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CERTAIN GENERATING FUNCTIONS OF GENERALISED HYPERGEOMETRIC 2D POLYNOMIALS FROM LIE-GROUP THEORETIC POINT OF VIEW

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Abstract: In this paper, we have obtained some novel generating functions of generalized hypergeometric $2D$ polynomials $(GH2DP)$ $U_{\mathsf{n}}(\beta;\gamma;x,y)$ by group theoretic method introduced by Louis Weisner.A suitable interpretation to the index n and parametres β,γ are given,we introduced five linear partial differential operators wich they generate a Lie-algebra. Futher, we have derived the well known generating functions of Laguerre polynomialof two variables as an application.

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1. Introduction

Generating functions play a very vital role in the investigation of various useful properties of the functions which they generate. The name generating function was introduced by Laplace in 1812 and since then it has been developed into various directions and found wide applications in different branches of science

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and technology. A generating function may be used to define a set of functions, to determine recurrence relation or to evaluate certain integrals etc.

The object of the present paper is to derive some generating functions, which are to believed to be new, of Generalised Hypergeometric $2D$ polynomials $(GH2DP)$, by suitably interpreting to the index n and the parameters β, γ simultaneously with the help of Weisner's group theoretic method. It may be mentioned that in the course of constructing a five dimensional Lie algebra we have obtained two operators such that, when operated on the polynomial under consideration, raise(lower) and lower(raise) the index and the parameter by one unit.

Many authors obtained generating functions for various orthogonal polynomials in the theory of special functions.

The generalized hypergeometric $2D$ polynomials $(GH2DP)U_n(\beta;\gamma;x,y)$ satisfies the following ordinary differential equation

$$y(x-y)D^2U_n(x,y) - [(n+\beta-1)x - (\gamma+2n-2)y]DU_n(x,y) - n(\gamma+n-1)U_n(x,y) = 0, \quad (1.1)$$

where $D = \dfrac{d}{dy}$ and $U_n(x,y) = U_n(\beta;\gamma;x,y)$.

2. Group Theoretic Discussion and Lie-Algebra

In order to use Weisner's method, we construct from (1.1) the following partial differential equation by replacing $\dfrac{d}{dy}$ by $\dfrac{\partial}{\partial y}$, n by $z\dfrac{\partial}{\partial z}$, β by $t\dfrac{\partial}{\partial t}$, γ by $s\dfrac{\partial}{\partial s}$ and $U_n(x,y)$ by $u(x,y,z,t,s)$:

$$\left[y(x-y)\frac{\partial^2}{\partial y^2} - z^2\frac{\partial^2}{\partial z^2} + (2y-x)z\frac{\partial^2}{\partial y\partial z} - tx\frac{\partial^2}{\partial y\partial t} + sy\frac{\partial^2}{\partial y\partial s} - sz\frac{\partial^2}{\partial z\partial s} + (x-2y)\frac{\partial}{\partial y}\right]u(x,y,z,t,s) = 0, \quad (2.1)$$

where

$$u(x,y,z,t,s) = U_n(\beta;\gamma;x,y)z^n t^\beta s^\gamma$$

is a solution of (2.1), since $U_n(\beta;\gamma;x,y)$ is a solution of (1.1).

Now by following differential recurrence relations:

$$DU_n(\beta;\gamma;x,y) = \frac{n}{y}U_n(\beta;\gamma;x,y) + \frac{n\beta x}{\gamma y}U_n - 1(\beta+1;\gamma+1;x,y), \quad (2.2)$$

$$\begin{aligned} DU_n(\beta;\gamma;x,y) =& \frac{(\beta-1)x-(\gamma+n-1)y}{y(x-y)}U_n(\beta;\gamma;x,y) \\ &+ \frac{(\gamma-1)}{y(x-y)}U_{n+1}(\beta-1;\gamma-1;x,y), \end{aligned} \tag{2.3}$$

where$D = \dfrac{d}{dy}$.

Let us introduce a set of infinitesimal partial differential operators,A_i,$i = 1,2,3,4,5$ as follows:

$$\begin{aligned} A_1 =& z\frac{\partial}{\partial z}, \\ A_2 =& t\frac{\partial}{\partial t}, \\ A_3 =& s\frac{\partial}{\partial s}, \\ A_4 =& \frac{yts}{xz}\frac{\partial}{\partial y} - \frac{ts}{x}\frac{\partial}{\partial z}, \\ A_5 =& \frac{zy(x-y)}{ts}\frac{\partial}{\partial y} + \frac{z^2y}{ts}\frac{\partial}{\partial z} - \frac{zx}{s}\frac{\partial}{\partial t} + \frac{yz}{t}\frac{\partial}{\partial s} - \frac{z(y-x)}{ts}. \end{aligned} \tag{2.4}$$

Then:

$$\begin{aligned} A_1\left[z^nt^\beta s^\gamma U_n(\beta;\gamma;x,y)\right] =& nz^nt^\beta s^\gamma U_n(\beta;\gamma;x,y), \\ A_2\left[z^nt^\beta s^\gamma U_n(\beta;\gamma;x,y)\right] =& \beta z^nt^\beta s^\gamma U_n(\beta;\gamma;x,y), \\ A_3\left[z^nt^\beta s^\gamma U_n(\beta;\gamma;x,y)\right] =& \gamma z^nt^\beta s^\gamma U_n(\beta;\gamma;x,y), \\ A_4\left[z^nt^\beta s^\gamma U_n(\beta;\gamma;x,y)\right] =& \frac{n\beta}{\gamma}z^{n-1}t^{\beta+1}s^{\gamma+1}U_{n-1}(\beta+1;\gamma+1;x,y), \\ A_5\left[z^nt^\beta s^\gamma U_n(\beta;\gamma;x,y)\right] =& (\gamma-1)z^{n+1}t^{\beta-1}s^{\gamma-1}U_{n+1}(\beta-1;\gamma-1;x,y). \end{aligned} \tag{2.5}$$

We now proceed to find the commutator relations satisfied by A_i, $i = 1,2,3,4,5$. Using the notation

$$[A,B]\,u = (AB-BA)u,$$

we receive the following commutator relations:

$$\begin{aligned} &[A_1,A_2] = 0, \quad [A_2,A_3] = 0, \quad [A_3,A_4] = A_4, \\ &[A_1,A_3] = 0, \quad [A_2,A_4] = 0, \quad [A_3,A_5] = -A_5, \\ &[A_1,A_4] = -A_4, \quad [A_2,A_5] = -A_5, \quad [A_3,A_4] = (-A_2+A_3-1), \\ &[A_1,A_5] = A_5. \end{aligned} \tag{2.6}$$

From the above commutator relations, we state the following theorem:

Theorem. *The set of operators* $A_i, i = 1, 2, 3, 4, 5$ *generates a five dimensional Lie algebra L.*

It can be shown that the partial differential operator L, given by

$$L = y(x-y)\frac{\partial^2}{\partial y^2} - z^2\frac{\partial^2}{\partial z^2} + (2y-x)z\frac{\partial^2}{\partial y\partial z} - tx\frac{\partial^2}{\partial y\partial t} + sy\frac{\partial^2}{\partial y\partial s} - sz\frac{\partial^2}{\partial z\partial s} + (x-2y)\frac{\partial}{\partial y},$$

can be expressed as follows:

$$x^{-1}yL = A_5A_4 + A_1A_2. \tag{2.7}$$

It can easily verified that the operators A_i, $i = 1, 2, 3, 4, 5$ commute with $x^{-1}yL$, i.e.

$$\left[x^{-1}yL, A_i\right] = 0, \quad i = 1, 2, 3, 4, 5.$$

Therefore the extended form of the group generated by the each of the operators A_i, $i = 1, 2, 3, 4, 5.$ are

$$e^{a_1A_1}f(x,y,z,t,s) = f(x,y,e^{a_1}z,t,s), \tag{2.9}$$
$$e^{a_2A_2}f(x,y,z,t,s) = f(x,y,z,e^{a_2}t,s), \tag{2.10}$$
$$e^{a_3A_3}f(x,y,z,t,s) = f(x,y,z,t,e^{a_3}s), \tag{2.11}$$
$$e^{a_4A_4}f(x,y,z,t,s) = f\left(x, \frac{xyz}{xz-a_4ts}, \frac{xz-a_4ts}{x}, t, s\right)), \tag{2.12}$$
$$e^{a_5A_5}f(x,y,z,t,s) = \frac{ts}{(y-x)za_5+st} \times f\left(x, \frac{yst}{(y-x)za_5+st}, \frac{z[(y-x)za_5+st]}{st-xza_5}, \frac{st-xza_5}{s}, \frac{s[(y-x)za_5+st]}{st-xza_5}\right), \tag{2.13}$$

where all a_i $(i = 1, 2, 3, 4, 5)$ are arbitrary constants and $f(x, y, z, t, s)$ is arbitrary function.

Hence

$$e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}f(x,y,z,t,s)$$

$$= \frac{ts}{(y-x)za_5 + st} f(x, \xi, \eta, \theta, \rho), \quad (2.14)$$

where:

$$\begin{aligned}
\xi =& \frac{xyzts}{[(y-x)z_a 5 + st][xz - a_4(st - xza_5)]}, \\
\eta =& \frac{e^{a_1}[(y-x)z_a 5 + st][xz - a_4(st - xza_5)]}{x(st - xza_5)}, \\
\theta =& \frac{e^{a_2} st - xza_5}{s}, \\
\rho =& \frac{e^{a_3} s[(y-x)z_a 5 + st]}{(st - xza_5)}.
\end{aligned}$$

3. Generating Functions

We receive, from (2.1), that $f(x, y, z, t, s) = z^n t^\beta s^\gamma U_n(\beta; \gamma; x, y)$ is a solution of the system:

$$\begin{cases} Lu = 0 \\ (A_1 - n)u = 0 \end{cases} \quad \begin{cases} Lu = 0 \\ (A_2 - \beta)u = 0 \end{cases} \quad \begin{cases} Lu = 0 \\ (A_3 - \gamma)u = 0 \end{cases}$$

$$\begin{cases} Lu = 0 \\ (A_1 + A_2 + A_3 - n - \beta - \gamma)u = 0 \end{cases}$$

From (2.8), we receive

$$S(x^{-1}yL)[z^\nu t^\beta s^\gamma U_\nu(\beta; \gamma; x, y)] = (x^{-1}yL)S[z^\nu t^\beta s^\gamma U_\nu(\beta; \gamma; x, y)],$$

where

$$S = e^{a_5 A_5} e^{a_4 A_4} e^{a_3 A_3} e^{a_2 A_2} e^{a_1 A_1}.$$

Hence the transformation $S[z^\nu t^\beta s^\gamma U_\nu(\beta; \gamma; x, y)]$ is also annulled by $(x^{-1}yL)$.

Choosing $a_i = 0,\ i = 1, 2, 3,\ a_4 = b,\ a_5 = c$, and writing $f(x, y, z, t, s)$ as $[z^\nu t^\beta s^\gamma U_\nu(\beta; \gamma; x, y)]$, i n(2.14), we obtain

$$\begin{aligned}
e^{cA_5} e^{bA_4} \left[z^\nu t^\beta s^\gamma U_\nu(\beta; \gamma; x, y) \right] =& \frac{ts}{(y-x)zc + st} \\
& \times \left[\frac{[(y-x)zc + st][xz - b(st - xzc)]}{[xz(st - xzc)]} \right]^\nu
\end{aligned}$$

$$\times \left[1 - \frac{xzc}{st}\right]^{\beta} \left[\frac{(y-x)zc + st}{(st - xzc)}\right]^{\gamma}$$
$$\times U_{\nu}\left[\beta; \gamma; x, \frac{xyzts}{[(y-x)zc + st][xz - b(st - xzc)}\right]$$
$$\times z^{\nu} t^{\beta} s^{\gamma}. \tag{3.1}$$

On the other hand

$$e^{cA_5} e^{bA_4}[z^{\nu} t^{\beta} s^{\gamma} U_{\nu}(\beta; \gamma; x, y)] = \sum_{l=0}^{\infty} \sum_{K=0}^{\infty} \frac{c^k A_5^k}{k!} \frac{b^l A_4^l}{l!} [U_{\nu}(\beta; \gamma; x, y) z^{\nu} t^{\beta} s^{\gamma}]$$
$$= \sum_{l=0}^{\infty} \sum_{K=0}^{\infty} \frac{c^k}{k!} \frac{b^l}{l!} (\frac{\nu\beta}{\gamma})_l (\gamma - 1 - l)_k (z^{-1} ts)^{l-k}$$
$$\times \left[U_{\nu-l+k}(\beta + l - k; \gamma + l - k; x, y) z^{\nu} t^{\beta} s^{\gamma}\right]. \tag{3.2}$$

Equating the expressios (3.1)and(3.2), we get

$$\frac{ts}{(y-x)zc + st} \left[\frac{[(y-x)zc + st][xz - b(st - xzc)]}{[xz(st - xzc)]}\right]^n$$
$$\left[1 - \frac{xzc}{st}\right]^{\beta} \left[\frac{(y-x)zc + st}{(st - xzc)}\right]^{\gamma}$$
$$U_n\left[\beta; \gamma; x, \frac{xyzts}{[(y-x)zc + st][xz - b(st - xzc)}\right] z^n t^{\beta} s^{\gamma}$$
$$= \sum_{l=0}^{\infty} \sum_{K=0}^{\infty} \frac{c^k}{k!} \frac{b^l}{l!} \left(\frac{\nu\beta}{\gamma}\right)_l (\gamma - l)_k (z^{-1} ts)^{l-k}$$
$$\left[U_{\nu-l+k}(\beta + l - k; \gamma + l - k; x, y) z^{\nu} t^{\beta} s^{\gamma}\right]. \tag{3.3}$$

This may be regarded as a new generating relation which in turn yields a good number of particular new/known generating relationby attributing different values of a_i, $i = 1, 2, 3, 4, 5$.

4. Particular Cases

1. when $b = 1$ and $c = 0$, replacing $z^{-1}ts$ by w we have

$$\left(\frac{x-w}{x}\right)^{\nu} U_{\nu}\left(\beta; \gamma; x, \frac{xy}{x-w}\right)$$

$$= \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{\nu\beta}{\gamma} \right) U_{\nu - l} \left(\beta + l; \gamma + l; x, y \right) w^{l}. \quad (4.1)$$

2. When $b = 0$ and $c = 1$, replacing $zt^{-1}s^{-1}$ by w, we get

$$[1 - (x - y)w]^{\nu+\gamma-1}(1 - wx)^{\beta-\gamma-\nu} U_{\nu} \left(\beta; \gamma; x, \frac{y}{1 - w(x - y)} \right)$$
$$= \sum_{l=0}^{\infty} \frac{1}{k!} (\gamma)_k U_{\nu+k} \left(\beta - k; \gamma - k; x, y \right) w^{k}. \quad (4.2)$$

5. Applications

From the relations (3.1) and (3.2), we can derive the following generating functions for Lagaurre polynomials of two variables:

$$\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} L_{\nu-l}^{(\alpha+l)}(x, y) w^l = L_{\nu}^{(\alpha)}(x + w, y),$$

$$\sum_{K=0}^{\infty} \frac{(-1)^k (1 + \alpha)_k}{k!} L_{\nu+k}^{(\alpha-k)}(x, y) w^k$$
$$= (1 + w(x - y))^{\alpha} exp(-wx) L_{\nu}^{(\alpha)}[x + wx(x - y); y].$$

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