SUPERCYCLICITY OF TUPLES IN SOME TOPOLOGIES

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Abstract: In this paper, we prove the supercyclicity for a tuple of operators in the strong operator topology and in the norm of Hilbert-Schmidt operators. Also, the same result for hypercyclicity of tuples is considered.

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1. Introduction

Let $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be an n -tuple of commuting operators acting on an infinite dimensional Banach space X . We will let $\mathcal{F} = \{T_1^{k_1} T_2^{k_2}, \dots, T_n^{k_n} : k_i \in \mathbb{Z}_+, i = 1, \dots, n\}$ be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector x is called a supercyclic (hypercyclic) vector for $\mathcal{T}$ if $\mathbb{C}Orb(\mathcal{T}, x)$ ($Orb(\mathcal{T}, x)$) is dense in X and in this case the tuple $\mathcal{T}$ is called supercyclic (hypercyclic). By $\mathcal{T}_d^{(k)}$ we will refer to the set of all k copies of an element of $\mathcal{F}$, i.e. $\mathcal{T}_d^{(k)} = \{S_1 \oplus \dots \oplus S_k : S_1 = \dots = S_k \in \mathcal{F}\}$. We say that $\mathcal{T}_d^{(k)}$ is hypercyclic provided there exist $x_1, \dots, x_k \in X$ such that $\{W(x_1 \oplus \dots \oplus x_k) : W \in \mathcal{T}_d^{(k)}\}$ is dense in the k copies

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of X , $X \oplus \dots \oplus X$, and similarly we say that $\mathcal{T}_d^{(k)}$ is supercyclic provided there exist $x_1, \dots, x_k \in X$ such that $\mathbf{C}\{W(x_1 \oplus \dots \oplus x_k) : W \in \mathcal{T}_d^{(k)}\}$ is dense in the k copies of X . We say that a tuple $\mathcal{T} = (T_1, T_2, \dots, T_n)$ is topologically transitive with respect to a tuple of nonnegative integer sequences

$$(\{k_{j(1)}\}_j, \{k_{j(2)}\}_j, \dots, \{k_{j(n)}\}_j),$$

if for every nonempty open subsets U, V of X there exists $j_0 \in \mathbb{N}$ such that $T_1^{k_{j_0(1)}} T_2^{k_{j_0(2)}} \dots T_n^{k_{j_0(n)}}(U) \cap V \neq \emptyset$. Also, we say that an n -tuple $\mathcal{T}$ is topologically transitive if it is topologically transitive with respect an n -tuple of nonnegative integer sequences. If X is separable, topological transitivity of $\mathcal{T}$ implies the hypercyclicity of $\mathcal{T}$. For several works see [1–3].

We begin with the following well known theorems that are the main tools we use to show that a tuple of operators is supercyclic.

Theorem 1.1. *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be a tuple of operators $T_1, T_2, \dots, T_n$. Then $\mathcal{T}$ is supercyclic, if and only if for any two non-void open sets U and V , there exist $m_i \geq 1$ for $i = 1, \dots, n$ and $\lambda \in \mathbf{C} \setminus \{0\}$ such that $\lambda T_1^{m_1} \dots T_n^{m_n}(U) \cap V \neq \emptyset$.*

Theorem 1.2. (Supercyclicity Criterion for Tuples of Operators) *Let $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be an n -tuple of operators acting on an infinite dimensional Banach space X . If there exist two dense subsets Y and Z in X , and strictly increasing sequences $\{m_{j_i}\}_j$ for $i = 1, \dots, n$, and a sequence of mappings $S_j : Z \rightarrow X$ such that:*

- 1) $T_1^{m_{j_1}} T_2^{m_{j_2}} \dots T_n^{m_{j_n}} S_j z \rightarrow z$ for every $z \in Z$,
 - 2) $\|T_1^{m_{j_1}} T_2^{m_{j_2}} \dots T_n^{m_{j_n}} y\| \|S_j z\| \rightarrow 0$ for every $y \in Y$ and every $z \in Z$,
- then $\mathcal{T}$ is supercyclic.

2. Main Results

We will investigate the supercyclicity for a tuple of operators by the three open sets.

Recall that if $\{e_i\}$ is a basis for a separable Hilbert space H , $A \in B(H)$ and $\|A\|_2 = [\sum_{i=1}^\infty \|Ae_i\|^2]^{\frac{1}{2}} < \infty$, then A is called a Hilbert-Schmidt operator. The set of Hilbert-Schmidt operators on H is denoted by $B_2(H)$.

Note that $B(H)$ and $B_2(H)$ respectively with strong operator topology and $\|\cdot\|_2$ -topology are separable. For this suppose that $\{e_i\}$ is the basis for the separable Hilbert space H , also suppose that S is a dense subset in H . Then

$S(H)$ will denote the set of all finite rank operators E such that there exists $N \in \mathbb{N}$ satisfying $E(e_n) = 0$ for $n \geq N$ and $E(e_n) \in S$ for $n < N$. Clearly, $S(H)$ is $\|\cdot\|_2$ -dense in $B_2(H)$ and if S is countable, then $S(H)$ is also countable.

For any operator $T \in B(H)$, the left multiplication operator $L_T : B(H) \rightarrow B(H)$ is defined by $L_T(S) = TS$ for all S in $B(H)$. If $\mathcal{T} = (T_1, T_2, \dots, T_n)$ is an n -tuple of operators acting on H , then by $\mathcal{L}_{T_1, \dots, T_n}$ we mean the tuple $(L_{T_1}, L_{T_2}, \dots, L_{T_n})$.

Lemma 2.1. *Let $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be an n -tuple of operators in $B(H)$. Then $\mathcal{T}_d$ acting on $\bigoplus_{n=1} H$ is unitary equivalent to $\mathcal{L}_{T_1, \dots, T_n}$ acting on $B_2(H)$.*

Proof. Let $\{h_i\}$ be a sequence in H , such that $\sum_{i=1} \|h_i\|^2 < \infty$. Put $h = \bigoplus_{i=1} h_i$ and define U_h by $U_h(x) = \sum_{i=1} \langle x, e_i \rangle h_i$ for every $x \in H$. Now consider the linear operator $R : \bigoplus_{n=1} H \rightarrow B_2(H)$ defined by $R(h) = U_h$. Note that for every $h = \bigoplus_{n=1} h_i$ in $\bigoplus_{n=1} H$ we get $\|U_h\|_2^2 = \sum_{i=1} \|U_h e_i\|^2 =$

$\sum_{i=1} \|h_i\|^2 < \infty$. This implies that R is an isometry. Now we show that R is also surjective. For this suppose that A is an arbitrary Hilbert-Schmidt operator and put $h_i = Ae_i$ for any $i \geq 1$. Then $\sum_{i=1} \|h_i\|^2 = \sum_{i=1} \|Ae_i\|^2 = \|A\|_2^2 < \infty$.

Thus $h = \bigoplus_{i=1} h_i \in \bigoplus_{n=1} H$ and $R(h) = A$. Now clearly we can see that $R^{-1} L_{T_1}^{k_1} L_{T_2}^{k_2} \dots L_{T_n}^{k_n} R = \bigoplus_{i=1} T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}$ and so the proof is complete. $\square$

Theorem 2.2. *Let $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be an n -tuple of operators acting on H . Then $\mathcal{T}$ satisfies the Supercyclicity Criterion if and only if $\mathcal{L}_{T_1, \dots, T_n}$ is SOT-supercyclic on $B(H)$.*

Proof. Note that $\mathcal{L}_{T_1, \dots, T_n}$ is SOT-supercyclic on $B(H)$ if and only if $\mathcal{L}_{T_1, \dots, T_n}$ is supercyclic on $B_2(H)$ with $\|\cdot\|_2$ -topology. Now Lemma 2.16 implies that $\mathcal{L}_{T_1, \dots, T_n}$ is supercyclic on $B_2(H)$ with $\|\cdot\|_2$ -topology if and only if for all integer $k \geq 1$, $\mathcal{T}_d^{(k)}$ is supercyclic on $\bigoplus_{i=1}^k H$ and this is true if and only if $\mathcal{T}$ satisfies the Supercyclicity Criterion. This completes the proof. $\square$

Corollary 2.3. *The tuple $\mathcal{T} = (T_1, \dots, T_n)$ satisfies the Supercyclicity Criterion if and only if for any two nonempty SOT-open sets U and V in $B(H)$, there exists some positive integer $k_1, \dots, k_n \geq 1$ and $\lambda \in \mathbb{C}$ such that $\lambda L_{T_1}^{k_1} L_{T_2}^{k_2} \dots L_{T_n}^{k_n} U \cap V \neq \emptyset$.*

Corollary 2.4. *If $\mathcal{T} = (T_1, \dots, T_n)$ satisfies the Supercyclicity Criterion, then $\mathcal{L}_{T_1, \dots, T_n}$ has a $\|\cdot\|_2$ -dense G_δ set of supercyclic vectors.*

Proof. Since each supercyclic operator on a complete metric space has a dense G_δ set of supercyclic vectors, and $B_2(H)$ is complete with $\|\cdot\|_2$ -topology, it is clear. $\square$

Theorem 2.5. Let $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be an n -tuple of operators acting on H . Then the following are equivalent:

i) $\mathcal{T}$ is supercyclic and for each nonempty open subset U and each neighborhood W of zero, $\lambda T_1^{m_1} T_2^{m_2} \dots T_n^{m_n} U \cap W \neq \phi$ and $\lambda T_1^{-m_1} T_2^{-m_2} \dots T_n^{-m_n} U \cap W \neq \phi$ for some integers $m_1, m_2, \dots, m_n \geq 1$ and $\lambda \in \mathbf{C} \setminus \{0\}$.

ii) For each pair U, V of nonempty open subsets of H , and each neighborhood W of zero, $\lambda T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} U \cap W \neq \phi$ and $\lambda T_1^{-k_1} T_2^{-k_2} \dots T_n^{-k_n} W \cap V \neq \phi$ for some integers $k_i \geq 1$, $i = 1, \dots, n$, and $\lambda \in \mathbf{C} \setminus \{0\}$.

Proof. (i) implies (ii): Let U, V and W be nonempty open sets in H with $0 \in W$. Since $\mathcal{T}$ is supercyclic, by Theorem 1.1, we have $U \cap \beta T_1^{-m_1} T_2^{-m_2} \dots T_n^{-m_n} V \neq \phi$ for some positive integers $m_1, m_2, \dots, m_n$ and $\beta \in \mathbf{C} \setminus \{0\}$. Let G be a neighborhood of zero that is contained in $W \cap \beta T_1^{-m_1} T_2^{-m_2} \dots T_n^{-m_n} W$. By condition (i), there exist some positive integer $k_1, k_2, \dots, k_n$ and $\lambda \in \mathbf{C} \setminus \{0\}$ such that

$$T_1^{-k_1} T_2^{-k_2} \dots T_n^{-k_n} G \cap \lambda(U \cap \beta T_1^{-m_1} T_2^{-m_2} \dots T_n^{-m_n} V) \neq \phi,$$

$$\lambda G \cap T_1^{-k_1} T_2^{-k_2} \dots T_n^{-k_n}(U \cap \beta T_1^{-m_1} T_2^{-m_2} \dots T_n^{-m_n} V) \neq \phi.$$

But

$$T_1^{-k_1} T_2^{-k_2} \dots T_n^{-k_n} G \cap \lambda(U \cap \beta T_1^{-m_1} T_2^{-m_2} \dots T_n^{-m_n} V)$$

is a subset of $T_1^{-k_1} T_2^{-k_2} \dots T_n^{-k_n} W \cap \lambda U$, thus $T_1^{-k_1} T_2^{-k_2} \dots T_n^{-k_n} W \cap \lambda U \neq \phi$. Also $\lambda G \cap T_1^{-k_1} T_2^{-k_2} \dots T_n^{-k_n}(U \cap \beta T_1^{-m_1} T_2^{-m_2} \dots T_n^{-m_n} V)$ is a subset of

$$\lambda \beta T_1^{-m_1} T_2^{-m_2} \dots T_n^{-m_n} W \cap \beta T_1^{-k_1} T_2^{-k_2} \dots T_n^{-k_n}(T_1^{-m_1} T_2^{-m_2} \dots T_n^{-m_n} V)$$

that is equal to $\beta T_1^{-m_1} T_2^{-m_2} \dots T_n^{-m_n}(\lambda W \cap T_1^{-k_1} T_2^{-k_2} \dots T_n^{-k_n} V)$. This implies that $T_1^{-k_1} T_2^{-k_2} \dots T_n^{-k_n} V \cap \lambda W \neq \phi$ and so (ii) holds. On the other hand, clearly (ii) implies (i) and this completes the proof. $\square$

By the techniques used in the proof of Theorems 2.2 and 2.5, we can prove the following theorem and so we omit the proof.

Theorem 2.6. Let $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be an n -tuple of operators acting on H . Then $\mathcal{T}$ satisfies the Hypercyclicity Criterion if and only if $\mathcal{L}_{T_1, \dots, T_n}$ is SOT-hypercyclic on $B(H)$ if and only if $\mathcal{L}_{T_1, \dots, T_n}$ is hypercyclic on $B_2(H)$ with $\|\cdot\|_2$ -topology.

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