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IMPROVEMENT OF HARMONIC AND CONTRA HARMONIC MEAN INEQUALITY CHAIN

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Abstract: Inequality involving the family of centroidal means and the family of Heinz type means is proved and using this the harmonic and contra harmonic mean inequality chain is improved.

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1. Introduction

For two distinct positive numbers a & b, we have the Arithmetic Mean = $A(a,b) = \frac{a+b}{2}$, Geometric mean = $G(a,b) = \sqrt{ab}$, Harmonic Mean = $H(a,b) = \frac{2ab}{a+b}$, Contra harmonic Mean = $C(a,b) = \frac{a^2+b^2}{a+b}$ and Heron mean = $h(a,b) = \frac{a+\sqrt{ab}+b}{3}$.

In [5], K. M. Nagaraja and et al, proved that $H(a,b) + C(a,b) = 2A(a,b)$ and also established some double inequalities involving means.

Several authors introduced and studied in depth the parameterized family of means such as Stolorsky's mean; functional means; Heinz means; etc., found some interesting results in ([2] -[16]).

The family of Heinz means are defined as,

$$H_v(a,b) = \frac{a^{1-v}b^v + a^v b^{1-v}}{2}; \qquad 0 \le v \le 1, a, b > 0.$$

For $v = 0, 1$, $H_v(a,b) = A(a,b)$ and for $v = 1/2$, $H_v(a,b) = G(a,b)$. The mean $H_v(a,b)$ expressed as a function of v is convex and attains minimum value at $v = 1/2$ and proved that $G(a,b) \le H_v(a,b) \le A(a,b)$ for $0 \le v \le 1$. The interesting results on family of heron means $h_\alpha(a,b) = (1-\alpha)G(a,b) + \alpha A(a,b)$ and Heinz mean were found in [2]. In [3, 4, 6], authors studied the family of heron means in different notations and established some inequalities with power mean = $M_r(a,b) = \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}}$, for $r \neq 0$, Identric mean = $I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ and Logarithmic mean = $L(a,b) = \frac{b-a}{lnb-lna}$.

For $0 \le v \le 1$, let $\alpha(v) = 1 - 4(v - v^2)$, the function $\alpha(v)$ is convex in nature and attains minimum value at $v = 1/2$ and maximum value at $v = 0, 1$. This work motivate us to develop this paper.

2. Definition and Results

In this section the family of Heinz type means are introduced, a lemma is proved and an inequality chain is improved.

For $0 \le v \le 1, \quad a, b > 0$,

$$H_v(a,b) = \frac{(a^2)^{1-v}(b^2)^v + (a^2)^v(b^2)^{1-v}}{a+b}$$

is called the family of Heinz type means. In [6, 13] authors studied some results related to centroidal mean which is denoted and defined by $CT(a,b) =$

$\frac{2}{3}\frac{a^2+ab+b^2}{a+b}$ which is equivalently written as $CT(a,b) = \frac{2}{3}C(a,b) + \frac{1}{3}H(a,b)$. The generalization of this mean for $0 \leq \alpha \leq 1$ is given by $CT_\alpha(a,b) = (1-\alpha)H(a,b) + \alpha C(a,b)$ is also called the family of centroidal means and is equal to centroidal mean for $\alpha = 2/3$.

Lemma 2.1. *The family of Heinz type means $H_v(a,b)$ and the family of centroidal means $CT_\alpha(a,b)$ satisfy the inequality,*

$$H_v(a,b) \leq CT_{\alpha(v)}(a,b); \quad 0 \leq v \leq 1, \quad \alpha(v) = 1 - 4(v - v^2)$$

Proof. In expanded form, the inequality $H_v(a,b) \leq CT_{\alpha(v)}(a,b)$ says that,

$$\frac{(a^2)^{1-v}(b^2)^v + (a^2)^v(b^2)^{1-v}}{a+b} \leq \left[4(v-v^2)\frac{2ab}{a+b} + (1-4(v-v^2))\frac{a^2+b^2}{a+b}\right]$$

Taking $a = e^x, b = e^y$, on simple manipulation leads to

$$\cosh(2v-1)(x-y) \leq 4(v-v^2) + (1-4(v-v^2))\cosh(x-y)$$

Let $\beta = 2v - 1$ and $x - y = X$, gives

$$\cosh \beta X \leq (1-\beta^2) + (\beta^2))\cosh X$$

on power series expansion above inequality becomes

$$\left(\beta^4\frac{X^4}{4!} + \beta^6\frac{X^6}{6!} + ...\right) \leq \beta^2\left(\frac{X^4}{4!} + \frac{X^6}{6!} + ...\right)$$

This is true for all X and $-1 \leq \beta \leq 1$. □

Theorem 1. *Suppose $a, b > 0$, then* $\left(1 + \frac{[lna-lnb]^2}{2}\right) H(a,b) \leq C(a,b)$.

Proof. Let $f(x) = \frac{(a^2)^{1-x}(b^2)^x+(a^2)^x(b^2)^{1-x}}{a+b}$, for $0 \leq x \leq 1$, by finding the first two derivative of $f(x)$ and simple calculations shows that for x in $(0,1)$,

$$f'(x) = 2(lna-lnb)\frac{(a^2)^{1-x}(b^2)^x - (a^2)^x(b^2)^{1-x}}{a+b} \quad \text{and} \quad f''(x) = 4(lna-lnb)^2 f(x)$$

So, for a given x in $(0,1/2)$, by Taylor's series expansion, there exists η in $(x,1/2)$ such that

$$f'(x) = H(a,b) + 4(lna-lnb)^2\frac{(a^2)^{1-\eta}(b^2)^\eta + (a^2)^\eta(b^2)^{1-\eta}}{2(a+b)}(x-1/2)^2$$

and by lemma **2.1**

$$H_v(a,b) \leq CT_{\alpha(v)}(a,b); \quad 0 \leq v \leq 1, \quad \alpha(v) = 1 - 4(v - v^2)$$

from above it follows that for x in $(0, 1/2)$ such that

$$H(a,b) + 4(lna - lnb)^2 \frac{(a^2)^{1-\eta}(b^2)^{\eta} + (a^2)^{\eta}(b^2)^{1-\eta}}{2(a+b)}(x - 1/2)^2 \leq CT_{\alpha(v)}$$

equivalently written as;

$$H(a,b) + 4(lna - lnb)^2 \frac{(a^2)^{1-\eta}(b^2)^{\eta} + (a^2)^{\eta}(b^2)^{1-\eta}}{2(a+b)}(x - 1/2)^2 \leq C(a,b)$$

Which is refinement or improvement of an inequality $H(a,b) \leq C(a,b)$. Hence the proof of the theorem **2.1** □

3. Application

A modified version of improved Arithmetic Geometric inequality (**1.2**) [11] is deduced from our result by substituting $a = \sqrt{u}$ and $b = \sqrt{v}$ in the form $\left(1 + \frac{(lnu - lnv)^2}{8}\right) G(u,v) \leq A(u,v)$. We have the well known inequality chain

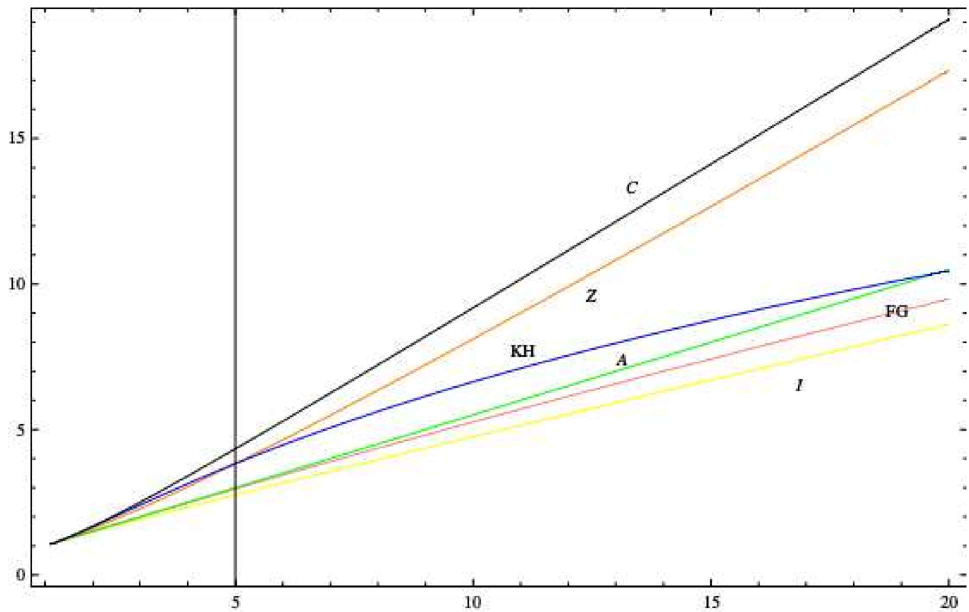
$$\frac{G^2}{C} \leq Z_1 \leq H \leq G \leq L \leq M_{\frac{1}{3}} \leq h \leq M_{\frac{2}{3}} \leq I \leq A \leq Z \leq C.$$

Where $Z_1 = a^{\frac{b}{a+b}} b^{\frac{a}{a+b}}$ and $Z = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}$. In [1], the author established the inequality $\left(1 + \frac{(lna - lnb)^2}{8}\right) G \leq A$ and by theorem **2.1**, the above inequality chain can be extended for $a > 1$ and $1 < b \leq 5.019618$ as;

$$\frac{G^2}{C} \leq Z_1 \leq H \leq G \leq L \leq M_{\frac{1}{3}} \leq h \leq M_{\frac{2}{3}} \leq I$$

$$\leq \left(1 + \frac{(lna - lnb)^2}{8}\right) G \leq A \leq Z \leq \left(1 + \frac{(lna - lnb)^2}{2}\right) H \leq C.$$

The graph depicts the inequality chain stated above, with $F = \left(1 + \frac{(lna - lnb)^2}{8}\right)$ and $K = \left(1 + \frac{(lna - lnb)^2}{2}\right)$. For $b > 5.019618$, Z exceeds KH.



(a) fig 1

Figure 1: The plot of two arguments: $a = 1.1$ and $b = x$

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