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DYNAMICS OF LESLIE-GOWER PREDATOR-PREY MODEL WITH ADDITIONAL FOOD FOR PREDATORS

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Abstract: We consider a Leslie-Gower predator prey model with additional food for predators. Here we investigate the dynamics of the model such as the permanence, determination of equilibrium points and their existence condition as well as their stability properties. It is shown that the model is permanence and has four equilibrium points, i.e., the extinction of both prey and predator point, the extinction of prey point, the extinction of predator and the coexistence point. The point of prey extinction and the coexistence point are conditionally stable while two other equilibrium points are always unstable. It is also shown that the additional food for predators may destabilize the extinction of prey point and at the same time stabilize the coexistence point. Such dynamical behavior agrees with our numerical results.

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Key Words: predator-prey model, Leslie-Gower model, permanence, additional food

1. Introduction

The interactions between prey population and predator population in an environment is often modeled by Lotka-Volterra [1]. In its development, Leslie [2] has modified the Lotka-Volterra predator-prey model into Leslie-Gower. The response function in this model is Holling type II, where the predation rate is not only dependent on prey population, but also on the amount of environmental protection. Aziz-Alaoui and Okiye [3] discussed the Leslie-Gower

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predator-prey model as stated below

$$\begin{aligned}\frac{dX}{dt} &= (1-X)X - \frac{\delta XY}{X+m},\\ \frac{dY}{dt} &= \beta\left(1-\frac{Y}{X+e}\right)Y,\end{aligned}$$

where $X = X(t)$ and $Y = Y(t)$ respectively represent prey and predator populations at time t.

The nature of actively moving predators becomes a reason that they can target on more than one prey or switch to other food sources. Considering the limited prey population will require additional food for predators, Srinivasu [4] has studied a Lotka-Volterra predator-prey model with additional food for predators. Later, Sen [5] developed Srinivasu's research [4] in terms of the handling time and the nutritional value of additional food. Many research on Lotka-Volterra predator-prey model with additional food have been conducted [6], [7]. According to Srinivasu [4], the additional food for predator results in a reduced predation target of predator to prey, so it helps the survival of prey population without disturbing predator's growth rate.

In this article, the Leslie-Gower predator-prey model [2] is reconsidered and modified by including the additional food to predators in terms of the handling time and the nutritional value of additional food. The Leslie-Gower predator-prey model with additional food for predators is stated as

$$\begin{aligned}\frac{dX}{dt} &=(1-X)X - \frac{\delta XY}{X+m+nA},\\ \frac{dY}{dt} &=\beta\left(1-\frac{Y}{X+e}\right)Y + \frac{\sigma nAY}{X+m+nA},\end{aligned} \tag{1}$$

where δ, m, n, A, β, e and σ are positive parameters. As mentioned by Prasad et. al. [6], the term nA represents quantity of additional food perceptible to the predator relative to prey.

2. Permanence

In this section we show the permanence of system (1) to ensure that all solutions of this system are bounded. For this purpose, we apply Lemma 1.

Lemma 1. (see Chen, [8]) *If $p > 0$, $q > 0$, $\frac{dX}{dt} \geq (p - qX)X$ when $t \geq t_0$, and $X(t_0) > 0$ then*

$$\lim_{t\to+\infty} \inf X(t) \geq \frac{p}{q}.$$

If $p > 0$, $q > 0$, $\frac{dX}{dt} \leq (p - qX)X$ *when* $t \geq t_0$, *and* $X(t_0) > 0$ *then*

$$\lim_{t\to+\infty} \sup X(t) \leq \frac{p}{q}.$$

Proposition 2. *Suppose that*

$$\beta(m + nA)^2 > \delta(\beta(m + nA) + \sigma nA)(1 + e),$$

and let $X(t)$ *and* $Y(t)$ *be any positive solution of system (1), then*

$$A_1 \leq \lim_{t\to+\infty} \inf X(t) \leq \lim_{t\to+\infty} \sup X(t) \leq B_1$$

and

$$A_2 \leq \lim_{t\to+\infty} \inf Y(t) \leq \lim_{t\to+\infty} \sup Y(t) \leq B_2,$$

where

$$A_1 = \frac{\beta(m + nA)^2 - \delta(\beta(m + nA) + \sigma nA)(1 + e)}{\beta(m + nA)^2},$$

$A_2 = e$, $B_1 = 1$, *and*

$$B_2 = \frac{(\beta(m + nA) + \sigma nA)(1 + e)}{\beta(m + nA)}.$$

Proof. Let's assume that $X(t)$,$Y(t)$ are any positive solution of system (1), then it is obvious that

$$\frac{dX(t)}{dt} \leq (1 - X(t))X(t),$$
$$\frac{dY(t)}{dt} \geq \beta\left(1 - \frac{Y(t)}{e}\right)Y(t).$$

From Lemma 1, we have that

$$\lim_{t\to+\infty} \sup X(t) \leq B_1 = 1 \text{ and } \lim_{t\to+\infty} \inf Y(t) \geq A_2 = e.$$

Hence, for arbitrary $\varepsilon > 0$, there exists $T_1 \geq 0$ such that

$$X(t) \leq 1 + \varepsilon, \text{for } t \geq T_1. \qquad (2)$$

Substituting (2) into the second equation of system (1) gives

$$\frac{dY(t)}{dt} \leq \left(\beta + \frac{\sigma nA}{m + nA} - \frac{\beta Y(t)}{1 + \varepsilon + e}\right)Y(t).$$

Based on Lemma 1 we get

$$\lim_{t\to+\infty} \sup Y(t) \leq \frac{(\beta(m+nA)+\sigma nA)(1+\varepsilon+e)}{\beta(m+nA)}.$$

If we take $\varepsilon \to 0$ then we obtain $\lim_{t\to+\infty} \sup Y(t) \leq B_2$. Then, for any $\varepsilon > 0$ there exists $T_2 \geq 0$ such that

$$Y(t) \leq \frac{(\beta(m+nA)+\sigma nA)(1+e)}{\beta(m+nA)} + \varepsilon, \text{for } t \geq T_2. \tag{3}$$

Furthermore, it is obvious that the first equation of system (1) leads to

$$\frac{dX(t)}{dt} \geq \left(1 - \frac{\delta Y(t)}{m+nA} - X(t)\right) X(t). \tag{4}$$

Combining (3) and (4) gives

$$\frac{dX(t)}{dt} \geq \left(\frac{(\beta(m+nA)^2 - \delta((\beta(m+nA)+\sigma nA)(1+e)+\varepsilon\beta(m+nA))}{\beta(m+nA)^2} - X(t)\right) X(t).$$

By assuming that

$$\beta(m+nA)^2 > \delta(\beta(m+nA)+\sigma nA)(1+e),$$

and using Lemma 1, we get

$$\lim_{t\to+\infty} \inf X(t) \geq \frac{\beta(m+nA)^2 - \delta((\beta(m+nA)+\sigma nA)(1+e)+\varepsilon\beta(m+nA))}{\beta(m+nA)^2}.$$

Since ε is arbitrary, we can take $\varepsilon \to 0$, Thus, $\lim_{t\to+\infty} \inf X(t) \geq A_1$. □

By denoting that

$$\alpha = min\{A_1, A_2\} \text{ and } \gamma = max\{B_1, B_2\},$$

Proposition 2 leads to

$$\alpha \leq min\{\lim_{t\to+\infty} \inf X(t), \lim_{t\to+\infty} \inf Y(t)\} \leq max\{\lim_{t\to+\infty} \sup X(t), \lim_{t\to+\infty} \sup Y(t)\} \leq \gamma.$$

This fact shows the permanence of system (1).

3. Equilibria and Existence

An equilibrium point of system (1) is the solution from the following system

$$
\begin{aligned}
&(1-X)X - \frac{\delta XY}{X+m} = 0,\\
&\beta\left(1-\frac{Y}{X+e}\right)Y + \frac{\sigma nAY}{X+m+nA} = 0.
\end{aligned}
$$

System (1) has four equilibrium points, namely the extinction of both prey and predator point $E_0(0,0)$, the extinction of predator point $E_1(0, e(1+\frac{\sigma nA}{\beta(m+nA)}))$, the extinction of prey point $E_2(1,0)$ and the survival of both prey and predator point $E_3(X^*, Y^*)$, with $Y^* = \frac{-(X^*)^2+(1-m-nA)X^*+m+nA}{\delta}$ must be positive and X^* is all possible positive roots of cubic equation

$$(X^*)^3 + 3v_1(X^*)^2 + 3v_2X^* + v_3 = 0, \tag{5}$$

where

$$
\begin{aligned}
v_1 &= \frac{\delta\beta + 2\beta m + 2\beta nA - \beta}{3\beta},\\
v_2 &= \frac{\delta\beta m + \delta\beta nA + \delta\beta e - 2\beta m + \beta m^2 + 2\beta mnA - 2nA\beta + \beta(nA)^2 + \delta\sigma nA}{3\beta},\\
v_3 &= \frac{\delta\beta em + \delta\beta enA - \beta m^2 - 2\beta mnA - \beta(nA)^2 + \delta\sigma nAe}{\beta}.
\end{aligned}
$$

Using the following transformation

$$s = X^* + v_1,$$

equation (5) can be written as

$$f(s) = s^3 + 3gs + h, \tag{6}$$

where $g = v_2 - v_1^2$ and $h = v_3 - 3v_1v_2 + 2v_1^3$. All possible positive roots of equation (6) can be evaluated by Cardan's method as stated in Lemma 3.

Lemma 3. *Cai et. al. [9]. The existence of positive root of equation (6) can be described as follows.*

1. *If $h < 0$ then equation (6) has a unique positive root.*

2. *If $h > 0$ and $g < 0$:*

   (a) *if $h^2 + 4g^3 = 0$, then equation (6) has a unique positive root of multiplicity two,*

   (b) *if $h^2 + 4g^3 < 0$, then equation (6) has two distinct positive roots.*

3. *If $h = 0$ and $g < 0$, then equation (6) has a unique positive root.*

If equation (6) has two positive roots, then the roots are

$$s_1 = \frac{\sqrt[3]{(-h + 4\sqrt{4g^3 + h^2})^2} - 4g}{2\sqrt[3]{(-h + 4\sqrt{4g^3 + h^2})^2}} \text{ and } s_2 = -\frac{s_1}{2} + \frac{\sqrt{s_1^3 + 4h}}{2\sqrt{s_1}},$$

respectively. Meanwhile, if equation (6) has only one positive root, then the root is $s_1 = \frac{\sqrt[3]{(-h+4\sqrt{4g^3+h^2})^2}-4g}{2\sqrt[3]{(-h+4\sqrt{4g^3+h^2})^2}}$.

4. Stability of the Equilibria

The local stability of each equilibrium point of system (1) is shown in Teorema 4.

Theorem 4. *Stability of Equilibria*

1. *Equilibrium point $E_0 = (0, 0)$ is unstable node.*

2. *Equilibrium point $E_1 = \left(0, e(1 + \frac{\sigma nA}{\beta(m+nA)})\right)$ is locally stable if*

$$\frac{\beta(m + nA)^2}{\delta(\beta(m + nA) + \sigma nA)} < e. \tag{7}$$

3. *Equilibrium point $E_2 = (1, 0)$ is a saddle point.*

4. *$E_3 = \left(X^*, \frac{-(X^*)^2+(1-m-nA)X^*+m+nA}{\delta}\right)$ is locally stable if*

$$(X^* + \beta)(X^* + m + nA) + \sigma nA > (1 - X^*)X^*, \tag{8}$$

and

$$\omega > \frac{\delta\sigma nA}{(X^* + m + nA)^2} \tag{9}$$

where $\omega = \frac{\beta(X^*+e)(X^*+m+nA)+\beta(1-X^*)(m+nA-e)}{(X^*+e)^2}$.

Proof. 1. The Jacobian matrix of system (1) at E_0 is

$$J(E_0) = \begin{bmatrix} 1 & 0 \\ 0 & \beta + \frac{\sigma n A}{m+nA} \end{bmatrix}.$$

Both eigenvalues of $J(E_0)$ are positive, therefore E_0 is unstable node.

2. The Jacobian matrix of system (1) at E_1 is $J(E_1) = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}$ where

$$a_{11} = 1 - \frac{\delta e \left(1 + \frac{\sigma n A}{\beta(m+nA)}\right)}{m+nA},$$

$$a_{21} = \frac{\beta \left(e\left(1 + \frac{\sigma n A}{\beta(m+nA)}\right)\right)^2}{e^2} - \frac{\sigma n A e \left(1 + \frac{\sigma n A}{\beta(m+nA}\right)}{(m+nA)^2},$$

$$a_{22} = \frac{-\beta(m+nA) - \sigma n A}{m+nA}.$$

The eigenvalues of $J(E_1)$ are

$$\lambda_1 = 1 - \frac{\delta e \left(1 + \frac{\sigma n A}{\beta(m+nA)}\right)}{m+nA} \text{ and } \lambda_2 = \frac{-\beta(m+nA) - \sigma n A}{m+nA} < 0.$$

Thus, E_1 is stable if $\frac{\beta(m+nA)^2}{\delta(\beta(m+nA)+\sigma n A)} < e$.

3. At E_2, the Jacobian matrix of system (1) is $J(E_2) = \begin{bmatrix} -1 & -\frac{\delta}{1+m+nA} \\ 0 & \beta + \frac{\sigma n A}{1+m+nA} \end{bmatrix}$. Clearly that one of eigenvalues of $J(E_2)$ is positive, so that E_2 is a saddle point.

4. The Jacobian matrix at E_3 is

$$J(E_3) = \begin{bmatrix} -X^* + \frac{\delta X^* Y^*}{(X^*+m+nA)^2} & -\frac{\delta X^*}{X^*+m+nA} \\ \frac{\beta (X^*)^2}{(X^*+e)^2} - \frac{\sigma n A Y^*}{(X^*+m+nA)^2} & -\frac{\beta Y^*}{X^*+e} \end{bmatrix}.$$

The trace and determinant of $J(E_3)$ are respectively given by

$$Trace(J(E_3)) = \frac{(1-X^*)X^* - (X^*+\beta)(X^*+m+nA) - \sigma n A}{X^*+m+nA}$$

and

$$Det(J(E_3)) = \omega - \frac{\delta\sigma nA}{(X^* + m + nA)^2}.$$

E_3 is stable if $Trace(J(E_3)) < 0$ and $Det(J(E_3)) > 0$. These conditions are equivalent to (8) and (9).

□

5. Numerical Simulations

To illustrate our analytical results, we show some numerical simulations using different parameter values.

Numerical simulation 1. First, we perform numerical simulation by taking $\delta = 0.7, m = 0.7, A = 6, \beta = 0.45, e = 0.8, \ = 3, n = 0.3$. It is found that $h < 0$, hence equation (6) has one positive roots$s^* > 0$. Equilibrium point E_3 does not exist because $s^* < v_1$. Therefore, there exist only three equilibrium points namely $E_0(0,0), E_1(0,4.64)$ and $E_2(1,0)$. Based on the calculation, it is noted that the parameter values satisfy condition (7). Thus, according to Theorema 4, E_1 is stable. This behavior is clearly seen from the phase portrait shown in Figure 1where all solutions with various initial values are convergent to E_1.

Numerical simulation 2. We observe from conditions (7-9) that changing the coefficient of additional food (n or A) may change the stability properties of E_1 and E_3. Indeed, increasing the value of n or A will also increase the value of left hand side of condition (7); meaning that it also increases the possibility of E_1 to be unstable. On the other hand, increasing the value of n or A will certainly increase the value of left hand side of conditions (8) and (9), and will simultaneously decrease the value of right hand side of condition (9). Hence, it increases the possibility of E_3 to be stable. To see this behavior, we perform a simulation using the same parameter values as in Simulation 1, except for $n = 0.8$. Notice that we have increase the coefficient of additional food for predator. These parameter values produce $h < 0$, meaning that equation (6) has unique $s^* > 0$. Since $s^* > v_1$, system (1) has four equilibrium points, namely $E_0(0,0), E_1(0,5.4545), E_2(1,0)$, and $E_3(0.1921, 6.5692)$. It is found that by changing the value of $n = 0.3$ into $n = 0.8$, condition (7) becomes unsatisfied, while conditions (8) and (9) are fulfilled. Thus, E_1 becomes unstable and E_3 is now stable. This behavior is obviously seen in Figure 2, where all solutions with various initial values converge to E_3.

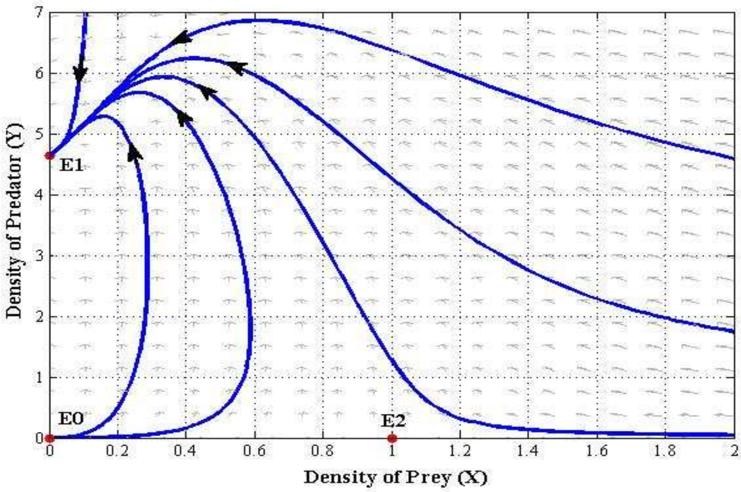

Figure 1: The portrait of model (1) with $\delta = 0.7, m = 0.7, A = 6, \beta = 0.45, e = 0.8, \ = 3, n = 0.3$

6. Conclusions

We have shown that the Leslie-Gower predator-prey model with additional food for predator is a permanence system, indicating that the solution of system is bounded. It is also found that the model has four equilibrium points, namely the extinction of both populations point (E_0), the extinction of predator point (E_1), the extinction of prey point (E_2), and both populations are able to survive (E_3). E_0 and E_2 are always unstable, whereas E_1 and E_3 are stable under certain conditions. Increasing the coefficient of additional food for predator (n) may stabilize equilibrium E_3 and at the same time destabilize equilibrium E_1.

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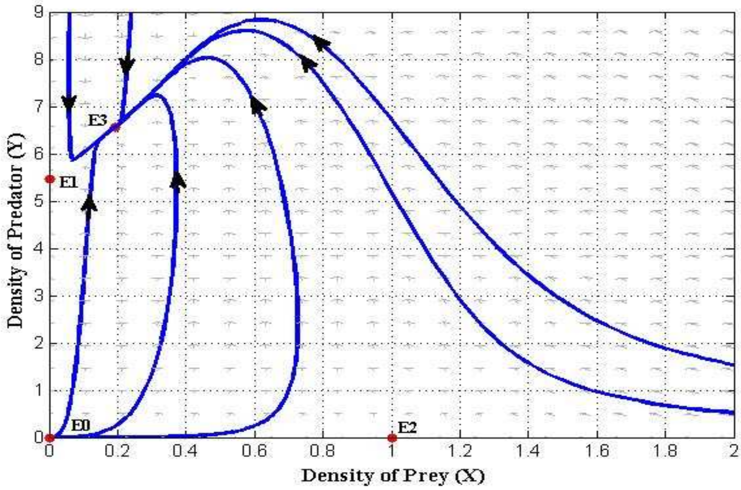

Figure 2: The portrait of model (1) with $\delta = 0.7, m = 0.7, A = 6, \beta = 0.45, e = 0.8, = 3, n = 0.8$

$10, 2016.$

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