T-SEPARATING SETS FOR COHERENT SEQUENCES

Martin Dowd
60 Mooring Ln.
Daly City, CA 94014, USA

Abstract: In a previous paper the author used methods of Witzany to give a lower bound for the smallest repeat point of a coherent sequence. Here the notion of a T-separating set is introduced, and the lower bound is improved.

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1. Introduction

In [4] some methods are introduced for constructing separating stationary sets for coherent sequences of normal ultrafilters. Some further results are given in [3]. Here these methods are improved on. Superschemes were introduced in [1], and an improved discussion is given in [2]. The methods here permit the use of superschemes in constructing separating stationary sets.

As noted in [3], by results of Mitchell there is a model \( L[U] \) such that in \( L[U] \), \( U \) is a coherent sequence of normal ultrafilters comprising all the normal ultrafilters. It is well-known that GCH holds in \( L[U] \).

Notation for coherent sequences will be as in [3]. Hereafter in this section it will be assumed that GCH holds and \( U \) is maximal, so that for a measurable cardinal \( \kappa \), \( \text{Dom}(U(\kappa)) = o(\kappa) \leq \kappa^{++} \).

2. Separating Sets

Say that \( S \) is a separating set for \( U(\kappa) \) at \( \alpha \) if \( S \in U(\kappa)(\alpha) \) but \( S \notin U(\kappa)(\beta) \).
for $\beta < \alpha$; such exists iff $\alpha$ is not a repeat point. Say that $S$ is T-separating if
in addition $S \in \mathcal{U}(\kappa)(\beta)$ for $\alpha \leq \beta < \text{Dom}(\mathcal{U}(\kappa))$. These may readily be seen
to exist for $\alpha$ up to a bound given in theorem 22.g of [3].

Given a measurable cardinal $\kappa$ and a function $f : \kappa \mapsto \text{Ord}$ let $D_f^{\geq} = \{ \lambda \in \text{Card} \cap \kappa : o(\lambda) \geq f(\lambda) \}$. For $\beta < \text{Dom}(\kappa)$, $U_\beta$ will be used as an abbreviation for $\mathcal{U}(\kappa)(\beta)$.

**Theorem 1.** Suppos $\alpha < \text{Dom}(\mathcal{U}(\kappa))$ and $f$ represents $\alpha$ on $[0, \text{Dom}(\mathcal{U}(\kappa)))$. Then $D_f^{\geq}$ is T-separating at $\alpha$.

**Proof.** $D_f^{\geq} \in U_\beta \iff [\alpha]_{U_\beta} \geq [f]_{U_\beta} \iff \beta \geq \alpha$. □

It may be easier to construct T-separating sets than representing functions on $[0, \text{Dom}(\mathcal{U}(\kappa)))$, and studying them in their own right is of interest.

**Theorem 2.** Suppose $S$ is a separating set at $\alpha$ and $\alpha + 1 < \text{Dom}(\mathcal{U}(\kappa))$. Then there is a T-separating set $S' \geq \alpha + 1$.

**Proof.** Let $S' = \{ \lambda \in \text{Card} \cap \kappa : \exists \eta < \text{Dom}(\mathcal{U}(\kappa))(\lambda \cap \eta \in \mathcal{U}(\lambda)(\eta)) \}$. Using coherence and the fact that $S$ is separating, $S' \in U_\beta \iff \exists \eta < \text{Dom}(\mathcal{U}(\kappa))(S \cap \eta \in \mathcal{U}(\kappa)(\eta)) \iff \beta > \alpha$. □

**Theorem 3.** Suppose $\eta < \kappa$, for $\xi < \eta$ $S_\xi$ is a T-separating set at $\alpha_\xi$, $\alpha = \sup_{\xi < \eta} \alpha_\xi$, and $\alpha < \text{Dom}(\mathcal{U}(\kappa))$. Let $S = \cap_{\xi < \eta} S_\xi$; then $S$ is a T-separating set at $\alpha$.

**Proof.** For $\beta \geq \alpha$, since $S_\xi \in U_\beta$ for $\xi < \eta$ and $U_\beta$ is $\kappa$-complete, $S \in U_\beta$. If $\beta < \alpha$ then $\beta < \alpha_\xi$ for some $\xi$, so $S_\xi \notin U_\beta$. so $S \notin U_\beta$ since $S \subseteq S_\xi$. □

**Theorem 4.** Suppose for $\xi < \kappa$ $S_\xi$ is a T-separating set at $\alpha_\xi$, $\alpha = \sup_{\xi < \kappa} \alpha_\xi$, and $\alpha < \text{Dom}(\mathcal{U}(\kappa))$. Let $S = \cap_{\xi < \kappa} S_\xi$; then $S$ is a T-separating set at $\alpha$.

**Proof.** For $\beta \geq \alpha$, since $S_\xi \in U_\beta$ for $\xi < \kappa$ and $U_\beta$ is normal, $S \in U_\beta$. If $\beta < \alpha$ then $\beta < \alpha_\xi$ for some $\xi$, so $S_\xi \notin U_\beta$. so $S \notin U_\beta$ since $S \subseteq S_\xi$ where $\mathcal{I}$ is the thin ideal. □

Recall from [2] that for $\kappa \in \text{Card}$ a scheme is a pair $\Sigma = \langle \sigma, \phi \rangle$ where $\sigma < \kappa^+$ and $\phi$ is a function whose domain is the set of limit ordinals $\alpha \leq \sigma$. For $\alpha \in \text{Dom}(\phi)$, $\phi(\alpha)$ is an increasing function with domain an ordinal $\eta \leq \kappa$, and whose range is an unbounded subset of $\alpha$. If $\text{cf}(\alpha) < \kappa$ then $\eta < \kappa$, and if $\text{cf}(\alpha) = \kappa$ then $\eta = \kappa$. 


A scheme is a recipe for an iteration. Given a scheme \( \Sigma \) with \( \sigma < \text{Dom}(\kappa) \) the subset \( S^\Sigma_\alpha \) may be defined inductively for \( \alpha \leq \sigma \) as follows.

0. \( S^\Sigma_0 = \text{Card} \cap \kappa \).
1. \( S^\Sigma_{\alpha+1} = (S^\Sigma_\alpha)' \), as in theorem 2.
2. \( (\alpha \in \text{Lim}, \text{cf}(\alpha) < \kappa) \cap \xi < \text{Dom}(\phi(\alpha)) S^\Sigma_\phi(\alpha)(\xi) \).
3. \( (\alpha \in \text{Lim}, \text{cf}(\alpha) = \kappa) \triangle \xi < \kappa S^\Sigma_\phi(\alpha)(\xi) \).

By theorems 2-4 \( \Sigma^\alpha \) is T-separating for all \( \alpha \leq \sigma \).

Suppose there is no T-separating set at \( \alpha \), and let \( \theta \) be the least such \( \alpha \). For another corollary of theorems 2-4, \( \text{cf}(\theta) = \kappa^+ \).

3. Superschemes

Recall from [2] that for \( \kappa \in \text{Card} \) a superscheme is a pair \( \Sigma = \langle \sigma, \phi \rangle \) where \( \sigma < \kappa^{++} \) and \( \phi \) is as for a scheme. To use a superscheme for an iteration a method must be specified for obtaining \( S_\alpha \) when \( \text{cf}(\alpha) = \kappa^+ \). Some discussion of this problem will be given here.

Suppose \( \alpha = \kappa^+ \), \( U \) is a normal ultrafilter on \( \kappa \), and \( M = \text{Ult}_U(\kappa) \). Since the well-orders of \( \kappa \), coded as subsets of \( \kappa \), are the same in \( V \) and \( M \), and the order type function is \( \Delta^1 \), \( (\kappa^+) \) in any \( \text{Ult}_U(V) \).

Another construction may be given by adapting a method of proposition 3.9 if [4]. As in [3] let \( C \) denote the map \( \alpha \mapsto j_{U(\kappa)(\alpha)}(\kappa) \). \( C_{U(\kappa)} \) may be written to indicate \( U \) and \( \kappa \).

**Theorem 5.** Suppose \( S \) is a separating set at \( \alpha \). For any \( \beta \in (\alpha, \text{min}(\text{Dom}(U(\kappa)), C(\alpha))) \) there is a T-separating set at \( \beta \).

**Proof.** As in the proof of theorem 2 let \( S_1 = \{ \lambda \in \text{Card} \cap \kappa : \exists \eta < \text{Dom}(\lambda)(S \cap \lambda \in U(\lambda)(\eta)) \} \), so that \( S_1 \in U_\beta \) iff \( \beta > \alpha \). For \( \lambda \in S_1 \) let \( f_\alpha(\lambda) \) be the least \( \eta \); then for \( \beta > \alpha \) the function \( f_\alpha \) represents \( \alpha \) in \( \text{Ult}(\kappa)(\beta) \). Choose \( g_\alpha : \kappa \mapsto \kappa \) such that \( [g_\alpha]_{U(\kappa)(\alpha)} = \beta \). For \( \lambda \in S_1 \) let \( \hat{g}(\lambda) = [g \upharpoonright \lambda]_{U(\lambda)(f_\alpha(\lambda))} \); otherwise let \( \hat{g}(\lambda) = 0 \). Let \( M \) denote \( \text{Ult}(\kappa)(\beta) \). Then \( g \in M \), and one verifies (see [4]) that \( [\hat{g}]_{U(\kappa)(\beta)} = \beta \). The set \( S_1 \cap D^\kappa_\beta \) is T-separating at \( \beta \).

Let \( C^{(1)} \) denote the fixed point enumerator of \( C \). If there is an \( \alpha \) such that
there is no T-separating set at \( \alpha \) let \( \theta \) denote the smallest such. As in theorem 22 of [3], \( C(\theta) = \theta \) and \( C^{(1)}(\kappa^+) \leq \theta \).

4. Iterating \( C \)

Let \( C(\sigma) \) denote the result of applying the fixed point operator to \( C \) \( \sigma \) times. If \( \text{Dom}(U(\kappa)) = \kappa^+ \) then for \( \sigma < \kappa^+ \) \( \text{Ran}(C^{(\sigma)}) \) is a club set, and the intersection of all these is empty. If \( \text{Dom}(U(\kappa)) < \kappa^+ \) then \( \text{Dom}(C^{(\sigma)}) \) decreases as \( \sigma \) increases, becoming \( \emptyset \) for some \( \sigma \leq \text{Dom}(U(\kappa)) \).

**Theorem 6.** Suppose \( \alpha < \text{Dom}(U(\kappa)) \), \( f_\alpha \) represents \( \alpha \) on \([0, \text{Dom}(U(\kappa))] \), and \( \alpha \in \text{Dom}(C^{(1)}) \). Then there is a T-separating set at \( C^{(1)}_{U(\kappa)}(\alpha) \).

Proof. Let \( S = \{ \lambda : o(\lambda) \geq C^{(1)}_{U(\lambda)}(f_\alpha(\lambda)) \} \). Then \( S \in U_\beta \) iff \( \beta \geq C^{(1)}_{U(\kappa)}(\alpha) \).

The condition on \( \lambda \) holds iff either \( \exists \eta < \text{Dom}(U(\lambda))(\eta = C^{(1)}_{U(\lambda)}(f_\alpha(\lambda))) \), or \( \text{Dom}(U(\lambda)) \) is a limit ordinal and \( \forall \zeta < f_\alpha(\lambda) \exists \eta < \text{Dom}(U(\lambda))(\eta = C^{(1)}_{U(\lambda)}(\zeta)) \) and \( \forall \zeta < \text{Dom}(U(\lambda))(\zeta < \text{Dom}(U(\lambda))) \).

**Theorem 7.** Suppose \( \alpha < \text{Dom}(U(\kappa)) \), there is a T-separating set at \( \alpha \), and \( C^{(1)}_{U(\kappa)}(\alpha) < \text{Dom}(U(\kappa)) \). Then there is a T-separating set at \( C^{(1)}_{U(\kappa)}(\alpha) \).

Proof. The proof of theorem 6 may be modified. Let \( S_1 \) be a T-separating set at \( \alpha \). Let \( f^E_\alpha \) be an E-representing function for \( \alpha \), as defined in [3]. Let \( S_2 \) be the set \( S \), as in the proof of theorem 6, with \( f^E_\alpha \) used in place of \( f_\alpha \). Then \( S_1 \cap S_2 \) is T-separating at \( C^{(1)}(\alpha) \).

**Theorem 8.** Suppose \( \alpha < \text{Dom}(U(\kappa)) \), \( f_\alpha \) represents \( \alpha \) on \([0, \text{Dom}(U(\kappa))] \), \( f_\sigma \) represents \( \sigma \) on \([0, \text{Dom}(U(\kappa))] \), and \( \alpha \in \text{Dom}(C^{(\sigma)}) \). Then there is a T-separating set at \( C^{(\sigma)}_{U(\kappa)}(\alpha) \).

Proof. Let \( S = \{ \lambda : o(\lambda) \geq C^{(\sigma)}_{U(\lambda)}(f_\alpha(\lambda)) \} \). Then \( S \in U_\beta \) iff \( \beta \geq C^{(\sigma)}_{U(\kappa)}(\alpha) \).

The condition on \( \lambda \) holds iff either \( \exists \eta < \text{Dom}(U(\lambda))(\eta = C^{(\sigma)}_{U(\lambda)}(f_\alpha(\lambda))) \), or \( \text{Dom}(U(\lambda)) \) is a limit ordinal and \( \forall \zeta < f_\alpha(\lambda) \exists \eta < \text{Dom}(U(\lambda))(\eta = C^{(\sigma)}_{U(\lambda)}(\zeta)) \) and \( \forall \zeta < \text{Dom}(U(\lambda)) \forall \tau < f_\alpha(\lambda)(C^{(\tau)}_{U(\lambda)}(\zeta) < \text{Dom}(U(\lambda))) \).

**Theorem 9.** Suppose \( \alpha < \text{Dom}(U(\kappa)) \), there is a T-separating set at \( \alpha \), there is a T-separating set at \( \sigma \), and \( \alpha \in \text{Dom}(C^{(\sigma)}) \). Then there is a T-separating set at \( C^{(\sigma)}_{U(\kappa)}(\alpha) \).
Proof. The proof of theorem 8 may be modified. Let \( S_{1\alpha} \) be a T-separating set at \( \alpha \). Let \( f^E_{\alpha} \) be an E-representing function for \( \alpha \). Let \( S_{1\sigma} \) be a T-separating set at \( \sigma \). Let \( f^E_{\sigma} \) be an E-representing function for \( \sigma \). Let \( S_2 \) be the set \( S \), as in the proof of theorem 8, with \( f^E_{\alpha} \) used in place of \( f_{\alpha} \) and \( f^E_{\sigma} \) used in place of \( f_{\sigma} \). Then \( S_{1\alpha} \cap S_{1\sigma} \cap S_2 \) is T-separating at \( C^{(1)}(\alpha) \).

These methods can be pursued further. Whether there are methods which may be pursued in \( L[U] \) is a question of considerable interest.

References


