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T-SEPERATING SETS FOR COHERENT SEQUENCES

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Abstract: In a previous paper the author used methods of Witzany to give a lower bound for the smallest repeat point of a coherent sequence. Here the notion of a T-seperating set is introduced, and the lower bound is improved.

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1. Introduction

In [4] some methods are introduced for constructing seperating stationary sets for coherent sequences of normal ultrafilters. Some further results are given in [3]. Here these methods are improved on. Superschemes were introduced in [1], and an improved discussion is given in [2]. The methods here permit the use of superschemes in constructing seperating stationary sets.

As noted in [3], by results of Mitchell there is a model $L[\mathcal{U}]$ such that in $L[\mathcal{U}]$, $\mathcal{U}$ is a coherent sequence of normal ultrafilters comprising all the norml ultrafilters. It is well-known that GCH holds in $L[\mathcal{U}]$.

Notation for coherent sequences will be as in [3]. Hereafter in this section it will be assumed that GCH holds and $\mathcal{U}$ is maximal, so that for a measurable cardinal κ, $\mathrm{Dom}(\mathcal{U}(\kappa)) = o(\kappa) \leq \kappa^{++}$.

2. Seperating Sets

Say that S is a seperating set for $\mathcal{U}(\kappa)$ at α if $S \in \mathcal{U}(\kappa)(\alpha)$ but $S \notin \mathcal{U}(\kappa)(\beta)$

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for $\beta < \alpha$; such exists iff α isnot a repeat point. Say that S is T-seperating if in addition $S \in \mathcal{U}(\kappa)(\beta)$ for $\alpha \leq \beta < \mathrm{Dom}(\mathcal{U}(\kappa))$. These may readily be seen to exist for α up to a bound given in theorem 22.g of [3].

Given a measurable cardinal κ and a function $f : \kappa \mapsto \mathrm{Ord}$ let $D^{\geq}_f = \{\lambda \in \mathrm{Card} \cap \kappa : o(\lambda) \geq f(\lambda)$. For $\beta < \mathrm{Dom}(\kappa)$, U_β will be used as an abbreviation for $\mathcal{U}(\kappa)(\beta)$.

Theorem 1. *Suppos $\alpha < Dom(\mathcal{U}(\kappa))$ and f represents α on $[0, Dom(\mathcal{U}(\kappa)))$. Then $D^{\geq}_f$ is T-seperating at α.*

Proof. $D^{\geq}_f \in U_\beta$ iff $[o]_{U_\beta} \geq [f]_{U_\beta}$ iff $\beta \geq \alpha$. □

It may be easier to construct T-seperating sets than representing functions on $[0, \mathrm{Dom}(\mathcal{U}(\kappa)))$, and studying them in their own right is of interest.

Theorem 2. *Suppose S is a seperating set at α and $\alpha + 1 < Dom(\mathcal{U}(\kappa))$. Then there is a T-seperating set S' at $\alpha + 1$.*

Proof. Let $S' = \{\lambda \in \mathrm{Card} \cap \kappa : \exists \eta < \mathrm{Dom}(\mathcal{U}(\lambda))(S \cap \lambda \in \mathcal{U}(\lambda)(\eta))$. Using coherence and the fact that S is seperating, $S' \in U_\beta$ iff $\models_{\mathrm{Ult}_{U_\beta}(V)} \exists \eta < \mathrm{Dom}(\mathcal{U}(\kappa))(S \in \mathcal{U}(\kappa)(\eta))$ iff $\beta > \alpha$. □

Theorem 3. *Suppose $\eta < \kappa$, for $\xi < \eta$ S_ξ is a T-seperating set at α_ξ, $\alpha = \sup_{\xi<\eta} \alpha_\xi$, and $\alpha < Dom(\mathcal{U}(\kappa))$. Let $S = \cap_{\xi<\eta} S_\xi$; then S is a T-seperating set at α.*

Proof. For $\beta \geq \alpha$, since $S_\xi \in U_\beta$ for $\xi < \eta$ and U_β is κ-complete, $S \in U_\beta$. If $\beta < \alpha$ then $\beta < \alpha_\xi$ for some ξ, so $S_\xi \notin U_\beta$. so $S \notin U_\beta$ since $S \subseteq S_\xi$. □

Theorem 4. *Suppose for $\xi < \kappa$ S_ξ is a T-seperating set at α_ξ, $\alpha = \sup_{\xi<\kappa} \alpha_\xi$, and $\alpha < Dom(\mathcal{U}(\kappa))$. Let $S = \triangledown_{\xi<\kappa} S_\xi$; then S is a T-seperating set at α.*

Proof. For $\beta \geq \alpha$, since $S_\xi \in U_\beta$ for $\xi < \kappa$ and U_β is normal, $S \in U_\beta$. If $\beta < \alpha$ then $\beta < \alpha_\xi$ for some ξ, so $S_\xi \notin U_\beta$. so $S \notin U_\beta$ since $S \subseteq_{\mathcal{I}} S_\xi$ where $\mathcal{I}$ is the thin ideal. □

Recall from [2] that for $\kappa \in \mathrm{Card}$ a scheme is a pair $\Sigma = \langle \sigma, \phi \rangle$ where $\sigma < \kappa^+$ and ϕ is a function whose domain is the set of limit ordinals $\alpha \leq \sigma$. For $\alpha \in \mathrm{Dom}(\phi)$, $\phi(\alpha)$ is an increasing function with domain an ordinal $\eta \leq \kappa$, and whose range is an unbounded subset of α. If $\mathrm{cf}(\alpha) < \kappa$ then $\eta < \kappa$, and if $\mathrm{cf}(\alpha) = \kappa$ then $\eta = \kappa$.

A scheme is a recipe for an iteration. Given a scheme Σ with $\sigma < \mathrm{Dom}(\kappa)$ the subset S^Σ_α may be defined inductively for $\alpha \leq \sigma$ as follows.

0. $S^\Sigma_0 = \mathrm{Card} \cap \kappa$.
1. $S^\Sigma_{\beta+1} = (S^\Sigma_\beta)'$, as in theorem 2.
2. $(\alpha \in \mathrm{Lim},\ \mathrm{cf}(\alpha) < \kappa)\ \cap_{\xi < \mathrm{Dom}(\phi(\alpha))} S^\Sigma_{\phi(\alpha)(\xi)}$.
3. $(\alpha \in \mathrm{Lim},\ \mathrm{cf}(\alpha) = \kappa)\ \triangle \xi < \kappa S^\Sigma_{\phi(\alpha)(\xi)}$.

By theorems 2-4 Σ^Σ_α is T-seperating for all $\alpha \leq \sigma$.

Suppose there is an α there is no T-seperating set at α, and let θ be the least such α. For another corollary of theorems 2-4, $\mathrm{cf}(\theta) = \kappa^+$.

3. Superschemes

Recall from [2] that for $\kappa \in \mathrm{Card}$ a superscheme is a pair $\Sigma = \langle \sigma, \phi \rangle$ where $\sigma < \kappa^{++}$ and ϕ is as for a scheme. To use a superscheme for an iteration a method must be specified for obtaining S_α when $\mathrm{cf}(\alpha) = \kappa^+$. Some discussion of this problem will be given here.

Suppose $\alpha = \kappa^+$, U is a normal ultrafilter on κ, and $M = \mathrm{Ult}_U(V)$. Since the well-orders of κ, coded as subsets of κ, are the same in V and M, and the order type function is Δ^1, $(\kappa^+)^M = \kappa^+$. Thus, the function $\lambda \mapsto \lambda^+$ represents κ^+ mod any normal ultrafilter on κ, and a T-seperating set may be constructed using theorem 1.

Alternatively, referring to [3] let $\preceq$ be a Δ^1_∞ WPS with $\Omega(\prec) = \kappa^+$ (in fact there is a $\Sigma^1_1 \preceq$). According to theorem 10 of [3], $\lambda \mapsto \Omega(\prec_\lambda)$ (defined on $D_\preceq$) represents κ^+ in any $\mathrm{Ult}_U(V)$.

Another construction may be given by adapting a method of proposition 3.9 if [4]. As in [3] let C denote the map $\alpha \mapsto j_{\mathcal{U}(\kappa)(\alpha)}(\kappa)$. $C_{\mathcal{U}(\kappa)}$ may be written to indicate $\mathcal{U}$ and κ.

Theorem 5. *Supopse S is a seperating set at α. For any $\beta \in (\alpha, \min(Dom(\mathcal{U}(\kappa)), C(\alpha)))$ there is a T-seperating set at β.*

Proof. As in the proof of theorem 2 let $S_1 = \{\lambda \in \mathrm{Card} \cap \kappa : \exists \eta < \mathrm{Dom}(\mathcal{U}(\lambda))(S \cap \lambda \in \mathcal{U}(\lambda)(\eta))\}$, so that $S_1 \in U_\beta$ iff $\beta > \alpha$, For $\lambda \in S_1$ let $f_\alpha(\lambda)$ be the least η; then for $\beta > \alpha$ the function $f\alpha$ represents α in $\mathrm{Ult}(\kappa)(\beta)$. Choose $g_n u : \kappa \mapsto \kappa$ such that $[g_\beta]_{\mathcal{U}(\kappa)(\alpha)} = \beta$. For $\lambda \in S_1$ let $\tilde{g}(\lambda) = [g \restriction \lambda]_{\mathcal{U}(\lambda)(f_\alpha(\lambda))}$; otherwise let $\tilde{g}(\lambda) = 0$. Let M denote $\mathcal{U}(\kappa)(\beta)$. Then $g \in M$, and one verifies (see [4]) that $[\tilde{g}]_{\mathcal{U}}(\kappa)(\beta) = \beta$. The set $S_1 \cap D^{\geq}_{\tilde{g}}$ is T-seperating at β. ∎

Let $C^{(1)}$ denote the fixed point enumerator of C. If there is an α such that there is no T-seperating set at α let θ denote the smallest such. As in theorem 22 of [3], $C(\theta) = \theta$ and $C^{(1)}(\kappa^+) \le \theta$.

4. Iterating *C*

Let $C^{(\sigma)}$ denote the result of applying the fixed point operator to C σ times. If $\mathrm{Dom}(\mathcal{U}(\kappa)) = \kappa^{++}$ then for $\sigma < \kappa^{++}$ $\mathrm{Ran}(C^{(\sigma)})$ is a club set, and the intersection of all these is empty. If $\mathrm{Dom}(\mathcal{U}(\kappa)) < \kappa^{++}$ then $\mathrm{Dom}(C^{(\sigma)})$ decreases as σ increases, becoming $\emptyset$ for some $\sigma \le \mathrm{Dom}(\mathcal{U}(\kappa))$.

Theorem 6. *Suppose $\alpha < Dom(\mathcal{U}(\kappa))$, f_α represents α on $[0, Dom(\mathcal{U}(\kappa)))$, and $\alpha \in Dom(C^{(1)})$. Then there is a T-seperating set at $C^{(1)}_{\mathcal{U}(\kappa)}(\alpha)$.*

Proof. Let $S = \{\lambda : o(\lambda) \ge C^{(1)}_{\mathcal{U}(\lambda)}(f_\alpha(\lambda))\}$. Then $S \in U_\beta$ iff $\beta \ge C^{(1)}_{\mathcal{U}(\kappa)}(\alpha)$. The condition on λ holds iff either $\exists\eta < \mathrm{Dom}(\mathcal{U}(\lambda))(\eta = C^{(1)}_{\mathcal{U}(\lambda)}(f_\alpha(\lambda)))$, or $\mathrm{Dom}(\mathcal{U}(\lambda))$ is a limit ordinal and $\forall\zeta < f_\alpha(\lambda)\exists\eta < \mathrm{Dom}(\mathcal{U}(\lambda))(\eta = C^{(1)}_{\mathcal{U}(\lambda)}(\zeta))$ and $\forall\zeta < \mathrm{Dom}(\mathcal{U}(\lambda))(C_{\mathcal{U}(\lambda)}(\zeta) < \mathrm{Dom}(\mathcal{U}(\lambda)))$. □

Theorem 7. *Supoose $\alpha < Dom(\mathcal{U}(\kappa))$, there is a T-seperating set at α, and $C^{(1)}_{\mathcal{U}(\kappa)}(\alpha) < Dom(\mathcal{U}(\kappa))$. Then there is a T-seperating set at $C^{(1)}_{\mathcal{U}(\kappa)}(\alpha)$.*

Proof. The proof of theorem 6 may be modified. Let S_1 be a T-seperating set at α. Let f^E_α be an E-representing function for α, as defined in [3]. Let S_2 be the set S, as in the proof of theorem 6, with f^E_α used in place of f_α. Then $S_1 \cap S_2$ is T-seperating at $C^{(1)}(\alpha)$. □

Theorem 8. *Suppose $\alpha < Dom(\mathcal{U}(\kappa))$, f_α represents α on $[0, Dom(\mathcal{U}(\kappa)))$, f_σ represents σ on $[0, Dom(\mathcal{U}(\kappa)))$, and $\alpha \in Dom(C^{(\sigma)})$. Then there is a T-seperating set at $C^{(\sigma)}_{\mathcal{U}(\kappa)}(\alpha)$.*

Proof. Let $S = \{\lambda : o(\lambda) \ge C^{(f_\sigma(\lambda))}_{\mathcal{U}(\lambda)}(f_\alpha(\lambda))\}$. Then $S \in U_\beta$ iff $\beta \ge C^{(\sigma)}_{\mathcal{U}(\kappa)}(\alpha)$. The condition on λ holds iff either $\exists\eta < \mathrm{Dom}(\mathcal{U}(\lambda))(\eta = C^{(\sigma)}_{\mathcal{U}(\lambda)}(f_\alpha(\lambda)))$, or $\mathrm{Dom}(\mathcal{U}(\lambda))$ is a limit ordinal and $\forall\zeta < f_\alpha(\lambda)\exists\eta < \mathrm{Dom}(\mathcal{U}(\lambda))(\eta = C^{(f_\sigma(\lambda))}_{\mathcal{U}(\lambda)}(\zeta))$ and $\forall\zeta < \mathrm{Dom}(\mathcal{U}(\lambda))\forall\tau < f_\sigma(\lambda)(C^{(\tau)}_{\mathcal{U}(\lambda)}(\zeta) < \mathrm{Dom}(\mathcal{U}(\lambda)))$. □

Theorem 9. *Supoose $\alpha < Dom(\mathcal{U}(\kappa))$, there is a T-seperating set at α, there is a T-seperating set at σ, and $\alpha \in Dom(C^{(\sigma)})$. Then there is a T-seperating set at $C^{(\sigma)}_{\mathcal{U}(\kappa)}(\alpha)$.*

Proof. The proof of theorem 8 may be modified. Let $S_{1\alpha}$ be a T-seperating set at α. Let f_α^E be an E-representing function for α. Let $S_{1\sigma}$ be a T-seperating set at σ. Let f_σ^E be an E-representing function for σ. Let S_2 be the set S, as in the proof of theorem 8, with f_α^E used in place of f_α and f_σ^E used in place of f_σ. Then $S_{1\alpha} \cap S_{1\sigma} \cap S_2$ is T-seperating at $C^{(1)}(\alpha)$. □

These methods can be pursued further. Whether there are methods which may be pursued in $L[\mathcal{U}]$ is a question of considerable interest.

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