International Journal of Pure and Applied Mathematics
Volume 115 No. 1 2017, 43-58
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v115i1.4

POSSIBLE PATHS IN AUTOCATALYTIC AND FUZZY AUTOCATALYTIC SETS

Zainab Mahamud[1], Tahir Ahmad[2 §]
[1,2]Department of Mathematical Science
Faculty of Science
Universiti Teknologi Malaysia
81310 Skudai, Johor, MALAYSIA
[2]Centre for Sustainable Nanomaterials
Ibnu Sina Institute for Scientific and Industrial Research
Universiti Teknologi Malaysia
81310 UTM,Skudai, Johor, MALAYSIA

Abstract: Graph theory has been used to model some systems. Graph and fuzzy set have led to the concept of fuzzy graph. An Autocatalytic Set (ACS) is a concept formally introduced in chemistry as a set of catalytically integrated molecules. Fuzzy Autocatalytic Set (FACS) is a merge between fuzzy, graph and autocatalytic set (ACS). In this paper, some new characteristics on autocatalytic and fuzzy autocatalytic sets are presented. These characteristics are on possible paths in autocatalytic and fuzzy autocatalytic sets.

AMS Subject Classification: 03E72, 05C20, 05C38, 05C50
Key Words: graph, fuzzy graph, autocatalytic set, fuzzy autocatalytic set

1. Introduction

Graph has been used to model some of systems [1]. Graph theory is one of mathematical disciplines [2] which provides tool for studying interconnection

Received: December 28, 2016
Revised: April 10, 2017
Published: June 26, 2017

§Correspondence author

among elements in natural and man-made systems [3] and [4]. There are many branches of mathematics that have connections with graph theory such as group theory, matrix theory, probability and topology. Graph has numerous applications such as in systems analysis, operational research, transportation and economics [5].

The definition of a graph is given as below.

Definition 1. [6] (Graph) A graph is a pair of sets (V, E) where V is the set of vertices and E is the set of edges.

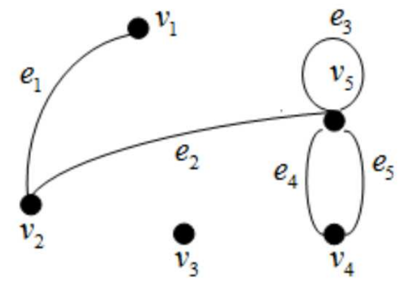


Figure 1: A sample of a graph

Some fundamental concepts for graph are given as below.

Definition 2. [7] (Path) A path in a graph is a walk in which no vertices are repeated.

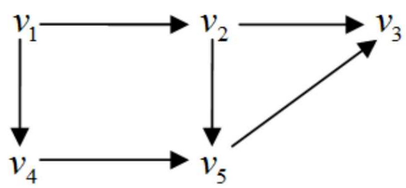


Figure 2

Figure 2 shows the link (v_1, v_3) has three paths namely $(v_1 \to v_2 \to v_3)$, $(v_1 \to v_2 \to v_5 \to v_3)$, $(v_1 \to v_4 \to v_5 \to v_3)$

Definition 3. [7] (Cycles and non-cycles graph) A cycle is a non-trivial closed walk in which no vertices are repeated except the first and last vertices while a non-cycle is a walk with non repeated vertices.

Definition 4. [8] (Directed graph) A directed graph is a pair (V, E) where V is a finite set of nodes and E is a set of edges in form of an ordered pairs (u, v) where $(u, v) \in V \times V$ and $u \neq v$.

Figure 3: (a) Cycle walk (b) Non-cycle walk

Figure 4: (a) Directed graph (b) Undirected graph

Definition 5. [9] (Subgraph) Let H be a graph with vertex set $V(H)$ and edge set $E(H)$, and similarly let G be a graph with vertex set $V(G)$ and edge set $E(G)$.

Then, we say that H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Figure 5: (a) Graph H (b) Graph G

Figure 5 illustrates graph H is a subgraph of graph G. Next section presents some concepts on fuzzy graph.

2. Fuzzy Graph

The concept of fuzzy set was first introduced in 1965 by Professor Lofti Zadeh. He utilized the concept of grade membership value. Fuzzy set has been used to model uncertainties in many systems [10]. Since then, many researchers have worked on properties and applications of fuzzy sets [11].

A fuzzy set is a set where there are some uncertainties in the set [5]. These uncertainties are presented in grade values in the interval [0, 1] while a crisp set is a set where its element is either 0 or 1 only. A crisp set can be treated as a special case of a fuzzy set with its membership value is $\{0, 1\}$.

Further, A. Rosenfeld introduced the concept of fuzzy graph in 1975 as below.

Definition 6. [12] (Fuzzy graph) Fuzzy graph $\mathrm{G} = (\sigma, \mu)$ is a pair of functions $\sigma : \mathrm{S} \to [0,1]$ and $\mu : \mathrm{S} \times \mathrm{S} \to [0,1]$ for $\forall \mathrm{x}, \mathrm{y} \in \mathrm{S}$, we have $\mu(\mathrm{x}, \mathrm{y}) \leq \sigma(\mathrm{x}) \wedge \sigma(\mathrm{y})$.

After that, Yeh and Bang [13] presented some special cases of graph fuzziness in 1997. Blue et.al. [5] introduced the taxonomy of fuzzy graph to generalize various types of fuzziness. Tahir et.al. refined the taxonomy as below.

Definition 7 [11]. A fuzzy graph G_F is a graph which satisfies one of the fuzziness (G_F^i of the i^{th} type) or any of its combination.

Type 1: $G_F^1 = \{G_{1_F}, G_{2_F}, G_{3_F}, ..., G_{n_F}\}$ where fuzziness is on G_{i_F} for $i = 1, 2, 3, \ldots, n$.

Type 2: $G_F^2 = \{V, E_F\}$ where the edge set is fuzzy.

Type 3: $G_F^3 = \{V, E(t_F, h_F)\}$ where both the vertex and edge set are crispy, but the edge has fuzzy head and tail.

Type 4: $G_F^4 = \{V_F, E\}$ where the vertex set is fuzzy.

Type 5: $G_F^5 = \{V, E(w_F)\}$ where both the vertex and edge sets are crisp, but the edges has fuzzy weights.

In the next section, the concept of fuzzy autocatalytic set is presented.

3. Fuzzy Autocatalytic Set

An Autocatalytic Set (ACS) is a concept introduced in 1859 by Darwin to understand the origins of species and Kauffman extended for artificial life in 1991 [14]. The concept of autocatalytic set is formally introduced in 1971 by Kauffman [15] as a set of catalytically integrated molecules. The term autocatalysis refers to a process whereby some entity facilitates the chemical reactions of another [16].

In 1998, Jain and Krishna [17] presented ACS in terms of graph. An autocatalytic set is described by a directed graph with nodes that represent species and the directed links represent catalytic interactions among them. A link from node j to node i indicates that species j is a catalyst for i.

Definition 8 [17]. An autocatalytic set is a subgraph, each of whose nodes has at least one incoming link from a node belonging to the same subgraph.

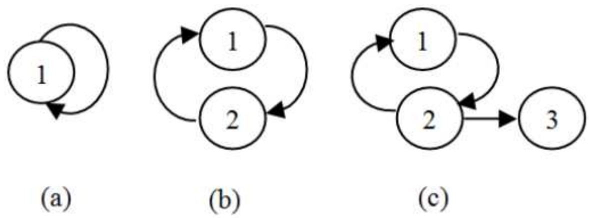


Figure 6: Some ACS

A graph with s nodes is completely specified by an $s \times s$ matrix, $C = (c_{ij})$ which is called the adjacency matrix of the graph. A graph is transformed to a square adjacency matrix, C where its components are given as

$$C_{ij} = \begin{cases} 1 & \text{if} \quad (v_j, v_i) \in E \\ 0 & \text{if} \quad (v_j, v_i) \notin E \end{cases}$$

The merger of fuzzy with autocatalytic set has instigated a new concept namely Fuzzy Autocatalytic Set (FACS). The formal definition of a FACS is given as follow.

Definition 9 [11]. Fuzzy Autocatalytic Set (FACS) is a sub graph each of whose nodes have at least one incoming link with membership value $\mu(e_i) \in (0, 1], \forall e_i \in E$ from a vertices belonging to the same sub graph.

Since then, fuzzy autocatalytic set has been discussed by several researchers [2], [11] and [18]. Next section presents some new characteristics of ACS and FACS.

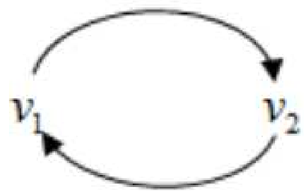
4. Some Characteristics on ACS and FACS

In this section, some new characteristics of ACS and FACS are presented in the form of lemmas, theorem and corollaries.

Lemma 1. *Given $G(V, E)$ is an autocatalytic set and $|V| = n$, then the generated non-cycle paths between two vertices is $n(n-1)$.*

Proof. Let $G(V, E)$ be an autocatalytic set and $V = \{v_1, v_2, v_3, ..., v_n\}$. Therefore, every vertex has incoming link or links and $|V| = n$.

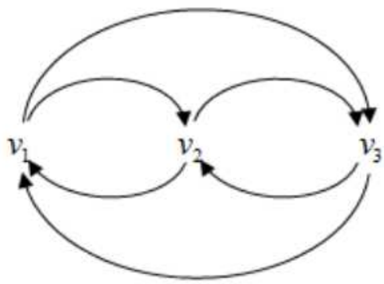
For $n = 2$;



$$\mathrm{E}' = \begin{pmatrix} (v_1, v_2) \\ (v_2, v_1) \end{pmatrix}.$$

The possible paths are $2 \times 1 = 2$:

For $n = 3$;

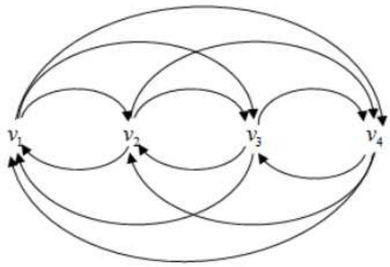


$$\mathrm{E}' = \begin{pmatrix} (v_1, v_2) & (v_1, v_3) \\ (v_2, v_1) & (v_2, v_3) \\ (v_3, v_1) & (v_3, v_2) \end{pmatrix}.$$

The possible paths are $3 \times 2 = 6$:

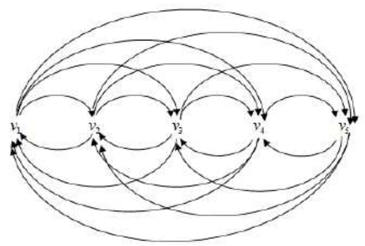
For $n = 4$;

$$\mathrm{E}' = \begin{pmatrix} (v_1, v_2) & (v_1, v_3) & (v_1, v_4) \\ (v_2, v_1) & (v_2, v_3) & (v_2, v_4) \\ (v_3, v_1) & (v_3, v_2) & (v_3, v_4) \\ (v_4, v_1) & (v_4, v_2) & (v_4, v_3) \end{pmatrix}.$$



The possible paths are $4 \times 3 = 12$:

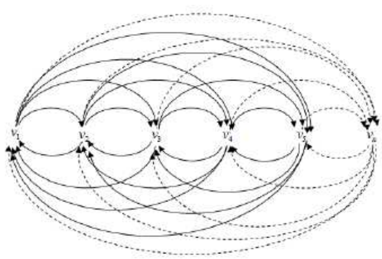
For $n = 5$;



$$\mathrm{E}' = \begin{pmatrix} (v_1, v_2) & (v_1, v_3) & (v_1, v_4) & (v_1, v_5) \\ (v_2, v_1) & (v_2, v_3) & (v_2, v_4) & (v_2, v_5) \\ (v_3, v_1) & (v_3, v_2) & (v_3, v_4) & (v_3, v_5) \\ (v_4, v_1) & (v_4, v_2) & (v_4, v_3) & (v_4, v_5) \\ (v_5, v_1) & (v_5, v_2) & (v_5, v_3) & (v_5, v_4) \end{pmatrix}.$$

The possible paths are $5 \times 4 = 20$.

The possibility of non-cycle paths for n vertices between two vertices is a set



$$\mathrm{E}' = \begin{pmatrix} (v_1, v_2) & (v_1, v_3) & (v_1, v_4) & \cdots & (v_1, v_n) \\ (v_2, v_1) & (v_2, v_3) & (v_2, v_4) & \cdots & (v_2, v_n) \\ (v_3, v_1) & (v_3, v_2) & (v_3, v_4) & \cdots & (v_3, v_n) \\ (v_4, v_1) & (v_4, v_2) & (v_4, v_3) & \cdots & (v_4, v_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (v_n, v_1) & (v_n, v_2) & (v_n, v_3) & \cdots & (v_n, v_{n-1}) \end{pmatrix}.$$

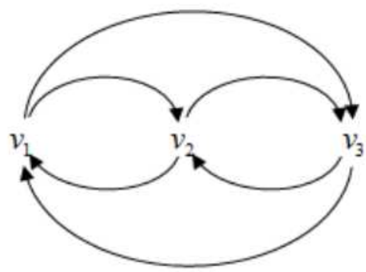
Hence

$$\begin{aligned} |E'| &= n(n-1) \\ &= \frac{n(n-1)(n-2)!}{(n-2)!} \\ &= \frac{n!}{(n-2)!}. \end{aligned}$$

Lemma 2. *Given $G(V,E)$ is an autocatalytic set and $|V| = n$, then the generated non-cycle paths between three vertices is $n(n-1)(n-2)$.*

Proof. Let $G(V,E)$ be an autocatalytic set and $V = \{v_1, v_2, v_3, ..., v_n\}$. Therefore, every vertex has incoming link or links and $|V| = n$.

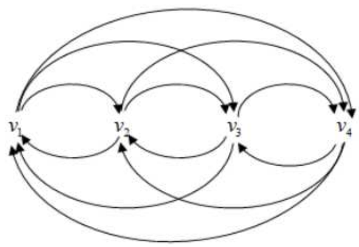
For $n = 3$;



$$\mathrm{E}' = \begin{pmatrix} (v_1, v_3, v_2) & (v_1, v_2, v_3) \\ (v_2, v_3, v_1) & (v_2, v_1, v_3) \\ (v_3, v_2, v_1) & (v_3, v_1, v_2) \end{pmatrix}$$

The possible paths are $3 \times 2 = 6$

For $n = 4$;

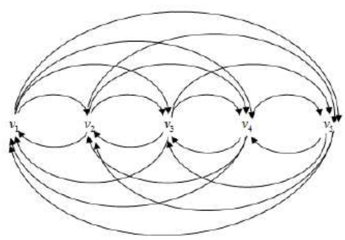


$E' =$

$$\begin{pmatrix} (v_1,v_3,v_2) & (v_1,v_2,v_3) & (v_1,v_2,v_4) \\ (v_2,v_3,v_1) & (v_2,v_1,v_3) & (v_2,v_1,v_4) \\ (v_3,v_2,v_1) & (v_3,v_1,v_2) & (v_3,v_1,v_4) \\ (v_4,v_2,v_1) & (v_4,v_1,v_2) & (v_4,v_1,v_3) \end{pmatrix} \cup$$
$$\begin{pmatrix} (v_1,v_4,v_2) & (v_1,v_4,v_3) & (v_1,v_3,v_4) \\ (v_2,v_4,v_1) & (v_2,v_4,v_3) & (v_2,v_3,v_4) \\ (v_3,v_4,v_1) & (v_3,v_4,v_2) & (v_3,v_2,v_4) \\ (v_4,v_3,v_1) & (v_4,v_3,v_2) & (v_4,v_2,v_3) \end{pmatrix}$$

The possible paths are $4 \times 3 \times 2 = 24$

For $n = 5$;



$E' =$

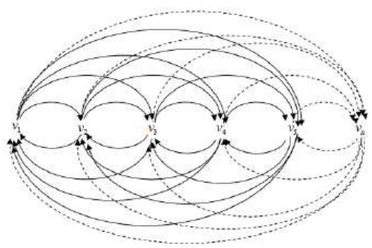
$$\begin{pmatrix} (v_1,v_3,v_2) & (v_1,v_2,v_3) & (v_1,v_2,v_4) & (v_1,v_2,v_5) \\ (v_2,v_3,v_1) & (v_2,v_1,v_3) & (v_2,v_1,v_4) & (v_2,v_1,v_5) \\ (v_3,v_2,v_1) & (v_3,v_1,v_2) & (v_3,v_1,v_4) & (v_3,v_1,v_5) \\ (v_4,v_2,v_1) & (v_4,v_1,v_2) & (v_4,v_1,v_3) & (v_4,v_1,v_5) \\ (v_5,v_2,v_1) & (v_5,v_1,v_2) & (v_5,v_1,v_3) & (v_5,v_1,v_4) \end{pmatrix} \cup$$

$$\begin{pmatrix} (v_1,v_4,v_2) & (v_1,v_4,v_3) & (v_1,v_3,v_4) & (v_1,v_3,v_5) \\ (v_2,v_4,v_1) & (v_2,v_4,v_3) & (v_2,v_3,v_4) & (v_2,v_3,v_5) \\ (v_3,v_4,v_1) & (v_3,v_4,v_2) & (v_3,v_2,v_4) & (v_3,v_2,v_5) \\ (v_4,v_3,v_1) & (v_4,v_3,v_2) & (v_4,v_2,v_3) & (v_4,v_2,v_5) \\ (v_5,v_3,v_1) & (v_5,v_3,v_2) & (v_5,v_2,v_3) & (v_5,v_2,v_4) \end{pmatrix} \cup$$

$$\begin{pmatrix} (v_1,v_5,v_2) & (v_1,v_5,v_3) & (v_1,v_5,v_4) & (v_1,v_4,v_5) \\ (v_2,v_5,v_1) & (v_2,v_5,v_3) & (v_2,v_5,v_4) & (v_2,v_4,v_5) \\ (v_3,v_5,v_1) & (v_3,v_5,v_2) & (v_3,v_5,v_4) & (v_3,v_4,v_5) \\ (v_4,v_5,v_1) & (v_4,v_5,v_2) & (v_4,v_5,v_3) & (v_4,v_3,v_5) \\ (v_5,v_4,v_1) & (v_5,v_4,v_2) & (v_5,v_4,v_3) & (v_5,v_3,v_4) \end{pmatrix}$$

The possible paths are $5 \times 4 \times 3 = 60$

The possibility of non-cycle paths for n vertices of three vertices is a set ;



$$\mathrm{E}' = \begin{pmatrix} (v_1,v_3,v_2) & (v_1,v_2,v_3) & (v_1,v_2,v_4) & (v_1,v_2,v_5) & \cdots & (v_1,v_2,v_n) \\ (v_2,v_3,v_1) & (v_2,v_1,v_3) & (v_2,v_1,v_4) & (v_2,v_1,v_5) & \cdots & (v_2,v_1,v_n) \\ (v_3,v_2,v_1) & (v_3,v_1,v_2) & (v_3,v_1,v_4) & (v_3,v_1,v_5) & \cdots & (v_3,v_1,v_n) \\ (v_4,v_2,v_1) & (v_4,v_1,v_2) & (v_4,v_1,v_3) & (v_4,v_1,v_5) & \cdots & (v_4,v_1,v_n) \\ (v_5,v_2,v_1) & (v_5,v_1,v_2) & (v_5,v_1,v_3) & (v_5,v_1,v_4) & \cdots & (v_5,v_1,v_n) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (v_n,v_2,v_1) & (v_n,v_1,v_2) & (v_n,v_1,v_3) & (v_n,v_1,v_4) & \cdots & (v_n,v_1,v_{n-1}) \end{pmatrix} \cup$$

$$\begin{pmatrix} (v_1,v_4,v_2) & (v_1,v_4,v_3) & (v_1,v_3,v_4) & (v_1,v_3,v_5) & \cdots & (v_1,v_3,v_n) \\ (v_2,v_4,v_1) & (v_2,v_4,v_3) & (v_2,v_3,v_4) & (v_2,v_3,v_5) & \cdots & (v_2,v_3,v_n) \\ (v_3,v_4,v_1) & (v_3,v_4,v_2) & (v_3,v_2,v_4) & (v_3,v_2,v_5) & \cdots & (v_3,v_2,v_n) \\ (v_4,v_3,v_1) & (v_4,v_3,v_2) & (v_4,v_2,v_3) & (v_4,v_2,v_5) & \cdots & (v_4,v_2,v_n) \\ (v_5,v_3,v_1) & (v_5,v_3,v_2) & (v_5,v_2,v_3) & (v_5,v_2,v_4) & \cdots & (v_5,v_2,v_n) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (v_n,v_3,v_1) & (v_n,v_3,v_2) & (v_n,v_2,v_3) & (v_n,v_2,v_4) & \cdots & (v_n,v_2,v_{n-1}) \end{pmatrix} \cup$$

$$\begin{pmatrix} (v_1,v_5,v_2) & (v_1,v_5,v_3) & (v_1,v_5,v_4) & (v_1,v_4,v_5) & \cdots & (v_1,v_4,v_n) \\ (v_2,v_5,v_1) & (v_2,v_5,v_3) & (v_2,v_5,v_4) & (v_2,v_4,v_5) & \cdots & (v_2,v_4,v_n) \\ (v_3,v_5,v_1) & (v_3,v_5,v_2) & (v_3,v_5,v_4) & (v_3,v_4,v_5) & \cdots & (v_3,v_4,v_n) \\ (v_4,v_5,v_1) & (v_4,v_5,v_2) & (v_4,v_5,v_3) & (v_4,v_3,v_5) & \cdots & (v_4,v_3,v_n) \\ (v_5,v_4,v_1) & (v_5,v_4,v_2) & (v_5,v_4,v_3) & (v_5,v_3,v_4) & \cdots & (v_5,v_3,v_n) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (v_n,v_4,v_1) & (v_n,v_4,v_2) & (v_n,v_4,v_3) & (v_n,v_3,v_4) & \cdots & (v_n,v_3,v_{n-1}) \end{pmatrix} \cup$$

$$\cdots \cup$$

$$\begin{pmatrix} (v_1,v_n,v_2) & (v_1,v_n,v_3) & (v_1,v_n,v_4) & (v_1,v_n,v_5) & \cdots & (v_1,v_{n-1},v_n) \\ (v_2,v_n,v_1) & (v_2,v_n,v_3) & (v_2,v_n,v_4) & (v_2,v_n,v_5) & \cdots & (v_2,v_{n-1},v_n) \\ (v_3,v_n,v_1) & (v_3,v_n,v_2) & (v_3,v_n,v_4) & (v_3,v_n,v_5) & \cdots & (v_3,v_{n-1},v_n) \\ (v_4,v_n,v_1) & (v_4,v_n,v_2) & (v_4,v_n,v_3) & (v_4,v_n,v_5) & \cdots & (v_4,v_{n-1},v_n) \\ (v_5,v_n,v_1) & (v_5,v_n,v_2) & (v_5,v_n,v_3) & (v_5,v_n,v_4) & \cdots & (v_5,v_{n-1},v_n) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (v_n,v_{n-1},v_1) & (v_n,v_{n-1},v_2) & (v_n,v_{n-1},v_3) & (v_n,v_{n-1},v_4) & \cdots & (v_n,v_{n-2},v_{n-1}) \end{pmatrix}.$$

Hence,

$$\begin{aligned} |E^{\square}| &= n(n-1)(n-2) \\ &= \frac{n(n-1)(n-2)(n-3)!}{(n-3)!} \\ &= \frac{n!}{(n-3)!}. \end{aligned}$$

This can be formalized as for n vertices, the generated non-cycle paths between k vertices is given as $\frac{n!}{(n-k)!}$. Therefore, generated non-cycle paths between n vertices of ACS is $n!$.

Lemma 3. *Given $G(V,E)$ is an autocatalytic set and $|V| = n$, then the generated non-cycle paths between n vertices is $n!$.*

Proof. By mathematical induction.

(i) Let $G(V,E)$ be an autocatalytic set and $V = \{v_1, v_2\}$. Therefore G forms an edge with two vertices such that $\{v_1, v_2\} \in V$. The probability of edges for G is a set

$$\text{E}(2) = \begin{pmatrix} (v_1, v_2) \\ (v_2, v_1) \end{pmatrix}$$

$|E(2)| = 2 = 2!$.

(ii) Assume $G(V,E)$ is an autocatalytic set with n vertices and $|E(n)| = n!$. Therefore, if $G^{'}(V,E)$ is an autocatalytic set with $n+1$ vertices, then

$$|E^{'}(n+1)| = |E^{'}\{v_1, v_2, v_3, v_n, ..., v_{n+1}\}|.$$

Since v_{n+1}can be arranged in the form of $(v_{n+1}, v_1, v_2, v_3, ..., v_n)$ or $(v_1, v_{n+1}, v_2, v_3, ..., v_n)$ or $(v_1, v_2, v_{n+1}, v_3, ..., v_n)$ until $(v_1, v_2, v_3, ..., v_n, v_{n+1})$.

Then

$$\begin{aligned} |E^{\prime}(n+1)| &= (n+1)|E^{\prime}(\{v_1, v_2, v_3, ..., v_n\})| \\ &= (n+1)n! \\ &= (n+1)!. \end{aligned}$$

Hence, the generated non-cycle paths between n vertices is given as $n!$. □

An inequality $|E(n)| \leq \sum_{k=2}^{n} \frac{n!}{(n-k)!}$ is formulated by using lemma 1, 2 and 3. The formal statement of this result is presented in the following theorem.

Theorem 1. *If $G(V,E)$ be an autocatalytic set, then the number of non-cycle path is at most $\sum_{k=2}^{n} \frac{n!}{(n-k)!}$.*

Proof. By mathematical induction.

(i) Let $G(V,E)$ be an autocatalytic set and $V = \{v_1, v_2\}$. The probability numbers of edges for G is a set

$$\text{E(2)} = \begin{pmatrix} (v_1, v_2) \\ (v_2, v_1) \end{pmatrix}$$

$|E(2)| = 2 = \frac{2!}{(2-2)!}$

(ii) Assume $G(V,E)$ is an autocatalytic set with n vertices and

$$|E(n)| \leq \sum_{k=2}^{n} \frac{n!}{(n-k)!}.$$

Therefore, for $G^{'}(V,E)$ with $n+1$ vertices,

$$|E^{'}(n+1)| = |E^{'}\{v_1, v_2, v_3, v_n, ..., v_{n+1}\}|.$$

Since v_{n+1} can be arranged in the form of $(v_{n+1}, v_1, v_2, v_3, ..., v_n)$ or $(v_1, v_{n+1}, v_2, v_3, ..., v_n)$ or $(v_1, v_2, v_{n+1}, v_3, ..., v_n)$ until $(v_1, v_2, v_3, ..., v_n, v_{n+1})$ and v_{n+1} can have non-cycle paths such that

$$\begin{aligned}|E^{\square}(n+1)| &= (n+1)|E^{\square}(\{v_1, v_2, v_3, ..., v_n\})| + (n+1)(n)\\ &\leq (n+1)\sum_{k=2}^{n}\frac{n!}{(n-k)!} + (n+1)(n)\\ &= (n+1)[n(n-1) + n(n-1)(n-2) + n(n-1)(n-2)(n-3)\\ &\quad + ... + (n(n-1)(n-2)(n-3)...6(5)(4)) +\\ &\quad (n(n-1)(n-2)(n-3)...5(4)(3)) + (n(n-1)(n-2)(n-3)\\ &\quad ...4(3)(2)) + (n(n-1)(n-2)(n-3)...3(2)(1))] + (n+1)(n)\end{aligned}$$

By switching expression $(n+1)(n)$ to the front, then

$$\begin{aligned}&= (n+1)(n) + (n+1)[n(n-1) + n(n-1)(n-2) +\\ &\quad n(n-1)(n-2)(n-3) + ... + (n(n-1)(n-2)(n-3)...6(5)(4))\\ &\quad + (n(n-1)(n-2)(n-3)...5(4)(3)) + (n(n-1)(n-2)(n-3)\\ &\quad ...4(3)(2)) + (n(n-1)(n-2)(n-3)...3(2)(1))]\\ &= \frac{(n+1)(n)(n-1)!}{(n-1)!} + \frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!} +\\ &\quad \frac{(n+1)(n)(n-1)(n-2)(n-3)!}{(n-3)!} + ... +\\ &\quad \frac{(n+1)(n)(n-1)(n-2)(n-3)...3(2)(1)}{3(2)(1)} +\\ &\quad \frac{(n+1)(n)(n-1)(n-2)(n-3)...3(2)(1)}{2(1)} +\\ &\quad \frac{(n+1)(n)(n-1)(n-2)(n-3)...3(2)(1)}{1!} +\\ &\quad \frac{(n+1)(n)(n-1)(n-2)(n-3)...3(2)(1)}{0!}\\ &= \frac{(n+1)!}{((n+1)-2)!} + \frac{(n+1)!}{((n+1)-3)!} + \frac{(n+1)!}{((n+1)-4)!} + ... +\\ &\quad \frac{(n+1)!}{((n+1)-(n-2))!} + \frac{(n+1)!}{((n+1)-(n-1))!} +\end{aligned}$$

$$\frac{(n+1)!}{((n+1)-(n))!}+\frac{(n+1)}{((n+1)-(n+1))!}$$
$$=\sum_{k=2}^{n+1}\frac{(n+1)!}{((n+1)-k)!}.$$

Hence, any autocatalytic set with n vertices has at most $\sum_{k=2}^{n}\frac{n!}{(n-k)!}$ paths. □

Corollary 1. *Let $G(V,E)$ be an autocatalytic set with $|V|=n$ and $H(V_H,E_H)$ is a proper subgraph of G, then*

$$|E_H(n)|<|E(n)|\leq\sum_{k=2}^{n}\frac{n!}{(n-k)!},$$

where $|E(n)|$ is the number of non-cycle paths.

Proof. Let G(V,E) be an autocatalytic set with $|V|=n$. Supposed $H(V_H,E_H)$ is a proper subgraph of G. By Theorem 1

$$|E_H(n)|<|E(n)|\leq\sum_{k=2}^{n}\frac{n!}{(n-k)!}.$$

□

Corollary 2. *If $G_F(V_H,E_H)$ is a fuzzy autocatalytic set, then*

$$|E_F(n)|\leq\sum_{k=2}^{n}\frac{n!}{(n-k)!},$$

where $E_F(n)$ is a set of non-cycle paths with n vertices.

Proof. Let $G_F(V_H,E_H)$ be a fuzzy autocatalytic set with $|V_F|=n$. Then, by using similar argument as in Theorem 1

$$|E_F(n)|\leq\sum_{k=2}^{n}\frac{n!}{(n-k)!},$$

where $E_F(n)$ is a set of non-cycle paths with n vertices. □

Corollary 3. *Let $G_F(V_H,E_H)$ be a fuzzy autocatalytic set with $|V_F|=n$ and $H(V_H^{'},E_H^{'})$ is a proper subgraph of G, then*

$$|E_F^{'}(n)|\leq\sum_{k=2}^{n}\frac{n!}{(n-k)!}.$$

Proof. Let $G_F(V_H, E_H)$ be a fuzzy autocatalytic set with $|V_F| = n$. Since H_F is a proper fuzzy subgraph of G_F, then by Corollary 2

$$|E_F^{'}(n)| < |E_F(n)| \leq \sum_{k=2}^{n} \frac{n!}{(n-k)!}.$$

□

5. Conclusion

We have shown that ACS and FACS with finite vertices have at most $\sum_{k=2}^{n} \frac{n!}{(n-k)!}$ generated non-cycle paths.

6. Acknowledgments

This work has been supported by FRGS Grant, Vote No 4F756.

References

[1] R.K. Balakrishnan Ranganathan, *A textbook of Graph Theory*, Springer Science and Business Media, New York, 2012.

[2] A. Azmirul and A. Tahir, On fuzzy autocatalytic set, *International Journal of Pure and Applied Mathematics*, **107**, No. 1 (2016), 59-68, **doi:** org/10.12732/ijpam.v107i1.6.

[3] Noor H. Ainy, A. Tahir and I. Razidah, Dynamic graph model of evaporation process in a boiler system, *Proceedings of the 2014 International Conference on Mathematical Methods, Mathematical Models and Simulation in Science and Engineering* (2014), 152-155.

[4] H. Noor Ainy, I. Razidah and A. Tahir, Transformation of fuzzy state space model of a boiler system: A graph theoretic approach, *World Scientific and Engineering Academy and Society (WSEAS) Transactions on Mathematics*, **9**, No. 9 (2010), 669-678.

[5] M. Blue, B. Bush and J. Puckett, *Applications of Fuzzy Logic to Graph Theory*, Energy and Environmental Analysis GroupTechnology and Safety Assessment Division, Los Alamos National Laboratory, 1997.

[6] R. Keijo, *Graph Theory*, Tampere University of Technology, Translation by Janne Tamminen, Kung-Chung Lee and Robert Piché, 2013.

[7] K. Nur Syahidah, *Matroid Structure of Dynamic Graph Model of Evaporation Process in a Boiler System*, Tampere University of Technology, Ijazah Sarjana, Universiti Teknologi Malaysia (UTM), 2015.

[8] F. Harary and G. Gupta, Dynamic graph models, *Mathematical and Computer Modelling*, **25** (1996), 79-87.

[9] S. Saha Ray, *Graph Theory with Algorithms and its Applications*, Springer, India, 2013.

[10] Q.O. Umilkeram and A. Tahir, Some properties of fuzzy quasi-metric space for FACS of fuzzy graph Type-3, *Malaysia Journal of Fundamental and Applied Sciences*, **11**, No. 3 (2015), 92-94.

[11] A. Tahir, B. Sabariah and A.A. Khairil, Modeling a clinical incineration process using fuzzy autocatalytic set, *Journal of Mathematical Chemistry*, **47** (2010), 1263-1273, **doi:** 10.1007/s10910-009-9650-1.

[12] A. Rosenfeid, Fuzzy sets and their applications to cognitive and decision processes, In: *Fuzzy Graphs* (Ed-s: L.A. Zadeh, K.S.M. Fu Shimura), Academic Press, New York (1975).

[13] R.T. Yeh, and S.Y. Bang, Fuzzy graphs and their applications to clustering analysis, In: *Fuzzy Relations* (Ed-s: L.A. Zadeh, K.S. Fu, M. Shimura), Academic Press, New York (1975).

[14] A.B. Sumarni, I. Razidah, and Noor H. Ainy, Graph dynamics representation of chemical reactions of a boiler, *IEEE Business Engineering and Industrial Applications Colloquium (BEIAC)* (2013), 906-910, **doi:** 10.1109/BEIAC.2013.6560268.

[15] S.A. Kauffman, Cellular Homeostasis, Epigenesist and replication in randomly aggregated macromolecular systems, *Journal of Cybernetics*, **1** (1971), 71-96, **doi:** 10.1080/01969727108545830.

[16] W. Hordijk, R.W. Peter and S. Mike, Autocatalytic sets and biological specificity, *Society for Mathematical Biology*, **76**, No. 1 (2013), 201-224, **doi:** 10.1007/s11538-013-9916-4.

[17] S. Jain, and S. Krishna, Autocatalytic sets and the growth of complexity in an evolutionary model, *Physical Review Letters*, **81**, No. 25 (1998), 5684-5687, **doi:** 10.1103/PhysRevLett.81.5684.

[18] A.B. Sumarni, A. Tahir, and B. Sabariah, Evaluation of steady state vector of fuzzy autocatalytic set of fuzzy graph Type-3 of an incineration process, *World Scientific and Engineering Academy and Society (WSEAS) Transactions on Mathematics*, **9**, No. 8 (2010), 658-667.