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NON-MSF MRA WAVELETS

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Abstract: In this article, we provide two classes of non-MSF MRA wavelets in $L^2(\mathbb{R}^2)$. The first arose through one-dimensional dyadic wavelet sets having two components, is an uncountable family, while the second one arose through a special kind of MRA wavelet sets having three components, is a countable family.

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1. Introduction

Bownik and Speegle [4] dealt with the converse of a result established by Chui and Shi [5], according to which for the dilation a which satisfies $a^j \notin \mathbb{Q}$, for all positive integers j, there exist only MSF-wavelets. Indeed, they provide a complete characterization of an expansive matrix that admits non-MSF wavelets. For an expansive matrix A, they obtain a non-MSF wavelet to exist provided there is a $p \in \mathbb{Z}\setminus\{0\}$ such that $(A^*)^p\, \mathbb{Z}^n \cap \mathbb{Z}^n \neq \{0\}$. Thus finding ways to create

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non-MSF wavelets has attracted various contributors in the field of wavelets. For instance, we refer to [2, 4, 10, 11, 12, 13]. The purpose of this paper is to provide non-MSF wavelets in $L^2(\mathbb{R}^2)$, considering specific wavelet sets in $\mathbb{R}^2$ arose from an MRA wavelet sets in $\mathbb{R}$ and its scaling set. We begin with necessary pre-requisites in Section 2. In Section 3, we obtain that the product of a one-dimensional dyadic MRA wavelet set with its scaling set is an A-wavelet set in $\mathbb{R}^2$, where $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. Considering wavelet sets having two components and a special class of MRA wavelet sets having three components, we provide non-MSF MRA A-wavelets in $L^2(\mathbb{R}^2)$, in Section 4.

2. Pre-Requisites

Throughout the paper, the symbols $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote, respectively, the set of natural numbers, the set of integers and the real line. The symbol A stands for an $n \times n$ expansive matrix such that $A\mathbb{Z}^n \subseteq \mathbb{Z}^n$, where $n \in \mathbb{N}$. The transpose of A is denoted by A^*. An element of $\mathbb{R}^n$ is represented by a column matrix. For a set E in the Euclidean space $\mathbb{R}^n$, the Lebesgue measure of E is denoted by $|E|$. The collection of all square integrable complex valued functions on $\mathbb{R}^n$, in which two functions are identified if they are equal almost everywhere (abbreviated, a.e.), is denoted by $L^2(\mathbb{R}^n)$. With the usual addition, the scalar multiplication and the inner product $\langle f, g\rangle$ of $f, g \in L^2(\mathbb{R}^n)$ defined by

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx,$$

$L^2(\mathbb{R}^n)$ becomes a Hilbert space. For a function $f \in L^2(\mathbb{R}^n)$, the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i<\xi,\, x>} dx.$$

A function ψ in $L^2(\mathbb{R}^n)$, is called an *orthonormal A-wavelet*, if the system $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$, where

$$\psi_{j,k}(x) = |detA|^{\frac{j}{2}} \psi\left(A^j x - k\right), \qquad x \in \mathbb{R}^n.$$

The following result characterizes an orthonormal A-wavelet.

Theorem 2.1. [4] *A unit vector ψ in $L^2(\mathbb{R}^n)$ is an orthonormal A-wavelet iff:*

(1) $\rho(\xi) \equiv \sum_{j\in\mathbb{Z}} |\hat{\psi}(A^{*j}\xi)|^2 = 1, \quad a.e.\ \ \xi \in \mathbb{R}^n$,

(2) $t_q(\xi) \equiv \sum_{j=0}^{\infty} \hat{\psi}(A^{*j}\xi)\overline{\hat{\psi}(A^{*j}(\xi + 2q\pi))} = 0, \quad a.e. \quad \xi \in \mathbb{R}^n, \quad q \in \mathbb{Z}^n \backslash A^*\mathbb{Z}^n$.

A method to obtain A-wavelets in $L^2(\mathbb{R}^n)$ arises from the celebrated notion known as the A-multiresolution analysis, which is described below:

Definition 2.2.[1] An *A-multiresolution analysis* (A-MRA) for $L^2(\mathbb{R}^n)$ is a sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R}^n)$ satisfying:

(1) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$;

(2) $f(\cdot) \in V_j$ iff $f(A\cdot) \in V_{j+1}$, for all $j \in \mathbb{Z}$;

(3) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;

(4) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;

(5) There exists a function $\varphi \in L^2(\mathbb{R}^n)$ such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}^n\}$ forms an orthonormal basis for V_0.

The function φ is called a *scaling function* of the A-MRA.

Definition 2.3. In case (5) in Definition 2.2 is replaced by (6):

(6) $\{\varphi(\cdot - k) : k \in \mathbb{Z}\} \subset V_0$, for all $\varphi \in V_0$, the family $(\{V_j\}_{j\in\mathbb{Z}}, \varphi)$ is called a *generalized multiresolution analysis* (GMRA).

The notion of a GMRA was introduced by L. Baggett, H. Medina and K. D. Merrill [1]. They have shown that GMRA's produce all wavelets.

It is known that the Lebesgue measure of the support of an n-dimensional orthonormal A-wavelet in $L^2(\mathbb{R}^n)$, is at least $(2\pi)^n$. Therefore, the study of A-wavelets with minimal support has attractive workers in wavelets. Such a wavelet is called a *minimally supported frequency* (MSF) *A-wavelet* [6, 7]. It has been obtained that for an MSF A-wavelet ψ, there exists a measurable set W of $\mathbb{R}^n$ with measure $(2\pi)^n$ such that $|\hat{\psi}| = \chi_W$. We call such a set W to be an *A-wavelet set*. An MSF A-wavelet ψ arising from an A-MRA is called an MRA MSF *A-wavelet* and the wavelet set associated with it is called an MRA MSF *A-wavelet set*.

The following theorem is a well known characterization which describes wavelet sets as precisely those sets that tile $\mathbb{R}^n$ by translations and dilations.

Theorem 2.4. [6] *A measurable set $W \subset \mathbb{R}^n$ is a wavelet set for dilation by an invertible real-valued matrix A iff:*

(1) $\dot{\bigcup}_{k\in\mathbb{Z}^n} (W + 2k\pi) = \mathbb{R}^n$, *a.e.*,

(2) $\dot{\bigcup}_{j\in\mathbb{Z}} (A^*)^j W = \mathbb{R}^n$, *a.e.*,

where $\dot{\bigcup}$ *denotes the disjoint union.*

Consider $E_1 = [-\pi,\ \pi]^2$ and $E_2 = [-\pi,\ \pi] \times [\pi,\ 2\pi] \cup [-\pi,\ \pi] \times [-2\pi,\ -\pi]$. The translation projection τ, and the dilation projection d are, respectively, defined as follows:

$$\tau : \mathbb{R}^2 \to E_1,\ \ \tau(x) = x + 2k\pi, \tag{1}$$

$$d : \mathbb{R}^2 \to E_2,\ \ d(x) = (A^*)^j x, \tag{2}$$

where k and j are unique members of $\mathbb{Z}^2$ and $\mathbb{Z}$ respectively such that $x + 2k\pi$, and $(A^*)^j x$ belong to E_1 and E_2, respectively.

Two measurable sets E and F of $\mathbb{R}^n$ are said to be 2π-*translation congruent* if there is a measurable bijection τ from E to F such that $\tau(t) - t \in 2\pi\mathbb{Z}^n$, for each $t \in E$. These sets are said to be A-*dilation congruent* if there is a measurable bijection δ from E to F such that $\delta(t) = (A^*)^m t \in F$, for an $m \in \mathbb{Z}$, where $t \in E$.

Assume, in addition, that $|detA| = 2$. Then for an MSF A-wavelet ψ in $L^2(\mathbb{R}^n)$, which arises from an A-MRA having φ as its scaling function, it is known that there is a measurable set S in $\mathbb{R}^n$ such that $|\hat{\varphi}| = \chi_S$. Also, for the scaling function φ of an A-MRA satisfying $|\hat{\varphi}| = \chi_S$, for some measurable set S in $\mathbb{R}^n$, there exists an MSF A-wavelet ψ associated with the A-MRA. Such a set is called an A-*scaling set* [3,9].

Stated below is a characterization of an A-scaling set:

Theorem 2.5. [3] *A measurable set S in $\mathbb{R}^n$ is an A-scaling set iff it satisfies the following*:

(1) $S \subset A^*S$,

(2) $W = A^*S \backslash S$ *is an A-wavelet set of* $\mathbb{R}^n$, *and*

(3)$\{S + 2k\pi : k \in \mathbb{Z}^n\}$ *is a measurable partition of* $\mathbb{R}^n$, a.e..

For a measurable set S satisfying (1) and (2), the following is shown to be an equivalent condition:

(4) $S = \cup_{j<0}(A^*)^j W$, for some A-wavelet set W.

3. MRA Wavelet Sets in $\mathbb{R}^2$

In this Section, we show that the product of a one-dimensional MRA dyadic wavelet set W and its associated scaling set S is an MRA A-wavelet set of $\mathbb{R}^2$, where $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. This result forms a basis for producing non-MSF MRA A-wavelets in this paper. We begin with the following theorem.

Theorem 1. *Let W be a one-dimensional MRA dyadic wavelet set whose associated scaling set is S. Then $S \times W$ is an MRA A-wavelet set in $\mathbb{R}^2$.*

Proof. In view of Theorem 2.4, it is sufficient to show that

1. $\dot{\bigcup}_{k\in\mathbb{Z}^2}((S\times W)+2k\pi)=\mathbb{R}^2$, a.e.,
2. $\dot{\bigcup}_{j\in\mathbb{Z}}(A^*)^j(S\times W)=\mathbb{R}^2$, a.e..

Notice that a measurable set in $\mathbb{R}^2$ having positive measure contains a rectangle modulo a set of measure zero. For (1), first we observe that $(S\times W)+2k\pi$, and $(S\times W)+2l\pi$ are disjoint a.e., where $k\neq l$. This is because of the fact that both S as well as W cover $\mathbb{R}$, a.e., by their integral translates. Now, assume, in contrary, that the disjoint union of 2π-$\mathbb{Z}^2$ translates of $S\times W$ fails to cover $\mathbb{R}^2$, a.e.. Then there is a rectangle, say, $A\times B$ modulo a set of measure zero in $\mathbb{R}^2$, which is not contained in $(S\times W)+2k\pi$, for any k in $\mathbb{Z}^2$. From this, it follows that B does not lie in any integral translate of the wavelet set W, which is a contradiction.

Next, from a direct computation, we have

$$(A^*)^j=\begin{cases}\begin{pmatrix}2^{j/2} & 0\\ 0 & 2^{j/2}\end{pmatrix}, & \text{when } j \text{ is even},\\ \begin{pmatrix}0 & 2^{(j-1)/2}\\ 2^{(j+1)/2} & 0\end{pmatrix}, & \text{when } j \text{ is odd}.\end{cases}$$

Since the integral dilates of W by 2 are disjoint, it follows that when distinct j and k are both either even or odd,

$$(A^*)^j(S\times W)\cap(A^*)^k(S\times W)=\phi. \tag{3}$$

In case, j is even and k is odd such that $j<k$, (3) follows by noting that $W=2S\backslash S$ and $S\subset 2S$. Other cases can be dealt with in a similar way. Likewise as in (1), we assume that $S\times W$ fails to cover $\mathbb{R}^2$ a.e, by integral dilates of A^*. Then, we find a rectangle $A\times B$, which does not lie in $(A^*)^j(S\times W)$ for any $j\in\mathbb{Z}$. For $l\in\mathbb{Z}$, consider $j=2l$, to conclude that $A\times B$ does not lie in $2^lS\times 2^lW$ for any $l\in\mathbb{Z}$, which in turn tells that B is not contained in 2^lW for any $l\in\mathbb{Z}$. Thus $\mathbb{R}^2$ is a disjoint union of $(A^*)^j(S\times W)$, a.e.. Thus, (2) holds. That $S\times W$ is an MRA A-wavelet set in $\mathbb{R}^2$ follows by noting that $S\times S$ is its scaling set. □

Considering the following two examples, we construct non-MSF MRA A-wavelets for $L^2(\mathbb{R}^2)$, where $A=\begin{pmatrix}0 & 2\\ 1 & 0\end{pmatrix}$, in the next Section. Ha, Kang, Lee and Seo [8] have characterized one-dimensional dyadic wavelet sets having two intervals, and also three intervals as follows:

(1) Wavelet sets having two intervals are precisely

$$W_a = [2a - 4\pi, a - 2\pi) \cup [a, 2a), \tag{4}$$

where $a \in (0,\ 2\pi)$. Furthermore, these all are MRA wavelet sets [8].

(2) Wavelet sets possessing three intervals are given by

$$W(j,p) \equiv I_{j,p} \cup J_{j,p} \cup K_{j,p}, \tag{5}$$

where

$$I_{j,p} \equiv \left[-2\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\pi\ ,\ -\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\pi \right],$$

$$J_{j,p} \equiv \left[\frac{2(p+1)\pi}{2^{j+1}-1}\ ,\ \frac{2(2p+1)\pi}{2^{j+1}-1} \right],$$

$$K_{j,p} \equiv \left[\frac{2^{j+1}(2p+1)\pi}{2^{j+1}-1}\ ,\ \frac{2^{j+2}(p+1)\pi}{2^{j+1}-1} \right],$$

and j, p are natural numbers such that $j \geq 2$ and $1 \leq p \leq 2^j - 2$. For $j \geq 2$, and an odd $p \in \mathbb{N}$, $W(j,p)$ is a non-MRA wavelet set [8], while for $p = 2^j - 2$, $W(j,p)$ is an MRA wavelet set [10].

Example 3.2. For $a \in (0, 2\pi)$, $W_a = [2a - 4\pi, a - 2\pi) \cup [a, 2a)$ is known to be a 2-dilation MRA wavelet set [8]. Since its scaling set S_a is $[a - 2\pi, a)$, by Theorem 3.1, it follows that $S_a \times W_a$ is an MRA A-wavelet set.

Example 3.3. Choosing $p = 2^j - 2$ and denoting $W(j,p)$ by W_j, we obtain the scaling set S_j of W_j as follows:

$$\begin{aligned} S_j = &\left[\frac{-2\pi}{2^{j+1}-1}, \frac{2(2^j-1)\pi}{2^{j+1}-1} \right] \cup \left[\frac{2(2^{j+1}-3)\pi}{2^{j+1}-1}, \frac{2^2(2^j-1)\pi}{2^{j+1}-1} \right] \cup \\ &\left[\frac{2^2(2^{j+1}-3)\pi}{2^{j+1}-1}, \frac{2^3(2^j-1)\pi}{2^{j+1}-1} \right] \cup \cdots \cup \left[\frac{2^j(2^{j+1}-3)\pi}{2^{j+1}-1}, \frac{2^{j+1}(2^j-1)\pi}{2^{j+1}-1} \right] \\ \equiv\ & S_{0j} \cup S_{1j} \cup S_{2j} \cup \cdots \cup S_{jj}, \end{aligned}$$

where $S_{ij} = \left[\frac{2^i(2^{j+1}-3)\pi}{2^{j+1}-1}, \frac{2^{i+1}(2^j-1)\pi}{2^{j+1}-1} \right]$, $i = 1, 2 \ldots j$ and $S_{0j} = \left[\frac{-2\pi}{2^{j+1}-1}, \frac{2(2^j-1)\pi}{2^{j+1}-1} \right]$.

From Theorem 3.1, $S_j \times W_j$, where $j \geq 2$, is an MRA A-wavelet set of $\mathbb{R}^2$.

4. Non-MSF MRA A-Wavelets in $L^2(\mathbb{R}^2)$

In this Section, we employ Examples 3.2 and 3.3 to construct non-MSF MRA A-wavelets for $L^2(\mathbb{R}^2)$.

We write

$$S_a \times W_a = I_{01} \cup I_{02},$$

where $I_{01} = [a-2\pi, a] \times [2(a-2\pi), (a-2\pi)]$, and $I_{02} = [a-2\pi, a] \times [a, 2a]$. Also, I_{01}^+ and I_{02}^+ denote, respectively, $[0, a] \times [2(a-2\pi), (a-2\pi)]$ and $[0, a] \times [a, 2a]$.

Under the above notation, for $m \in \mathbb{N}$ and $a \in (0, \pi)$ the following can immediately be obtained.

Lemma 4.1.

1. $(A^*)^{-m} I_{02}^+ - (0, 2\pi) \subset I_{01}^+$,
2. $I_{02}^+ - (A^*)^m (0, 2\pi) \subset (A^*)^m I_{01}^+$.

Denoting $I_{01} \cup I_{02} \cup (A^*)^{-m} I_{02}^+ \cup I_{02}^+ - (A^*)^m (0, 2\pi)$ by $E_{a,m}$, we have the following:

Lemma 4.2. (1) *If $\xi \in I_{02}^+$, then*

$$\begin{cases} \xi + 2k\pi \in E_{a,m}, & \text{iff } k = (0,0) \text{ or } -(A^*)^m (0,1), \\ (A^*)^l \xi \in E_{a,m}, & \text{iff } l = 0 \text{ or } -m. \end{cases}$$

(2) *If $\xi \in (A^*)^{-m} I_{02}^+$, then*

$$\begin{cases} \xi + 2k\pi \in E_{a,m}, & \text{iff } k = (0,0) \text{ or } (0,-1), \\ (A^*)^l \xi \in E_{a,m}, & \text{iff } l = 0 \text{ or } m. \end{cases}$$

(3) *If $\xi \in (A^*)^{-m} I_{02}^+ - (0, 2\pi)$, then*

$$\begin{cases} \xi + 2k\pi \in E_{a,m}, & \text{iff } k = (0,0) \text{ or } (0,1), \\ (A^*)^l \xi \in E_{a,m}, & \text{iff } l = 0 \text{ or } m. \end{cases}$$

(4) *If $\xi \in I_{02}^+ - (A^*)^m (0, 2\pi)$, then*

$$\begin{cases} \xi + 2k\pi \in E_{a,m}, & \text{iff } k = (0,0) \text{ or } (A^*)^m (0,1), \\ (A^*)^l \xi \in E_{a,m}, & \text{iff } l = 0 \text{ or } -m. \end{cases}$$

(5) *If $\xi \in I_{02} \setminus I_{02}^+$, then*

$$\begin{cases} \xi + 2k\pi \in E_{a,m}, & \text{iff } k = (0,0), \\ (A^*)^l \xi \in E_{a,m}, & \text{iff } l = 0. \end{cases}$$

(6) *If* $\xi \in I_{01}\backslash((A^*)^{-m}I_{02}^+ - (0,2\pi))$, *then*

$$\begin{cases} \xi + 2k\pi \in E_{a,m}, & \text{iff } \ k = (0,0), \\ (A^*)^l\xi \in E_{a,m}, & \text{iff } \ l = 0. \end{cases}$$

Proof. We prove only (1). Observing that:

(A) for a measurable set E in $\mathbb{R}^2$ and a $k \in \mathbb{Z}^2$, $\tau(E) = \tau(E + 2k\pi)$,

(B) for disjoint measurable sets E and F of $\mathbb{R}^2$ contained in a wavelet set W, $\tau(E) \cap \tau(F) = \phi$, and similar facts related to the dilation projection, we conclude the following:

$$\tau(I_{02}^+) = \tau(I_{02}^+ - (A^*)^m(0,2\pi)), \quad \tau((A^*)^{-m}I_{02}^+) = \tau((A^*)^{-m}I_{02}^+ - (0,2\pi)), \tag{6}$$

$$d(I_{02}^+) = d((A^*)^{-m}I_{02}^+), \quad d((A^*)^{-m}I_{02}^+ - (0,2\pi)) = d(I_{02}^+ - (A^*)^m(0,2\pi)). \tag{7}$$

Next, since $I_{01} \cup I_{02}$ is an $L^2(\mathbb{R}^2)$ A-wavelet set and $(A^*)^{-m}I_{02}^+ - (0,2\pi) \subset I_{01}$, we have

$$\tau(I_{02}^+) \cap \tau(I_{01}) = \phi, \tag{8}$$

and

$$\tau(I_{02}^+) \cap \tau((A^*)^{-m}I_{02}^+ - (0,2\pi)) = \phi. \tag{9}$$

From (6) and (9), we get

$$\tau(I_{02}^+) \cap \tau((A^*)^{-m}I_{02}^+) = \phi. \tag{10}$$

Therefore, if $\xi \in I_{02}^+$, then $\xi + 2k\pi \in E_{a,m}$ iff $k = (0,0)$, or $-(A^*)^m(0,1)$.

Similarly, we have

$$d(I_{02}^+) \cap d(I_{01}) = \phi, \tag{11}$$

and

$$d(I_{02}^+) \cap d((A^*)^{-m}I_{02}^+ - (0,2\pi)) = \phi. \tag{12}$$

Together with (7), (12) implies that

$$d(I_{02}^+) \cap d(I_{02}^+ - (A^*)^m(0,2\pi)) = \phi. \tag{13}$$

Thus, if $\xi \in I_{02}^+$, then $(A^*)^l\xi \in E_{a,m}$ iff $l = 0$, or $-m$. □

Theorem 4.3. *For each* $a \in (0,\pi)$ *and* $m \in \mathbb{N}$, *the function* $\psi_{a,m}$ *defined by*

$$\widehat{\psi}_{a,m} = \begin{cases} \frac{1}{\sqrt{2}} & if \quad \xi \in I_{02}^+ \cup (A^*)^{-m}I_{02}^+ \cup ((A^*)^{-m}I_{02}^+ - (0,2\pi)), \\ \frac{-1}{\sqrt{2}} & if \ \ \xi \in I_{02}^+ - (A^*)^m(0,2\pi), \\ 1 & if \quad \xi \in I_{01}\backslash((A^*)^{-m}I_{02}^+ - (0,2\pi)) \cup I_{0,2}^-, \\ 0 & otherwise, \end{cases}$$

is a non-MSF A-wavelet for $L^2(\mathbb{R}^2)$.

Proof. That the sets used to define $\hat{\psi}_{a,m}$ are pairwise disjoint is seen with the help of Lemmas 4.1, 4.2 and Theorem 2.4. Using Theorem 2.1, we show that $\psi_{a,m}$ is an orthonormal wavelet for $L^2(\mathbb{R}^2)$. Since $\hat{\psi}_{a,m}$ is not a characteristic function, $\psi_{a,m}$ comes out to be a non-MSF wavelet. Since

$$\begin{aligned}||\hat{\psi}_{a,m}||_2^2 &= \int_{\mathbb{R}^2} |\hat{\psi}_{a,m}(\xi)|^2 \, d\xi \\ &= \frac{1}{2}(|I_{0,2}^+| + |(A^*)^{-m} I_{0,2}^+| + |(A^*)^{-m} I_{0,2}^+| + |I_{0,2}^+|) \\ &\quad +|I_{0,1}| - |(A^*)^{-m} I_{0,2}^+| + |I_{0,2}^-| \\ &= |I_{0,1}| + |I_{0,2}^+| + |I_{0,2}^-| \\ &= |S_a \times W_a| = (2\pi)^2,\end{aligned}$$

it follows that $||\psi_{a,m}||_2 = 1$. Thus, $\psi_{a,m}$ is a unit vector in $L^2(\mathbb{R}^2)$.

(i) Since $\rho(A^*\xi)=\rho(\xi)$ for a.e. $\xi \in \mathbb{R}^2$, it suffices to show that $\rho(\xi)=1$, on $S_a \times W_a$. If $\xi \in I_{01}$, then either $\xi \in I_{01}\backslash((A^*)^{-m} I_{02}^+ - (0, 2\pi))$, or $\xi \in (A^*)^{-m} I_{02}^+ - (0, 2\pi)$. If $\xi \in (A^*)^{-m} I_{02}^+ - (0, 2\pi)$, then by the definition of $\psi_{a,m}$, $(A^*)^j\xi \in \text{supp } \hat{\psi}_{a,m}$ iff $j = 0$, or m. Hence

$$\rho(\xi) = |\hat{\psi}(\xi)|^2 + |\hat{\psi}((A^*)^m\xi)|^2 = 1.$$

If $\xi \in I_{01}\backslash((A^*)^{-m} I_{02}^+ - (0, 2\pi))$, then $(A^*)^j\xi \in \text{supp } \hat{\psi}_{a,m}$ iff $j = 0$. Hence

$$\rho(\xi) = |\hat{\psi}(\xi)|^2 = 1.$$

If $\xi \in I_{02}^+$, then $(A^*)^j\xi \in \text{supp } \hat{\psi}_{a,m}$ iff $j = 0$, or $-m$. Hence

$$\rho(\xi) = |\hat{\psi}(\xi)|^2 + |\hat{\psi}((A^*)^{-m}\xi)|^2 = 1.$$

If $\xi \in I_{02}^-$, then $(A^*)^j\xi \in \text{supp } \hat{\psi}_{a,m}$ iff $j = 0$. Hence

$$\rho(\xi) = |\hat{\psi}(\xi)|^2 = 1.$$

(ii) In view of $t_{-q}(\xi) = \overline{t_q(\xi - 2q\pi)}$, we show that $t_q(\xi) = 0$, a.e., and for all $\{q = (q_1, q_2) :\ q_2 \text{ is an odd positive integer}\}$.

The term $\hat{\psi}_{a,m}((A^*)^j\xi)\ \overline{\hat{\psi}_{a,m}((A^*)^j(\xi + 2q\pi))}$ is nonzero, when both $(A^*)^j\xi$ and $(A^*)^j(\xi + 2q\pi)$ lie in the support of $\hat{\psi}_{a,m}$. By Lemma 4.2, this is possible if either $(A^*)^j q = (0, 1)$, or $(A^*)^j q = (A^*)^m(0, 1)$. In the first case $(2^{\frac{j}{2}} q_1, 2^{\frac{j}{2}} q_2) = (0, 1)$, when j is even, and $(2^{\frac{j-1}{2}} q_1, 2^{\frac{j+1}{2}} q_2) = (0, 1)$, when j is odd. This is possible only if $j = 0$ and $q = (0, 1)$. In this case $(A^*)^j\xi \in (A^*)^{-m} I_{02}^+ - (0, 2\pi)$, $(A^*)^j(\xi + 2q\pi) \in (A^*)^{-m} I_{02}^+$. Hence $(A^*)^{j+m}\xi \in I_{0,2}^+ - (A^*)^m(0, 2\pi)$, and $(A^*)^{j+m}(\xi + 2q\pi) \in I_{02}^+$. Thus

$$t_q(\xi) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) = 0.$$

The second case can be treated similarly. This completes the proof. □

Noting that an MRA wavelet set W is disjoint from its scaling set S, we have

$$S_j \times W_j = (S_{0j} \cup S_{1j} \cup S_{2j} \cup \cdots \cup S_{jj}) \times (W_1 \cup W_2 \cup W_3)$$
$$= \bigcup_{i=0,1,2,\ldots j;\, k=1,2,3} S_{ij} \times W_k.$$

For brevity, we denote $S_{ij} \times W_k$ by I_{ik}.

With the above notation, for $m \in \mathbb{N}$ and $j \geq 2$, we have the following:

Lemma 4.4.

1. $(A^*)^{-m} I_{02} + 2(0, 2^j - 1)\pi \subset I_{03}$,
2. $I_{02} + (A^*)^m 2(0, 2^j - 1)\pi \subset (A^*)^m I_{03}$.

Denoting
$I_{01} \cup I_{02} \cup I_{03} \cup (A^*)^{-m} I_{02} \cup I_{02} + (A^*)^m 2(0, 2^j - 1)\pi \cup I_{11} \cup \cdots \cup I_{j1} \cup I_{j2} \cup I_{j3}$
by $E_{j,m}$, we have the following:

Lemma 4.5.

1. *If* $\xi \in I_{02}$, *then*

$$\begin{cases} \xi + 2k\pi \in E_{j,m}, & \text{iff } k = (0,0) \text{ or } (A^*)^m (0, 2^j - 1), \\ (A^*)^l \xi \in E_{j,m}, & \text{iff } l = 0 \text{ or } -m. \end{cases}$$

2. *If* $\xi \in (A^*)^{-m} I_{02}$, *then*

$$\begin{cases} \xi + 2k\pi \in E_{j,m}, & \text{iff } k = (0,0) \text{ or } (0, 2^j - 1), \\ (A^*)^l \xi \in E_{j,m}, & \text{iff } l = 0 \text{ or } m. \end{cases}$$

3. *If* $\xi \in (A^*)^{-m} I_{02} + 2(0, 2^j - 1)\pi$, *then*

$$\begin{cases} \xi + 2k\pi \in E_{j,m}, & \text{iff } k = (0,0) \text{ or } (0, -(2^j - 1)), \\ (A^*)^l \xi \in E_{j,m}, & \text{iff } l = 0 \text{ or } m. \end{cases}$$

4. *If* $\xi \in I_{02} + (A^*)^m 2(0, 2^j - 1)\pi$, *then*

$$\begin{cases} \xi + 2k\pi \in E_{j,m}, & \text{iff } k = (0,0) \text{ or } (A^*)^m (0, -(2^j - 1)), \\ (A^*)^l \xi \in E_{j,m}, & \text{iff } l = 0 \text{ or } -m. \end{cases}$$

5. *If* $\xi \in I_{03} \backslash ((A^*)^{-m} I_{02} + 2(0, 2^j - 1)\pi)$, *then*

$$\begin{cases} \xi + 2k\pi \in E_{j,m}, & \text{iff } k = (0,0), \\ (A^*)^l \xi \in E_{j,m}, & \text{iff } l = 0. \end{cases}$$

6. *If* $\xi \in I_{01} \cup I_{11} \cup \cdots \cup I_{j1} \cup I_{j2} \cup I_{j3}$, *then*

$$\begin{cases} \xi + 2k\pi \in E_{j,m}, & \text{iff } k = (0,0), \\ (A^*)^l \xi \in E_{j,m}, & \text{iff } l = 0. \end{cases}$$

Proof. It is similar to the proof of Lemma 4.2. □

Theorem 4.6. *For each* $j \geq 2$ *and* $m \in \mathbb{N}$, *the function* $\psi_{j,m}$ *defined by*

$$\widehat{\psi}_{j,m} = \begin{cases} \frac{1}{\sqrt{2}} & if \quad \xi \in I_{02} \cup (A^*)^{-m} I_{02} \cup ((A^*)^{-m} I_{02} + 2(0, 2^j - 1)\pi)), \\ \frac{-1}{\sqrt{2}} & if \quad \xi \in I_{0,2} + (A^*)^m 2(0, 2^j - 1)\pi), \\ 1 & if \quad \xi \in I_{03} \backslash ((A^*)^{-m} I_{02} + 2(0, 2^j - 1)\pi) \cup I_{01} \cup I_{11} \\ & \cup \cdots \cup I_{j1} \cup I_{j2} \cup I_{j3}, \\ 0 & otherwise, \end{cases}$$

is a non-MSF A-wavelet for $L^2(\mathbb{R}^2)$.

Proof. It is similar to the proof of Theorem 4.3. □

Remark 4.7. Since the construction in Theorems 4.3 and 4.6 are based on MRA A-wavelet sets, the non-MSF A-wavelets thus constructed are MRA, in addition [10].

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