and one evader, the movements of the players in $\mathbb{R}^2$ are described by the following equations:

$$\begin{aligned} \dot{x}_{11} &= u_{11}(t), \quad x_{11}(0) = x_{11}^0, \quad \int_0^\cdot u_{11}^2(t) dt \leq \rho_{11}^2, \\ \dot{x}_{12} &= u_{12}(t), \quad x_{12}(0) = x_{12}^0, \quad \int_0^\cdot u_{12}^2(t) dt \leq \rho_{12}^2, \end{aligned} \quad (1)$$

$$\begin{aligned} \dot{x}_{21} &= u_{21}(t), \quad x_{21}(0) = x_{21}^0, \quad \int_0^\cdot u_{21}^2(t) dt \leq \rho_{21}^2, \\ \dot{x}_{22} &= u_{22}(t), \quad x_{22}(0) = x_{22}^0, \quad \int_0^\cdot u_{22}^2(t) dt \leq \rho_{22}^2, \end{aligned} \quad (2)$$

$$\begin{aligned} \dot{y}_1 &= v_1(t), \quad y_1(0) = y_1^0, \quad \int_0^\cdot v_1^2(t) dt \leq \sigma_1^2, \\ \dot{y}_2 &= v_2(t), \quad y_2(0) = y_2^0, \quad \int_0^\cdot v_2^2(t) dt \leq \sigma_2^2, \end{aligned} \quad (3)$$

where $\rho_{ij}, \sigma_j \geq 0$ are given positive numbers, $x_i(t) = (x_{i1}(t), x_{i2}(t))$ and $y(t) = (y_1(t), y_2(t))$ are the positions of the players at time t while $u_i = (u_{i1}, u_{i2})$ and $v(t)$ are the control parameters of the pursuers and the evader respectively for $i = 1, 2$.

Definition 2.1. A measurable function $u_i(t) = (u_{i1}(t), u_{i2}(t))$, $t \geq 0$, is called admissible control of the pursuer x_i if

$$\int_0^\cdot u_{ij}^2(s) ds \leq \rho_{ij}^2, \quad i, j = 1, 2. \quad (4)$$

Definition 2.2. A measurable function $v(t) = (v_1(t), v_2(t))$, $t \geq 0$, is called admissible control of evader if

$$\int_0^\cdot v_j^2(s) ds \leq \sigma_j^2, \quad j = 1, 2. \quad (5)$$

Definition 2.3. A function of the form

$$V(t) = \begin{cases} (0, 0), & 0 \leq t \leq \varepsilon, \\ f(u_1(t - \varepsilon), u_2(t - \varepsilon)), & t > \varepsilon, \end{cases} \quad (6)$$

is called strategy of evader, where ε is a positive number, $f : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ is a continuous function, and $u_i(t) = (u_{i1}(t), u_{i2}(t))$, $t \geq 0$, $i = 1, 2$, are admissible controls of pursuers.

Definition 2.4. We say that evasion is possible in the game (1)–(6) if there exists a strategy of evader such that $x_i(t) \neq y(t)$, $i = 1, 2$, $t \geq 0$, for any admissible control of pursuers.

Problem. Find conditions under which evasion is possible in the game (1)–(6) and construct strategies for the evader.

3. Main Result

In this section, we formulate the main result of this paper. The polygon is called polygon of energy vectors. Horizontal axis represents resources of first coordinate of evader's control and vertical axis represents that of the second coordinate. Let $M = M_1 \cup M_2 \cup M_3$, where $M_1 = M_{11} \cup M_{12}$, $M_2 = M_{21} \cup M_{22}$ are defined as follows (Figure 1):

$$\begin{aligned} M_{11} &= \{(\xi, \eta) \mid \xi \geq \rho_{11}^2 + \rho_{21}^2, \eta \geq 0\}, \\ M_{12} &= \{(\xi, \eta) \mid \xi \geq 0, \eta \geq \rho_{12}^2 + \rho_{22}^2\}, \\ M_{21} &= \{(\xi, \eta) \mid \min\{\rho_{11}^2, \rho_{21}^2\} < \xi < \max\{\rho_{11}^2, \rho_{21}^2\}, \max\{\rho_{12}^2, \rho_{22}^2\} < \eta < \rho_{12}^2 + \rho_{22}^2\}, \\ M_{22} &= \{(\xi, \eta) \mid \max\{\rho_{11}^2, \rho_{21}^2\} < \xi < \rho_{11}^2 + \rho_{21}^2, \min\{\rho_{12}^2, \rho_{22}^2\} < \eta < \max\{\rho_{12}^2, \rho_{22}^2\}\}, \\ M_3 &= \{(\xi, \eta) \mid \max\{\rho_{11}^2, \rho_{21}^2\} < \xi < \rho_{11}^2 + \rho_{21}^2, \max\{\rho_{12}^2, \rho_{22}^2\} < \eta < \rho_{12}^2 + \rho_{22}^2\}. \end{aligned}$$

Figure 1: Polygon of energy vectors

First, we formulate lemmas for auxiliary differential games.

Let us given a differential game of one pursuer and one evader described by the following equations

$$\dot{x}(t) = u(t), \quad x(0) = x_0, \quad \int_0^\infty u^2(t)dt \leq \rho^2, \quad (7)$$

$$\dot{y}(t) = v(t), \quad y(0) = y_0, \quad \int_0^\infty v^2(t)dt \leq \sigma^2, \quad (8)$$

where $\rho, \sigma \geq 0$, $x, x_0, u, y, y_0, v \in \mathbb{R}$ with $x_0 \neq y_0$, $x(t)$ and $y(t)$ are the positions, $u(t)$ and $v(t)$, $t \geq 0$, are the control functions of the pursuer and the evader respectively.

Lemma 3.1. *If $\rho \leq \sigma$, then evasion is possible in the game (7)–(8).*

It is not difficult to show that the following strategy of evader

$$v(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon, \\ |u(t - \varepsilon)|e, & t > \varepsilon, \end{cases} \quad (9)$$

where

$$e = \frac{y_0 - x_0}{|y_0 - x_0|}, \quad 0 < \varepsilon < \frac{1}{4\rho^2}(y_0 - x_0)^2, \quad (10)$$

is admissible and ensures evasion.

Now consider a differential game of two pursuers and one evader described by the following equations

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad \int_0^\infty u_i^2(t)dt \leq \rho_i^2, \quad (11)$$

$$\dot{y}(t) = v(t), \quad y(0) = y_0, \quad \int_0^\infty v^2(t)dt \leq \sigma^2, \quad (12)$$

where $\rho_i, \sigma \geq 0$, $x_i, x_{i0}, u_i, y, y_0, v \in \mathbb{R}$ with $x_{i0} \neq y_0$, $u_i(t)$ is the control function of the i -th pursuer, $i = 1, 2$, and $v(t)$ is that of the evader.

Lemma 3.2. *If $\sigma^2 \geq \rho_1^2 + \rho_2^2$, then evasion is possible in the game (11)–(12).*

Let $\rho = \sqrt{\rho_1^2 + \rho_2^2}$ and $\rho_i \leq \rho, i = 1, 2$. Set

$$v(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon, \\ \sqrt{u_1^2(t - \varepsilon) + u_2^2(t - \varepsilon)}, & t > \varepsilon, y^0 \geq \max\{x_1(\varepsilon), x_2(\varepsilon)\}, \\ -\sqrt{u_1^2(t - \varepsilon) + u_2^2(t - \varepsilon)}, & t > \varepsilon, y^0 < \min\{x_1(\varepsilon), x_2(\varepsilon)\}, \end{cases} \quad (13)$$

as admissible strategy of the evader where

$$0 < \varepsilon < \frac{1}{4\rho^2} \min_{i=1,2} (y^0 - x_i^0)^2.$$

Theorem 3.3. Let $P_1 = \left\{ (z_1, z_2) \left| \min_{i=1,2} x_{i1} \leq z_1 \leq \max_{i=1,2} x_{i1} \right. \right\}$,
 $P_2 = \left\{ (z_1, z_2) \left| \min_{i=1,2} x_{i2} \leq z_2 \leq \max_{i=1,2} x_{i2} \right. \right\}$ and $P = P_1 \cup P_2$. If $(\sigma_1^2, \sigma_2^2) \in M_1$ and $y^0 \notin P$, then evasion is possible in the game (1)–(6).

Proof. First let $(\sigma_1^2, \sigma_2^2) \in M_{11}$ and $y^0 \notin P_2$.

We have $\sigma_1^2 \geq \rho_{11}^2 + \rho_{21}^2$. By Lemma 3.2, admissible strategy of the evader

$$v_1(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon, \\ \sqrt{u_{11}^2(t-\varepsilon) + u_{21}^2(t-\varepsilon)}, & t > \varepsilon, y_1^0 \geq \max\{x_{11}(\varepsilon), x_{21}(\varepsilon)\}, \\ -\sqrt{u_{11}^2(t-\varepsilon) + u_{21}^2(t-\varepsilon)}, & t > \varepsilon, y_1^0 < \min\{x_{11}(\varepsilon), x_{21}(\varepsilon)\}, \end{cases} \quad (14)$$

where

$$0 < \varepsilon < \frac{1}{4\rho_1^2} \min_{j=1,2} (y_1^0 - x_{j1}^0)^2, \quad (15)$$

guarantees that $x_{11}(t) \neq y_1(t)$ and therefore evasion is possible.

Let now $(\sigma_1^2, \sigma_2^2) \in M_{12}$ and $y^0 \notin P_1$. We have $\sigma_2^2 \geq \rho_{12}^2 + \rho_{22}^2$. Lemma 3.2 ensures that the strategy

$$v_2(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon, \\ \sqrt{u_{12}^2(t-\varepsilon) + u_{22}^2(t-\varepsilon)}, & t > \varepsilon, y_2^0 \geq \max\{x_{12}(\varepsilon), x_{22}(\varepsilon)\}, \\ -\sqrt{u_{12}^2(t-\varepsilon) + u_{22}^2(t-\varepsilon)}, & t > \varepsilon, y_2^0 < \min\{x_{12}(\varepsilon), x_{22}(\varepsilon)\}, \end{cases} \quad (16)$$

is admissible and evasion is possible, where ε satisfy the following inequality

$$0 < \varepsilon < \frac{1}{4\rho_2^2} \min_{j=1,2} (y_2^0 - x_{j2}^0)^2.$$

This completes the proof of Theorem 3.3. □

Theorem 3.4. If $(\sigma_1^2, \sigma_2^2) \in M_2$, then evasion is possible for any initial positions of the players in the game (1)–(6).

Proof. First let $(\sigma_1^2, \sigma_2^2) \in M_{21}$. Then we have

$$\min\{\rho_{11}^2, \rho_{21}^2\} < \sigma_1^2 < \max\{\rho_{11}^2, \rho_{21}^2\}, \quad (17)$$

$$\max\{\rho_{12}^2, \rho_{22}^2\} < \sigma_2^2 < \rho_{12}^2 + \rho_{22}^2. \quad (18)$$

Those inequalities imply that

$$\sigma_1^2 > \rho_{21}^2, \quad \sigma_2^2 > \rho_{12}^2$$

or

$$\sigma_1^2 > \rho_{11}^2, \quad \sigma_2^2 > \rho_{22}^2.$$

Without any loss of generality, we assume that

$$\sigma_1^2 > \rho_{21}^2, \quad \sigma_2^2 > \rho_{12}^2. \quad (19)$$

Set

$$v_1(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon, \\ \alpha(t) + |u_{21}(t - \varepsilon)|, & t > \varepsilon, \quad y_1^0 \geq x_{21}(\varepsilon), \\ -(\alpha(t) + |u_{21}(t - \varepsilon)|), & t > \varepsilon, \quad y_1^0 < x_{21}(\varepsilon), \end{cases} \quad (20)$$

and

$$v_2(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon, \\ \alpha(t) + |u_{12}(t - \varepsilon)|, & t > \varepsilon, \quad y_2^0 \geq x_{12}(\varepsilon), \\ -(\alpha(t) + |u_{12}(t - \varepsilon)|), & t > \varepsilon, \quad y_2^0 < x_{12}(\varepsilon), \end{cases} \quad (21)$$

as strategy of the evader where

$$\alpha(t) = \begin{cases} 1, & \varepsilon < t \leq \delta + \varepsilon, \\ 0, & t > \delta + \varepsilon, \end{cases} \quad (22)$$

and numbers $\varepsilon, \delta > 0$ satisfy the following inequalities

$$\delta < \min \left\{ 1, (\sigma_1 - \rho_{21})^2, (\sigma_2 - \rho_{12})^2, \frac{|y_0 - x_{10}|^2}{4[(1 + 2\rho_{12})^2 + (1 + \rho_{11} + \rho_{21})^2]}, \frac{|y_0 - x_{20}|^2}{4[(1 + 2\rho_{21})^2 + (1 + \rho_{12} + \rho_{22})^2]} \right\}, \quad (23)$$

and

$$0 < \varepsilon < \min \left\{ \frac{|y_0 - x_{10}|^2}{4(\rho_{11}^2 + \rho_{12}^2)}, \frac{|y_0 - x_{20}|^2}{4(\rho_{21}^2 + \rho_{22}^2)}, \frac{\delta^2}{4\rho_{12}^2}, \frac{\delta^2}{4\rho_{21}^2} \right\}. \quad (24)$$

Controls (20) and (21) are admissible since by (23),

$$\begin{aligned}
 \int_0^\delta v_1^2(s)ds &= \left(\int_0^\varepsilon + \int_\varepsilon^\delta \right) v_1^2(s)ds \\
 &= \int_\varepsilon^\delta [\alpha(s) + |u_{21}(s - \varepsilon)|]^2 ds \\
 &= \int_\varepsilon^{\delta+\varepsilon} 1 ds + 2 \int_\varepsilon^{\delta+\varepsilon} |u_{21}(s - \varepsilon)| ds + \int_\varepsilon^{\delta+\varepsilon} u_{21}^2(s - \varepsilon) ds \\
 &\leq \delta + 2\sqrt{\delta}\rho_{21} + \rho_{21}^2 \\
 &= (\sqrt{\delta} + \rho_{21})^2 < \sigma_1^2,
 \end{aligned}$$

and similarly, we have

$$\begin{aligned}
 \int_0^\delta v_2^2(s)ds &\leq \delta + 2\sqrt{\delta}\rho_{12} + \rho_{12}^2 \\
 &= (\sqrt{\delta} + \rho_{12})^2 < \sigma_2^2.
 \end{aligned}$$

Now estimate the distance between the evader and first pursuer for the in strategy (21).

Let $0 \leq t \leq \varepsilon$. Then by (24),

$$\begin{aligned}
 |y(t) - x_1(t)| &= \left| y_0 + \int_0^t v(s)ds - x_{10} - \int_0^t u_1(s)ds \right| \\
 &\geq |y_0 - x_{10}| - \left| \int_0^t u_1(s)ds \right| \\
 &\geq |y_0 - x_{10}| - \sqrt{\varepsilon} \sqrt{\rho_{11}^2 + \rho_{12}^2} \\
 &> \frac{|y_0 - x_{10}|}{2}.
 \end{aligned}$$

In particular, $|y(\varepsilon) - x_1(\varepsilon)| = |y_0 - x_{10}| > \frac{|y_0 - x_{10}|}{2}$.

Estimate the distance between the players.

By (20) and (21), $v_2(t) = \alpha(t) + |u_{12}(t - \varepsilon)|$, $t > \varepsilon$, where the function $\alpha(t)$ is defined by (22).

(i) When $\varepsilon < t \leq \delta + \varepsilon$, using triangle inequality, we obtain

$$\begin{aligned}
 |y(t) - x_1(t)| &= \sqrt{(y_1(t) - x_{11}(t))^2 + (y_2(t) - x_{12}(t))^2} \\
 &= \left[\left(y_1^0 + \int_\varepsilon^t (1 + |u_{21}(s - \varepsilon)|) ds - x_{11}(\varepsilon) - \int_\varepsilon^t u_{11}(s) ds \right)^2 \right.
 \end{aligned}$$

$$\begin{aligned}
& + \left(y_2^0 + \int_{\varepsilon}^t (1 + |u_{12}(s - \varepsilon)|) ds - x_{12}(\varepsilon) - \int_{\varepsilon}^t u_{12}(s) ds \right)^2 \Big]^{1/2} \\
& \geq \sqrt{(y_1^0 - x_{11}(\varepsilon))^2 + (y_2^0 - x_{12}(\varepsilon))^2} - \sqrt{A^2 + B^2}, \tag{25}
\end{aligned}$$

where

$$\begin{aligned}
A &= \int_{\varepsilon}^t (1 + |u_{21}(s - \varepsilon)|) ds - \int_{\varepsilon}^t u_{11}(s) ds, \\
B &= \int_{\varepsilon}^t (1 + |u_{12}(s - \varepsilon)|) ds - \int_{\varepsilon}^t u_{12}(s) ds.
\end{aligned}$$

Following that,

$$\begin{aligned}
A &\leq \delta + \int_{\varepsilon}^{\delta+\varepsilon} |u_{21}(s - \varepsilon)| ds + \int_{\varepsilon}^{\delta+\varepsilon} |u_{11}(s)| ds \\
&\leq \delta + \sqrt{\delta} \rho_{21} + \sqrt{\delta} \rho_{11} \\
&= \sqrt{\delta} (\sqrt{\delta} + \rho_{11} + \rho_{21}) \\
&\leq \sqrt{\delta} (1 + \rho_{11} + \rho_{21}),
\end{aligned}$$

and

$$\begin{aligned}
B &\leq \delta + \int_{\varepsilon}^{\delta+\varepsilon} |u_{12}(s - \varepsilon)| ds + \int_{\varepsilon}^{\delta+\varepsilon} |u_{12}(s)| ds \\
&\leq \delta + \sqrt{\delta} \rho_{12} + \sqrt{\delta} \rho_{12} \\
&= \sqrt{\delta} (\sqrt{\delta} + 2\rho_{12}) \\
&\leq \sqrt{\delta} (1 + 2\rho_{12}).
\end{aligned}$$

Therefore by (24) and (25),

$$\begin{aligned}
|y(t) - x_1(t)| &\geq \frac{|y_0 - x_1(\varepsilon)|}{2} - \sqrt{\delta(1 + \rho_{11} + \rho_{21})^2 + \delta(1 + 2\rho_{12})^2} \\
&\geq \frac{|y_0 - x_{10}|}{2} - \sqrt{\delta(1 + \rho_{11} + \rho_{21})^2 + \delta(1 + 2\rho_{12})^2} > 0.
\end{aligned}$$

(ii) Let now $t > \delta + \varepsilon$. Without any loss of generality, we assume that $y_2^0 \geq x_{12}(\varepsilon)$. Then,

$$y_2(t) - x_{12}(t) = y_2^0 - x_{12}(\varepsilon) + \int_{\varepsilon}^t (\alpha(s) + |u_{12}(s - \varepsilon)|) ds - \int_{\varepsilon}^t u_{12}(s) ds$$

$$\begin{aligned} &\geq \int_{\varepsilon}^{\delta+\varepsilon} 1 \, ds + \int_{\varepsilon}^t |u_{12}(s-\varepsilon)| \, ds - \int_{\varepsilon}^t |u_{12}(s)| \, ds \\ &\geq \delta - \sqrt{\varepsilon}\rho_{12} - \sqrt{\varepsilon}\rho_{12} > 0. \end{aligned}$$

This result shows that $y_2(t) > x_{12}(t)$ for $t > \delta + \varepsilon$, meaning that evasion is possible.

Similarly, it can be shown that evasion is possible when $y_2^0 < x_{12}(\varepsilon)$ where

$$v_2(t) = -(\alpha(t) + |u_{12}(t-\varepsilon)|), \quad t > \varepsilon, \quad y_2^0 < x_{12}(\varepsilon).$$

In short, strategy (21) of the evader is admissible and evasion is possible from the first pursuer. A similar way can be used for strategy (20) to prove that the evader is able to avoid the capture by the second pursuer.

Let now $(\sigma_1^2, \sigma_2^2) \in M_{22}$. Then we have

$$\max\{\rho_{11}^2, \rho_{21}^2\} < \sigma_1^2 < \rho_{11}^2 + \rho_{21}^2, \quad (26)$$

$$\min\{\rho_{12}^2, \rho_{22}^2\} < \sigma_2^2 < \max\{\rho_{12}^2, \rho_{22}^2\}. \quad (27)$$

Without any loss of generality, we assume that

$$\sigma_1^2 > \rho_{11}^2, \quad \sigma_2^2 > \rho_{22}^2. \quad (28)$$

Set

$$v_1(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon, \\ \alpha(t) + |u_{11}(t-\varepsilon)|, & t > \varepsilon, \quad y_1^0 \geq x_{11}(\varepsilon), \\ -(\alpha(t) + |u_{11}(t-\varepsilon)|), & t > \varepsilon, \quad y_1^0 < x_{11}(\varepsilon), \end{cases} \quad (29)$$

and

$$v_2(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon, \\ \alpha(t) + |u_{22}(t-\varepsilon)|, & t > \varepsilon, \quad y_2^0 \geq x_{22}(\varepsilon), \\ -(\alpha(t) + |u_{22}(t-\varepsilon)|), & t > \varepsilon, \quad y_2^0 < x_{22}(\varepsilon), \end{cases} \quad (30)$$

as strategies of the evader where the function $\alpha(t)$ is defined by (22).

Using the same arguments applied to (21), one can show that strategies (29) and (30) assure the inequalities $x_{11}(t) \neq y_1(t)$ and $x_{22}(t) \neq y_2(t)$ for all $t \geq 0$ when

$$\delta < \min \left\{ 1, (\sigma_1 - \rho_{11})^2, (\sigma_2 - \rho_{22})^2, \frac{|y_0 - x_{10}|^2}{4[(1 + 2\rho_{11})^2 + (1 + \rho_{11} + \rho_{22})^2]}, \right. \\ \left. \frac{|y_0 - x_{20}|^2}{4[(1 + 2\rho_{22})^2 + (1 + \rho_{11} + \rho_{21})^2]} \right\}, \quad (31)$$

and

$$0 < \varepsilon < \min \left\{ \frac{|y_0 - x_{10}|^2}{4(\rho_{11}^2 + \rho_{12}^2)}, \frac{|y_0 - x_{20}|^2}{4(\rho_{21}^2 + \rho_{22}^2)}, \frac{\delta^2}{4\rho_{11}^2}, \frac{\delta^2}{4\rho_{22}^2} \right\}. \quad (32)$$

Hence, evasion is possible from both the pursuers.

This ends the proof of Theorem 3.4. $\square$

Theorem 3.5. *Let the initial positions of the players be any. If $(\sigma_1^2, \sigma_2^2) \in M_3$, then evasion is possible in the game (1)–(6).*

Proof. Since $(\sigma_1^2, \sigma_2^2) \in M_3$, then we have

$$\begin{aligned} \max\{\rho_{11}^2, \rho_{21}^2\} &< \sigma_1^2 < \rho_{11}^2 + \rho_{21}^2, \\ \max\{\rho_{12}^2, \rho_{22}^2\} &< \sigma_2^2 < \rho_{12}^2 + \rho_{22}^2. \end{aligned}$$

Now, since $\sigma_1^2 > \max\{\rho_{11}^2, \rho_{21}^2\}$ and $\sigma_2^2 > \max\{\rho_{12}^2, \rho_{22}^2\}$, without any loss of generality, we consider

$$\sigma_1^2 > \rho_{11}^2, \sigma_2^2 > \rho_{22}^2. \quad (33)$$

Then in the similar manner, we can see that $x_{11}(t) \neq y_1(t)$ and $x_{22}(t) \neq y_2(t)$ for all $t \geq 0$. In summary, evasion is possible from both the pursuers, that is $x_1(t) \neq y_1(t)$ and $x_2(t) \neq y_2(t)$ for all $t \geq 0$.

This completes the proof of Theorem 3.5. $\square$

4. Conclusion

An evasion differential game of two pursuers and one evader with coordinate-wise integral constraints has been studied. Coordinate-wise integral constraints are imposed on control functions of players. We have found sufficient conditions of evasion. Then we have constructed strategies for the evader in explicit form.

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