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NEW CLASSES OF MAPPINGS AND SEPARATION AXIOMS INDUCED BY P_S-OPEN sets

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Abstract: An operation γ on $P_SO(X)$ is a mapping $\gamma\colon P_SO(X) \to P(X)$ such that $U \subseteq \gamma(U)$ for every $U \in P_SO(X)$. A nonempty set A of X is said to be P_S^γ-open if for each $x \in A$, there exists an P_S-open set U such that $x \in U$ and $\gamma(U) \subseteq A$ [2]. This paper is to continue the study of an operation γ on $P_SO(X)$ and the concept of P_S^γ-open sets of (X, τ). Using this operation and this set, we introduce the concept of P_S^γ-generalized closed sets and then investigate some of its properties. In addition, more separation axioms P_S^γ-T_n' spaces ($n \in \{0, \frac{1}{2}, 1, 2\}$) have been investigated. Finally, some main characterizations of P_S-(γ, β)-continuous mappings with P_S^β-closed graphs have been obtained.

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Key Words: P_S^γ-open sets, P_S^γ-g.closed sets, P_S^γ-T_n' spaces ($n \in \{0, \frac{1}{2}, 1, 2\}$), P_S-(γ, β)-continuous mappings, P_S^β-closed graphs

1. Introduction

The concept of preopen sets and semiopen sets was defined respectively by Mashhour et al. [12] and Levine [10]. While the concept of P_S-open set was introduced by Khalaf and Asaad [9]. Kasahara [8] defined the concept of an

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operation on τ and investigated the concept of α-closed graphs of mappings. After the work of Kasahara, Jankovic [7] introduced the concept of operation-closures of α and defined mapping with strongly closed graph. Ogata [13] introduced and studied the concept of operation-open sets, and used it to study operation-separation axioms and operation-mappings. In recent years, many concepts of operation γ in a topological space (X, τ) have been developed. An, Cuong and Maki [1] developed an operation γ on the collection of all preopen subsets of (X, τ) to introduce the notion of pre-γ-open sets and studied some of their properties. Asaad [2] defined and investigated the concept of the mapping γ on the collection of all P_S-open ($P_SO(X)$) subsets of (X, τ), and defined the notion of P_S^γ-open sets and studied some of their topological properties. He also defined some separation axioms and mappings (see [[2], [3]]) by utilizing the operation γ on $P_SO(X)$ and the set P_S^γ-open. The aim of this paper is to continue the study of an operation γ on $P_SO(X)$ and the concept of P_S^γ-open sets of (X, τ). The notion of P_S^γ-generalized closed sets and its different characterizations are given in this paper. In Section 4, P_S^γ-T'_n spaces for $n \in \{0, \frac{1}{2}, 1, 2\}$ are studied and investigated. In the last two sections, some characterizations of P_S-(γ, β)-continuous mappings with P_S^β-closed graphs have been obtained.

2. Preliminaries

Throughout this paper, the space (X, τ) and (Y, σ) represent nonempty spaces on which no separation axioms are assumed, unless otherwise mentioned, and they are simply written as X and Y, respectively, when no confusion arises. The closure and the interior of a set A of a space X are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a space X is said to be preopen [12] (respectively, semiopen [10]) if $A \subseteq Int(Cl(A))$ (respectively, $A \subseteq Cl(Int(A))$). The complement of semiopen set is said to be semiclosed [4]. A preopen subset A of (X, τ) is called P_S-open [9] if for each $x \in A$, there exists a semiclosed set F such that $x \in F \subseteq A$. The complement of a P_S-open set is P_S-closed [9]. The P_S-closure of a subset A of X is defined as the intersection of all P_S-closed sets containing A and it is denoted by $P_SCl(A)$ [9]. The family of all P_S-open (resp. preopen) subsets of a space (X, τ) is denoted by $P_SO(X)$ (resp. $PO(X)$).

An operation γ_p on $PO(X)$ is a mapping $\gamma_p \colon PO(X) \to P(X)$ such that $U \subseteq \gamma_p(U)$ for each $U \in PO(X)$, where $P(X)$ is the power set of X and $\gamma_p(U)$ denotes the value of γ_p at U. A nonempty set A of X is said to be pre γ_p-open if for each $x \in A$, there exists a preopen set U such that $x \in U$ and $\gamma_p(U) \subseteq A$ [1].

A point $x \in X$ is in the pre γ_p-closure of a set $A \subseteq X$ if $\gamma_p(U) \cap A \neq \phi$ for each preopen set U containing x. The family of all pre γ_p-open subsets of a space (X, τ) is denoted by $PO(X)_{\gamma_p}$. The set of all pre γ_p-closure points of A is called pre γ_p-closure of A and is denoted by $pCl_{\gamma_p}(A)$ [1]. A subset A of (X, τ) with an operation γ_p on $PO(X)$ is said to be pre γ_p-g.closed if $pCl_{\gamma_p}(A) \subseteq U$ whenever $A \subseteq U$ and U is a pre γ_p-open of (X, τ) [1]. A topological space (X, τ) with an operation γ_p on $PO(X)$ is said to be pre γ_p-$T_{\frac{1}{2}}$[1] if every pre γ_p-g.closed set in X is pre γ_p-closed. A topological space (X, τ) with an operation γ_p on $PO(X)$ is said to be pre γ_p-T_0 [1] if for any two distinct points x, y in X, there exists a preopen set U such that $x \in U$ and $y \notin \gamma_p(U)$ or $y \in U$ and $x \notin \gamma_p(U)$. A topological space (X, τ) with an operation γ_p on $PO(X)$ is said to be pre γ_p-T_1 [1] if for any two distinct points x, y in X, there exist two preopen sets U and V containing x and y respectively such that $y \notin \gamma_p(U)$ and $x \notin \gamma_p(V)$. A topological space (X, τ) with an operation γ_p on $PO(X)$ is said to be pre γ_p-T_2 [1] if for any two distinct points x, y in X, there exist two preopen sets U and V containing x and y respectively such that $\gamma_p(U) \cap \gamma_p(V) = \phi$.

An operation γ on $P_SO(X)$ is a mapping $\gamma\colon P_SO(X) \to P(X)$ such that $U \subseteq \gamma(U)$ for every $U \in P_SO(X)$, where $\gamma(U)$ is the value of γ at U. A nonempty set A of X is said to be P_S^γ-open if for each $x \in A$, there exists an P_S-open set U such that $x \in U$ and $\gamma(U) \subseteq A$ [2]. The complement of an P_S^γ-open set of X is P_S^γ-closed. The family of all P_S^γ-open subsets of a space (X, τ) is denoted by $P_S^\gamma O(X)$. The union of any class of P_S^γ-open sets in (X, τ) is P_S^γ-open. The intersection of any class of P_S^γ-closed sets in X is P_S^γ-closed. A topological space (X, τ) is said to be P_S^γ-regular if for each $x \in X$ and for each P_S-open set U containing x, there exists an P_S-open set W such that $x \in W$ and $\gamma(W) \subseteq U$. An operation γ on $P_SO(X)$ is said to be P_S-regular [2] if for each $x \in X$ and for every pair of P_S-open sets U_1 and U_2 such that both containing x, there exists an P_S-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2)$. Recall that a topological space (X, τ) is called locally indiscrete [5] if every open subset of X is closed. If (X, τ) is locally indiscrete, then the relation between the concept of P_S^γ-open set and γ-open set are identical [2]. Recall that a topological space (X, τ) is called semi-T_1 [11] if for each pair of distinct points x, y in X, there exist two semiopen sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$. If (X, τ) is semi-T_1, then the relation between the concept of P_S^γ-open set and pre γ_p-open set are identical [2].

In order to make the contents of this paper as self contained as possible, we briefly describe certain definitions, notations and some properties.

Definition 2.1. [2] Let A be any subset of a topological space (X, τ) and γ

be an operation on $P_SO(X)$. The P_S^γ-closure of A is defined as the intersection of all P_S^γ-closed sets of X containing A and it is denoted by $P_S^\gamma Cl(A)$. That is,

$$P_S^\gamma Cl(A) = \bigcap\{F : A \subseteq F,\, X\backslash F \in P_S^\gamma O(X)\}.$$

Theorem 2.2. *[2] Let A be any subset of a topological space (X, τ) and γ be an operation on $P_SO(X)$. Then $x \in P_S^\gamma Cl(A)$ if and only if $A \cap U \neq \phi$ for every P_S^γ-open set U of X containing x.*

Lemma 2.3. *[2] $P_S^\gamma Cl(A)$ is P_S^γ-closed set in X, A is P_S^γ-closed if and only if $P_S^\gamma Cl(A) = A$, and $P_S^\gamma Cl(P_S^\gamma Cl(A)) = P_S^\gamma Cl(A)$ for any $A \subseteq X$.*

Lemma 2.4. *[2] A topological space (X, τ) is P_S^γ-regular if and only if $P_S^\gamma O(X) = P_SO(X)$.*

Lemma 2.5. *[9] Let (X, τ) be a topological space. Then the following statements are true:*

1. *If X is locally indiscrete, then $P_SO(X) = \tau$.*
2. *If X is semi-T_1, then $P_SO(X) = PO(X)$.*

Definition 2.6. The point $x \in X$ is in the P_S-closure$_\gamma$ of a set A if $\gamma(U) \cap A \neq \phi$ for each P_S-open set U containing x. The set of all P_S-closure$_\gamma$ points of A is called P_S-closure$_\gamma$ of A and is denoted by $P_SCl_\gamma(A)$.

Lemma 2.7. *The following statements are true for any subsets A and B of a topological space (X, τ) with an operation γ on $P_SO(X)$.*

1. *$P_SCl_\gamma(A)$ is P_S-closed set in X.*
2. *$A \subseteq P_SCl(A) \subseteq P_SCl_\gamma(A) \subseteq P_S^\gamma Cl(A)$.*
3. *$P_SCl_\gamma(\phi) = \phi$ and $P_SCl_\gamma(X) = X$.*
4. *A is P_S^γ-closed if and only if $P_SCl_\gamma(A) = A$.*
5. *If $A \subseteq B$, then $P_SCl_\gamma(A) \subseteq P_SCl_\gamma(B)$.*
6. *$P_SCl_\gamma(A \cap B) \subseteq P_SCl_\gamma(A) \cap P_SCl_\gamma(B)$.*
7. *$P_SCl_\gamma(A) \cup P_SCl_\gamma(B) \subseteq P_SCl_\gamma(A \cup B)$.*

Proof. Straightforward. □

Theorem 2.8. *For any subsets A, B of a topological space (X, τ). If γ is an P_S-regular operation on $P_SO(X)$, then $P_SCl_\gamma(A) \cup P_SCl_\gamma(B) = P_SCl_\gamma(A \cup B)$.*

Proof. Let $x \notin P_SCl_\gamma(A) \cup P_SCl_\gamma(B)$. Then there exist P_S-open sets U_1 and U_2 such that $x \in U_1$, $x \in U_2$, $A \cap \gamma(U_1) = \phi$ and $A \cap \gamma(U_2) = \phi$. Since γ is an P_S-regular operation on $P_SO(X)$, then there exists an P_S-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2)$. Thus, we have

$$(A \cup B) \cap \gamma(W) \subseteq (A \cup B) \cap (\gamma(U_1) \cap \gamma(U_2)).$$

This implies that $(A \cup B) \cap \gamma(W) = \phi$ since $(A \cup B) \cap (\gamma(U_1) \cap \gamma(U_2)) = \phi$. This means that $x \notin P_SCl_\gamma(A \cup B)$ and hence $P_SCl_\gamma(A \cup B) \subseteq P_SCl_\gamma(A) \cup P_SCl_\gamma(B)$. Using Lemma 2.7 (7), we have the equality. □

Definition 2.9. An operation γ on $P_SO(X)$ is said to be P_S-open if for each $x \in X$ and for every P_S-open set U containing x, there exists an P_S^γ-open set W containing x such that $W \subseteq \gamma(U)$.

Theorem 2.10. *Let A be any subset of a topological space (X, τ). If γ is an P_S-open operation on $P_SO(X)$, then $P_SCl_\gamma(A) = P_S^\gamma Cl(A)$, $P_SCl_\gamma(P_SCl_\gamma(A))$ $= P_SCl_\gamma(A)$ and $P_SCl_\gamma(A)$ is P_S^γ-closed set in X.*

Proof. First we need to show that $P_S^\gamma Cl(A) \subseteq P_SCl_\gamma(A)$ since by Lemma 2.7 (2), we have $P_SCl_\gamma(A) \subseteq P_S^\gamma Cl(A)$. Now let $x \notin P_SCl_\gamma(A)$, then there exists an P_S-open set U containing x such that $A \cap \gamma(U) = \phi$. Since γ is an P_S-open on $P_SO(X)$, then there exists an P_S^γ-open set W containing x such that $W \subseteq \gamma(U)$. So $A \cap W = \phi$ and hence by Theorem 2.2, $x \notin P_S^\gamma Cl(A)$. Therefore, $P_S^\gamma Cl(A) \subseteq P_SCl_\gamma(A)$. Hence $P_SCl_\gamma(A) = P_S^\gamma Cl(A)$. Moreover, using the above result and by Lemma 2.3, we get $P_SCl_\gamma(P_SCl_\gamma(A)) = P_SCl_\gamma(A)$ and by Lemma 2.7 (4), we obtain $P_SCl_\gamma(A)$ is P_S^γ-closed set in X. □

Theorem 2.11. *Let A be any subset of a topological space (X, τ) and γ be an operation on $P_SO(X)$. Then the following statements are equivalent:*

1. *A is P_S^γ-open set.*
2. *$P_SCl_\gamma(X \backslash A) = X \backslash A$.*
3. *$P_S^\gamma Cl(X \backslash A) = X \backslash A$.*
4. *$X \backslash A$ is P_S^γ-closed set.*

Proof. Clear. □

3. P_S^γ-g Closed Sets

Definition 3.1. A subset A of a topological space (X,τ) with an operation γ on $P_SO(X)$ is said to be P_S^γ-generalized closed (in short P_S^γ-g.closed) if $P_SCl_\gamma(A) \subseteq U$ whenever $A \subseteq U$ and U is an P_S^γ-open set in X.

Lemma 3.2. *Let (X,τ) be a topological space and γ be an operation on $P_SO(X)$. A set A in (X,τ) is P_S^γ-g.closed if and only if $A \cap P_S^\gamma Cl(\{x\}) \neq \phi$ for every $x \in P_SCl_\gamma(A)$.*

Proof. Suppose A is P_S^γ-g.closed set in X and suppose (if possible) that there exists an element $x \in P_SCl_\gamma(A)$ such that $A \cap P_S^\gamma Cl(\{x\}) = \phi$. This follows that $A \subseteq X\backslash P_S^\gamma Cl(\{x\})$. Since $P_S^\gamma Cl(\{x\})$ is P_S^γ-closed implies $X\backslash P_S^\gamma Cl(\{x\})$ is P_S^γ-open and A is P_S^γ-g.closed set in X. Then, we have that $P_SCl_\gamma(A) \subseteq X\backslash P_S^\gamma Cl(\{x\})$. This means that $x \notin P_SCl_\gamma(A)$. This is a contradiction. Hence $A \cap P_S^\gamma Cl(\{x\}) \neq \phi$.

Conversely, let $U \in P_S^\gamma O(X)$ such that $A \subseteq U$. To show that $P_SCl_\gamma(A) \subseteq U$. Let $x \in P_SCl_\gamma(A)$. Then by hypothesis, $A \cap P_S^\gamma Cl(\{x\}) \neq \phi$. So there exists an element $y \in A \cap P_S^\gamma Cl(\{x\})$. Thus $y \in A \subseteq U$ and $y \in P_S^\gamma Cl(\{x\})$. By Theorem 2.2, $\{x\} \cap U \neq \phi$. Hence $x \in U$ and so $P_SCl_\gamma(A) \subseteq U$. Therefore, A is P_S^γ-g.closed set in (X,τ). □

Theorem 3.3. *Let A be a subset of topological space (X,τ) and γ be an operation on $P_SO(X)$. If A is P_S^γ-g.closed, then $P_SCl_\gamma(A)\backslash A$ does not contain any non-empty P_S^γ-closed set.*

Proof. Let F be a non-empty P_S^γ-closed set in X such that $F \subseteq P_SCl_\gamma(A)\backslash A$. Then $F \subseteq X\backslash A$ implies $A \subseteq X\backslash F$. Since $X\backslash F$ is P_S^γ-open set and A is P_S^γ-g.closed set, then $P_SCl_\gamma(A) \subseteq X\backslash F$. That is $F \subseteq X\backslash P_SCl_\gamma(A)$. Hence $F \subseteq X\backslash P_SCl_\gamma(A) \cap P_SCl_\gamma(A)\backslash A \subseteq X\backslash P_SCl_\gamma(A) \cap P_SCl_\gamma(A) = \phi$. This shows that $F = \phi$. This is contradiction. Therefore, $F \nsubseteq P_SCl_\gamma(A)\backslash A$. □

Theorem 3.4. *If $\gamma\colon P_SO(X) \to P(X)$ is an P_S-open operation, then the converse of the Theorem 3.3 is true.*

Proof. Let U be an P_S^γ-open set in (X,τ) such that $A \subseteq U$. Since $\gamma\colon P_SO(X) \to P(X)$ is an P_S-open operation, then by Theorem 2.10, $P_SCl_\gamma(A)$ is P_S^γ-closed set in X. Thus, we have $P_SCl_\gamma(A) \cap X\backslash U$ is an P_S^γ-closed set in (X,τ). Since $X\backslash U \subseteq X\backslash A$, $P_SCl_\gamma(A) \cap X\backslash U \subseteq P_SCl_\gamma(A)\backslash A$. Using the assumption of the converse of the Theorem 3.3, $P_SCl_\gamma(A) \subseteq U$. Therefore, A is P_S^γ-g.closed set in (X,τ). □

Corollary 3.5. *Let A be an P_S^γ-g.closed subset of topological space (X,τ) and let γ be an operation on $P_SO(X)$. Then A is P_S^γ-closed if and only if $P_SCl_\gamma(A)\backslash A$ is P_S^γ-closed set.*

Proof. Let A be an P_S^γ-closed set in (X,τ). Then by Lemma 2.7 (4), $P_SCl_\gamma(A) = A$ and hence $P_SCl_\gamma(A)\backslash A = \phi$ which is P_S^γ-closed set.

Conversely, suppose $P_SCl_\gamma(A)\backslash A$ is P_S^γ-closed and A is P_S^γ-g.closed. Then by Theorem 3.3, $P_SCl_\gamma(A)\backslash A$ does not contain any non-empty P_S^γ-closed set and since $P_SCl_\gamma(A)\backslash A$ is P_S^γ-closed subset of itself, then $P_SCl_\gamma(A)\backslash A = \phi$ implies $P_SCl_\gamma(A) \cap X\backslash A = \phi$. Hence $P_SCl_\gamma(A) = A$. This follows from Lemma 2.7 (4) that A is P_S^γ-closed set in (X,τ). □

Theorem 3.6. *Let (X,τ) be a topological space and γ be an operation on $P_SO(X)$. If A is P_S^γ-g.closed and P_S^γ-open subset of X, then A is P_S^γ-closed.*

Proof. Since A is P_S^γ-g.closed and P_S^γ-open set in X, then $P_SCl_\gamma(A) \subseteq A$ and hence by Lemma 2.7 (4), A is P_S^γ-closed. □

Theorem 3.7. *In any topological space (X,τ) with an operation γ on $P_SO(X)$. For an element $x \in X$, the set $X\backslash\{x\}$ is P_S^γ-g.closed or P_S^γ-open.*

Proof. Suppose that $X\backslash\{x\}$ is not P_S^γ-open. Then X is the only P_S^γ-open set containing $X\backslash\{x\}$. This implies that $P_SCl_\gamma(X\backslash\{x\}) \subseteq X$. Thus $X\backslash\{x\}$ is an P_S^γ-g.closed set in X. □

Corollary 3.8. *In any topological space (X,τ) with an operation γ on $P_SO(X)$. For an element $x \in X$, either the set $\{x\}$ is P_S^γ-closed or the set $X\backslash\{x\}$ is P_S^γ-g.closed.*

Proof. Suppose $\{x\}$ is not P_S^γ-closed, then $X\backslash\{x\}$ is not P_S^γ-open. Hence by Theorem 3.7, $X\backslash\{x\}$ is P_S^γ-g.closed set in X. □

Definition 3.9. Let A be any subset of a topological space (X,τ) and γ be an operation on $P_SO(X)$. Then the $P_S^\gamma O(X)$-kernel of A is denoted by $P_S^\gamma O(X)$-ker(A) and is defined as follows:

$$P_S^\gamma O(X)\text{-ker}(A) = \cap\{U : A \subseteq U \text{ and } U \in P_S^\gamma O(X)\}$$

In other words, $P_S^\gamma O(X)$-ker(A) is the intersection of all P_S^γ-open sets of (X,τ) containing A.

Theorem 3.10. *Let $A \subseteq (X,\tau)$ and γ be an operation on $P_SO(X)$. Then A is P_S^γ-g.closed if and only if $P_SCl_\gamma(A) \subseteq P_S^\gamma O(X)$-ker(A).*

Proof. Suppose that A is P_S^γ-g.closed. Then $P_SCl_\gamma(A) \subseteq U$, whenever $A \subseteq U$ and U is P_S^γ-open. Let $x \in P_SCl_\gamma(A)$. Then by Lemma 3.2, $A \cap P_S^\gamma Cl(\{x\}) \neq \phi$. So there exists a point z in X such that $z \in A \cap P_S^\gamma Cl(\{x\})$ implies that $z \in A \subseteq U$ and $z \in P_S^\gamma Cl(\{x\})$. By Theorem 2.2, $\{x\} \cap U \neq \phi$. Hence we show that $x \in P_S^\gamma O(X)$-ker(A). Therefore, $P_SCl_\gamma(A) \subseteq P_S^\gamma O(X)$-ker$(A)$.

Conversely, let $P_SCl_\gamma(A) \subseteq P_S^\gamma O(X)$-ker$(A)$. Let U be any P_S^γ-open set containing A. Let x be a point in X such that $x \in P_SCl_\gamma(A)$. Then $x \in P_S^\gamma O(X)$-ker(A). Namely, we have $x \in U$, because $A \subseteq U$ and $U \in P_S^\gamma O(X)\}$. That is $P_SCl_\gamma(A) \subseteq P_S^\gamma O(X)$-ker$(A) \subseteq U$. Therefore, A is P_S^γ-g.closed set in X. □

4. P_S^γ-T_n' Spaces for $n \in \{0, \frac{1}{2}, 1, 2\}$

In this section, we introduce some types of P_S^γ- separation axioms called P_S^γ-T_n' for $n \in \{0, \frac{1}{2}, 1, 2\}$. Some basic properties of these spaces are investigated.

Definition 4.1. A topological space (X, τ) with an operation γ on $P_SO(X)$ is said to be:
(i) P_S^γ-T_0' if for any two distinct points x, y in X, there exists an P_S-open set U such that either $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.
(ii) P_S^γ-T_0 [2] if for each pair of distinct points x, y in X, there exists a P_S^γ-open set U containing one of the points but not the other.

Definition 4.2. A topological space (X, τ) with an operation γ on $P_SO(X)$ is said to be:
(i) P_S^γ-T_1' if for any two distinct points x, y in X, there exist two P_S-open sets U and V containing x and y respectively such that $y \notin \gamma(U)$ and $x \notin \gamma(V)$. (ii) P_S^γ-T_1 [2] if for each pair of distinct points x
$, y$ in X, there exist two P_S^γ-open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

Definition 4.3. A topological space (X, τ) with an operation γ on $P_SO(X)$ is said to be:
(i) P_S^γ-T_2' if for any two distinct points x, y in X, there exist two P_S-open sets U and V containing x and y respectively such that $\gamma(U) \cap \gamma(V) = \phi$.
(ii) P_S^γ-T_2 [2] if for each pair of distinct points x, y in X, there exist disjoint P_S^γ-open sets U and V containing x and y respectively.

Definition 4.4. A topological space (X, τ) with an operation γ on $P_SO(X)$ is said to be P_S^γ-$T_{\frac{1}{2}}'$ if every P_S^γ-g.closed set in X is P_S^γ-closed set.

Theorem 4.5. *For any topological space (X, τ) with an operation γ on $P_SO(X)$. Then (X, τ) is P_S^γ-$T'_{\frac{1}{2}}$ if and only if for each element $x \in X$, the set $\{x\}$ is P_S^γ-closed or P_S^γ-open.*

Proof. Let X be an P_S^γ-$T'_{\frac{1}{2}}$ space and let $\{x\}$ is not P_S^γ-closed set in (X, τ). By Corollary 3.8, $X\backslash\{x\}$ is P_S^γ-g.closed. Since (X, τ) is P_S^γ-$T'_{\frac{1}{2}}$, then $X\backslash\{x\}$ is P_S^γ-closed set which means that $\{x\}$ is P_S^γ-open set in X.

Conversely, let F be any P_S^γ-g.closed set in the space (X, τ). We have to show that F is P_S^γ-closed (that is $P_SCl_\gamma(F) = F$ (by Lemma 2.7 (4))). It is sufficient to show that $P_SCl_\gamma(F) \subseteq F$. Let $x \in P_SCl_\gamma(F)$. By hypothesis $\{x\}$ is P_S^γ-closed or P_S^γ-open for each $x \in X$. So we have two cases:

Case (1): If $\{x\}$ is P_S^γ-closed set. Suppose $x \notin F$, then $x \in P_SCl_\gamma(F)\backslash F$ contains a non-empty P_S^γ-closed set $\{x\}$. A contradiction since F is P_S^γ-g.closed set and according to the Theorem 3.3. Hence $x \in F$. This follows that $P_SCl_\gamma(F) \subseteq F$ and hence $P_SCl_\gamma(F) = F$. This means from by Lemma 2.7 (4) that F is P_S^γ-closed set in (X, τ). Thus (X, τ) is P_S^γ-$T'_{\frac{1}{2}}$ space.

Case (2): If $\{x\}$ is P_S^γ-open set. Then by Theorem 2.2, $F \cap \{x\} \neq \phi$ which implies that $x \in F$. So $P_SCl_\gamma(F) \subseteq F$. Thus by Lemma 2.7 (4), F is P_S^γ-closed. Therefore, (X, τ) is P_S^γ-$T'_{\frac{1}{2}}$ space. □

Theorem 4.6. *Suppose that γ is an P_S-open operation on $P_SO(X)$. A space (X, τ) is an P_S^γ-T'_0 if and only if $P_SCl_\gamma(\{x\}) \neq P_SCl_\gamma(\{y\})$, for every pair x, y of X with $x \neq y$.*

Proof. **Necessary Part.** Let x, y be any two distinct points of an P_S^γ-T'_0 space (X, τ). Then by definition, we assume that there exists an P_S^γ-open set U such that $x \in U$ and $y \notin \gamma(U)$. Since γ is an P_S-open operation on $P_SO(X)$, then there exists an P_S^γ-open set W such that $x \in W$ and $W \subseteq \gamma(U)$. Hence $y \in X\backslash\gamma(U) \subseteq X\backslash W$. Since $X\backslash W$ is an P_S^γ-closed set in (X, τ). Then we obtain that $P_SCl_\gamma(\{y\}) \subseteq X\backslash W$ and therefore $P_SCl_\gamma(\{x\}) \neq P_SCl_\gamma(\{y\})$.

Sufficient Part. Suppose for any $x, y \in X$ with $x \neq y$, we have $P_SCl_\gamma(\{x\}) \neq P_SCl_\gamma(\{y\})$. Now, we assume that there exists $z \in X$ such that $z \in P_SCl_\gamma(\{x\})$, but $z \notin P_SCl_\gamma(\{y\})$. If $x \in P_SCl_\gamma(\{y\})$, then $\{x\} \subseteq P_SCl_\gamma(\{y\})$, which implies that $P_SCl_\gamma(\{x\}) \subseteq P_SCl_\gamma(\{y\})$ (by Lemma 2.7 (5)). This implies that $z \in P_SCl_\gamma(\{y\})$. This contradiction shows that $x \notin P_SCl_\gamma(\{y\})$. This means that by Definition 2.6, there exists an P_S-open set U such that $x \in U$ and $\gamma(U) \cap \{y\} = \phi$. Thus, we have that $x \in U$ and $y \notin \gamma(U)$. It gives that the space (X, τ) is P_S^γ-T'_0. □

Theorem 4.7. *[2] A space (X, τ) is P_S^γ-T_0 if and only if $P_S^\gamma Cl(\{x\}) \neq P_S^\gamma Cl(\{y\})$, for every pair $x, y \in X$ with $x \neq y$.*

Theorem 4.8. *Suppose that γ is an P_S-open operation on $P_S O(X)$. A space (X, τ) is P_S^γ-T_0' if and only if (X, τ) is P_S^γ-T_0.*

Proof. This follows from Theorem 4.6, Theorem 4.7 and the fact that $P_S Cl_\gamma(A) = P_S^\gamma Cl(A)$ for any $A \subseteq X$ holds under the assumption that γ is an P_S-open operation on $P_S O(X)$ (see Theorem 2.10). □

Theorem 4.9. *For a topological space (X, τ) with an operation γ on $P_S O(X)$. Then the following conditions are true:*

1. *(X, τ) is P_S^γ-T_1'.*
2. *For every point $x \in X$, the set $\{x\}$ is P_S^γ-closed.*
3. *(X, τ) is P_S^γ-T_1.*

Proof. (1) $\Rightarrow$ (2) Let x be a point of an P_S^γ-T_1' space (X, τ). Then for any point $y \in X$ such that $x \neq y$, there exists an P_S-open set V_y such that $y \in V_y$ but $x \notin \gamma(V_y)$. Thus, $y \in \gamma(V_y) \subseteq X \backslash \{x\}$. This implies that $X \backslash \{x\} = \cup\{\gamma(V_y) : y \in X \backslash \{x\}\}$. It is shown that $X \backslash \{x\}$ is P_S^γ-open set in (X, τ). Hence $\{x\}$ is P_S^γ-closed set in (X, τ).

(2) $\Rightarrow$ (1) Let $x, y \in X$ such that $x \neq y$. By hypothesis, we get $X \backslash \{y\}$ and $X \backslash \{x\}$ are P_S^γ-open sets such that $x \in X \backslash \{y\}$ and $y \in X \backslash \{x\}$. Therefore, there exist P_S-open sets U and V such that $x \in U$, $y \in V$, $\gamma(U) \subseteq X \backslash \{y\}$ and $\gamma(V) \subseteq X \backslash \{x\}$. So, $y \notin \gamma(U)$ and $x \notin \gamma(V)$. This implies that (X, τ) is P_S^γ-T_1'.

(2) $\Leftrightarrow$ (3) See [[2], Theorem 4.7 (1)]. □

Theorem 4.10. *For any topological space (X, τ) and any operation γ on $P_S O(X)$, the following properties hold.*

1. *Every P_S^γ-T_n space is P_S^γ-T_n', where $n \in \{2, 0\}$.*
2. *Every P_S^γ-T_2' space is P_S^γ-T_1'.*
3. *Every P_S^γ-T_1' space is P_S^γ-$T_{\frac{1}{2}}'$.*
4. *Every P_S^γ-$T_{\frac{1}{2}}'$ space is P_S^γ-T_0.*
5. *Every P_S^γ-T_n' space is pre γ_p-T_n, where $n \in \{0, \frac{1}{2}, 1, 2\}$.*
6. *Every P_S^γ-T_n' space is P_S-T_n, where $n \in \{0, 1, 2\}$.*

Proof. The proofs are obvious by their definitions. □

Lemma 4.11. *If* (X, τ) *is* P_S^γ-T_n *space, then* (X, τ) *is* P_S^γ-T_{n-1} *for* $n = 1, 2$.

Remark 4.12. Every P_S^γ-open set is pre γ_p-open, but the converse is not true in general. For instance, let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a, b\}, \{c\}\}$. Then $PO(X)$ = all subsets of X and $P_SO(X) = \tau$. Define an operation $\gamma_p\colon PO(X) \to P(X)$ by $\gamma_p(A) = A$ for all $A \in PO(X)$. Here, $P_S^\gamma O(X) = P_SO(X)$ and $PO(X)_{\gamma_p} = PO(X)$. Therefore, the set $\{a\}$ is pre γ_p-open, but it is not P_S^γ-open.

Remark 4.13. By Theorem 4.10, Remark 4.12 and [1], we obtain the following diagram of implications. Moreover, the following Examples 4.14, 4.15, 4.16, 4.17, and 4.18 below show that the reverse implications in Diagram 1 are not true in general.

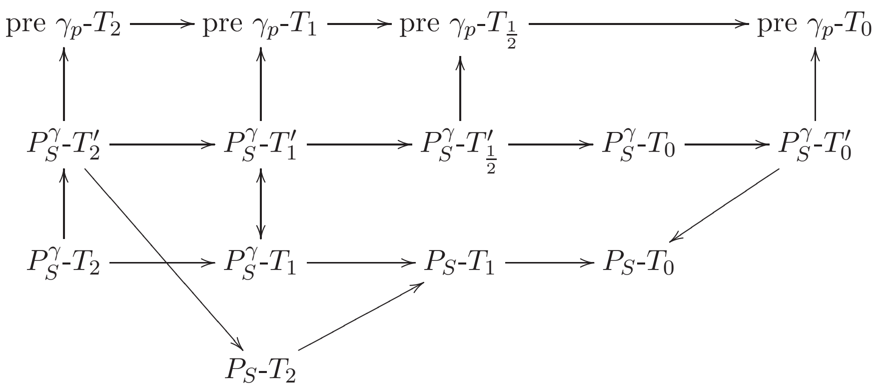

Diagram 1. The relations between various types of operation-separation axioms

Example 4.14. The space (X, τ) in the example as showed in Remark 4.12 is pre γ_p-T_n, but it is not P_S^γ-T_n' for $n \in \{0, \frac{1}{2}, 1, 2\}$.

Example 4.15. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Define an operation $\gamma\colon P_SO(X) \to P(X)$ by $\gamma(A) = A$ for all $A \in P_SO(X)$. Thus, $P_S^\gamma O(X) = P_SO(X) = \{\phi, X, \{b\}, \{a, c\}\}$. Then the space (X, τ) is P_S^γ-T_0', but it is not P_S^γ-$T_{\frac{1}{2}}'$.

Example 4.16. Let $X = \{a, b, c\}$ and $\tau = P(X) = P_SO(X)$. Let $\gamma\colon P_SO(X) \to P(X)$ be an operation on $P_SO(X)$ defined as follows:

For every set $A \in P_S O(X)$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a\} \text{ or } \{b\} \text{ or } \{a,b\} \text{ or } \{b,c\} \\ X & \text{otherwise} \end{cases}$$

Clearly, $P_S^\gamma O(X) = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$. Thus, the space (X,τ) is P_S^γ-$T'_{\frac{1}{2}}$, but it is not P_S^γ-T'_1.

Example 4.17. Suppose $X = \{a,b,c\}$ and τ be the discrete topology on X. Define an operation γ on $P_S O(X)$ as follows: For every $A \in P_S O(X)$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a,b\} \text{ or } \{a,c\} \text{ or } \{b,c\} \\ X & \text{otherwise} \end{cases}$$

Then (X,τ) is P_S^γ-T'_1 space, but (X,τ) is not P_S^γ-T'_2.

Example 4.18. Consider $X = \{a,b,c\}$ and τ be the discrete topology on X. Define an operation γ on $P_S O(X)$ as follows: For every $A \in P_S O(X)$

$$\gamma(A) = \begin{cases} \{\text{a, b}\} & \text{if } A = \{b\} \\ \{\text{b, c}\} & \text{if } A = \{c\} \text{ or } \{b,c\} \\ X & \text{otherwise} \end{cases}$$

Thus, the space (X,τ) is P_S^γ-T'_0, but it is not P_S^γ-T_0.

Lemma 4.19. *The following are holds for any topological space (X,τ):*

1. *If (X,τ) is locally indiscrete, then (X,τ) is P_S^γ-T'_n if and only if it is γ-T_n, where $n \in \{0, \frac{1}{2}, 1, 2\}$.*
2. *If (X,τ) is semi-T_1, then (X,τ) is P_S^γ-T'_n if and only if it is pre γ_p-T_n, where $n \in \{0, \frac{1}{2}, 1, 2\}$.*

Proof. The part (1) follows directly from Lemma 2.5 (1), and the part (2) follows directly from Lemma 2.5 (2). □

5. P_S-(γ, β)-Continuous Mappings

Throughout Section 5 and Section 6, let $\gamma\colon P_S O(X) \to P(X)$ and $\beta\colon P_S O(Y) \to P(Y)$ be operations on $P_S O(X)$ and $P_S O(Y)$ respectively. In this section, we introduce a new class of mappings called P_S-(γ, β)-continuous. Some characterizations and properties of this mapping are investigated.

Definition 5.1. A mapping $f\colon (X,\tau) \to (Y,\sigma)$ is said to be P_S-(γ,β)-continuous if for each $x \in X$ and each P_S-open set V containing $f(x)$, there exists an P_S-open set U containing x such that $f(\gamma(U)) \subseteq \beta(V)$.

Theorem 5.2. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be an P_S-(γ,β)-continuous mapping, then,*

1. *$f(P_SCl_\gamma(A)) \subseteq P_SCl_\beta(f(A))$, for every $A \subseteq (X,\tau)$.*
2. *$f^{-1}(F)$ is P_S^γ-closed set in (X,τ), for every P_S^β-closed set F of (Y,σ).*

Proof. (1) Let $y \in f(P_SCl_\gamma(A))$ and V be any P_S-open set containing y. Then by hypothesis, there exists $x \in X$ and P_S-open set U containing x such that $f(x) = y$ and $f(\gamma(U)) \subseteq \beta(V)$. Since $x \in P_SCl_\gamma(A)$, we have $\gamma(U) \cap A \neq \phi$. Hence $\phi \neq f(\gamma(U) \cap A) \subseteq f(\gamma(U)) \cap f(A) \subseteq \beta(V) \cap f(A)$. This implies that $y \in P_SCl_\beta(f(A))$. Therefore, $f(P_SCl_\gamma(A)) \subseteq P_SCl_\beta(f(A))$.

(2) Let F be any P_S^β-closed set of (Y,σ). By using (1), we have

$$f(P_SCl_\gamma(f^{-1}(F))) \subseteq P_S\text{-}Cl_\beta(F) = F.$$

Therefore, $P_SCl_\gamma(f^{-1}(F)) = f^{-1}(F)$. Hence $f^{-1}(F)$ is P_S^γ-closed set in (X,τ). □

Theorem 5.3. *In Theorem 5.2, the properties of P_S-(γ,β)-continuity of f, (1) and (2) are equivalent to each other if either the space (Y,σ) is P_S^β-regular or the operation β is P_S-open.*

Proof. It follows from the proof of Theorem 5.2 that we know the following implications: "P_S-(γ,β)-continuity of f" $\Rightarrow$ (1) $\Rightarrow$ (2). Thus, when the space (Y,σ) is P_S^β-regular, we prove the implication: (2) $\Rightarrow$ P_S-(γ,β)-continuity of f. Let $x \in X$ and let $V \in P_SO(Y)$ such that $f(x) \in V$. Since (Y,σ) is an P_S^β-regular space, then by Lemma 2.4, $V \in P_S^\beta O(Y)$. By using (2) of Theorem 5.2, $f^{-1}(V) \in P_S^\gamma O(X)$ such that $x \in f^{-1}(V)$. So there exists an P_S-open set U such that $x \in U$ and $\gamma(U) \subseteq f^{-1}(V)$. This implies that $f(\gamma(U)) \subseteq V \subseteq \beta(V)$. Therefore, f is P_S-(γ,β)-continuous.

Now, when β is an P_S-open operation, we show the implication: (2) $\Rightarrow$ P_S-(γ,β)-continuity of f. Let $x \in X$ and let $V \in P_SO(Y)$ such that $f(x) \in V$. Since β is an P_S-open operation, then there exists $W \in P_S^\beta O(Y)$ such that $f(x) \in W$ and $W \subseteq \beta(V)$. By using (2) of Theorem 5.2, $f^{-1}(W) \in P_S^\gamma O(X)$ such that $x \in f^{-1}(W)$. So there exists an P_S-open set U such that $x \in U$ and $\gamma(U) \subseteq f^{-1}(W) \subseteq f^{-1}(\beta(V))$. This implies that $f(\gamma(U)) \subseteq \beta(V)$. Hence f is P_S-(γ,β)-continuous. □

Definition 5.4. A mapping $f\colon (X,\tau) \to (Y,\sigma)$ is said to be

1. P_S-(γ,β)-closed if the image of each P_S^{γ}-closed set of X is P_S^{β}-closed in Y.

2. P_S-(id,β)-closed if the image of each P_S-closed set of X is P_S^{β}-closed in Y.

Theorem 5.5. *Suppose that a mapping $f\colon (X,\tau) \to (Y,\sigma)$ is both P_S-(γ,β)-continuous and P_S-(id,β)-closed, then:*

1. *For every P_S^{γ}-g.closed set A of (X,τ), the image $f(A)$ is P_S^{β}-g.closed in (Y,σ).*

2. *For every P_S^{β}-g.closed set B of (Y,σ), the inverse set $f^{-1}(B)$ is P_S^{γ}-g.closed in (X,τ).*

Proof. (1) Let G be any P_S^{β}-open set in (Y,σ) such that $f(A) \subseteq G$. Since f is P_S-(γ,β)-continuous mapping, then by using Theorem 5.2 (2), $f^{-1}(G)$ is P_S^{γ}-open set in (X,τ). Since A is P_S^{γ}-g.closed and $A \subseteq f^{-1}(G)$, we have $P_SCl_{\gamma}(A) \subseteq f^{-1}(G)$, and hence $f(P_SCl_{\gamma}(A)) \subseteq G$. Thus, by Lemma 2.7 (1), $P_SCl_{\gamma}(A)$ is P_S-closed set and since f is P_S-(id,β)-closed, then $f(P_SCl_{\gamma}(A))$ is P_S^{β}-closed set in Y. Therefore, $P_S\text{-}Cl_{\beta}(f(A)) \subseteq P_SCl_{\beta}(f(P_SCl_{\gamma}(A))) = f(P_SCl_{\gamma}(A)) \subseteq G$. This implies that $f(A)$ is P_S^{β}-g.closed in (Y,σ).

(2) Let H be any P_S^{γ}-open set of a space (X,τ) such that $f^{-1}(B) \subseteq H$. Let $C = P_SCl_{\gamma}(f^{-1}(B)) \cap (X \backslash H)$, then by Lemma 2.7 (1), C is P_S-closed set in (X,τ). Since f is P_S-(id,β)-closed mapping. Then $f(C)$ is P_S^{β}-closed in (Y,σ). Since f is P_S-(γ,β)-continuous mapping, then by using Theorem 5.2 (1), we have $f(C) = f(P_SCl_{\gamma}(f^{-1}(B))) \cap f(X\backslash H) \subseteq P_SCl_{\beta}(B) \cap f(X\backslash H) \subseteq P_SCl_{\beta}(B) \cap (Y\backslash B) = P_SCl_{\beta}(B)\backslash B$. This implies from Theorem 3.3 that $f(C) = \phi$, and hence $C = \phi$. So $P_SCl_{\gamma}(f^{-1}(B)) \subseteq H$. Therefore, $f^{-1}(B)$ is P_S^{γ}-g.closed in (X,τ). □

Theorem 5.6. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be an injective, P_S-(γ,β)-continuous and P_S-(id,β)-closed mapping. If (Y,σ) is P_S^{β}-$T'_{\frac{1}{2}}$, then (X,τ) is P_S^{γ}-$T'_{\frac{1}{2}}$.*

Proof. Let G be any P_S^{γ}-g.closed set of (X,τ). Since f is P_S-(γ,β)-continuous and P_S-(id,β)-closed mapping. Then by Theorem 5.5 (1), $f(G)$ is P_S^{β}-g.closed in (Y,σ). Since (Y,σ) is P_S^{β}-$T'_{\frac{1}{2}}$, then $f(G)$ is P_S^{β}-closed in Y. Again, since f is P_S-(γ,β)-continuous, then by Theorem 5.2 (2), $f^{-1}(f(G))$ is P_S^{γ}-closed in X.

Hence G is P_S^γ-closed in X since f is injective. Therefore, (X,τ) is an P_S^γ-$T'_{\frac{1}{2}}$ space. □

Theorem 5.7. *Let $f\colon (X,\tau)\to(Y,\sigma)$ be a surjective, P_S-(γ,β)-continuous and P_S-(id,β)-closed mapping. If (X,τ) is P_S^γ-$T'_{\frac{1}{2}}$, then (Y,σ) is P_S^β-$T'_{\frac{1}{2}}$.*

Proof. Let H be an P_S^β-g.closed set of (Y,σ). Since f is P_S-(γ,β)-continuous and P_S-(id,β)-closed mapping. Then by Theorem 5.5 (2), $f^{-1}(H)$ is P_S^γ-g.closed in (X,τ). Since (X,τ) is P_S^γ-$T'_{\frac{1}{2}}$, then we have, $f^{-1}(H)$ is P_S^γ-closed set in X. Again, since f is P_S-(id,β)-closed mapping, then $f(f^{-1}(H))$ is P_S^β-closed in Y. Therefore, H is P_S^β-closed in Y since f is surjective. Hence (Y,σ) is P_S^β-$T'_{\frac{1}{2}}$ space. □

Theorem 5.8. *If a mapping $f\colon (X,\tau)\to(Y,\sigma)$ is injective P_S-(γ,β)-continuous and the space (Y,σ) is P_S^β-T'_2, then the space (X,τ) is P_S^γ-T'_2.*

Proof. Let x_1 and x_2 be any distinct points of a space (X,τ). Since f is an injective mapping and (Y,σ) is P_S^β-T'_2. Then there exist two P_S-open sets U_1 and U_2 in Y such that $f(x_1)\in U_1$, $f(x_2)\in U_2$ and $\beta(U_1)\cap\beta(U_2)=\phi$. Since f is P_S-(γ,β)-continuous, there exist P_S-open sets V_1 and V_2 in X such that $x_1\in V_1$, $x_2\in V_2$, $f(\gamma(V_1))\subseteq\beta(U_1)$ and $f(\gamma(V_2))\subseteq\beta(U_2)$. Therefore $\beta(U_1)\cap\beta(U_2)=\phi$. Hence (X,τ) is P_S^γ-T'_2. □

Theorem 5.9. *If a mapping $f\colon (X,\tau)\to(Y,\sigma)$ is injective P_S-(γ,β)-continuous and the space (Y,σ) is P_S^β-T'_n, then the space (X,τ) is P_S^γ-T'_n for $n\in\{0,1\}$.*

Proof. The proof is similar to Theorem 5.8. □

Definition 5.10. A mapping $f\colon (X,\tau)\to(Y,\sigma)$ is said to be P_S-(γ,β)-homeomorphism if f is bijective, P_S-(γ,β)-continuous and f^{-1} is P_S-(β,γ)-continuous.

Theorem 5.11. *Suppose that a mapping $f\colon (X,\tau)\to(Y,\sigma)$ is an P_S-(γ,β)-homeomorphism. If (X,τ) is P_S^γ-$T'_{\frac{1}{2}}$, then (Y,σ) is P_S^β-$T'_{\frac{1}{2}}$.*

Proof. Let $\{y\}$ be any singleton set of (Y,σ). Then there exists an element x of X such that $y=f(x)$. So by hypothesis and Theorem 4.5, we have $\{x\}$ is P_S^γ-closed or P_S^γ-open set in X. By using Theorem 5.2, $\{y\}$ is P_S^β-closed or P_S^β-open set. Hence the space by Theorem 4.5, (Y,σ) is P_S^β-$T'_{\frac{1}{2}}$. □

6. Mappings with P_S^β-Closed Graphs

For a mapping $f\colon (X,\tau) \to (Y,\sigma)$, the subset $\{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f and is denoted by $G(f)$ [6]. In this section, we further investigate general operator approaches of closed graphs of mappings. Let $\rho\colon (\tau \times \sigma)_g \to P(X \times Y)$ be an operation on $(\tau \times \sigma)_g$.

Definition 6.1. The graph $G(f)$ of $f\colon (X,\tau) \to (Y,\sigma)$ is called P_S^β-closed if for each $(x,y) \in (X \times Y)\backslash G(f)$, there exist P_S-open sets $U \subseteq X$ and $V \subseteq Y$ containing x and y, respectively, such that $(U \times \beta(V)) \cap G(f) = \phi$.

The proof of the following lemma follows directly from the above definition.

Lemma 6.2. *A mapping $f\colon (X,\tau) \to (Y,\sigma)$ has P_S^β-closed graph if and only if for each $(x,y) \in (X \times Y)\backslash G(f)$, there exist $U \in P_SO(X)$ containing x and $V \in P_SO(Y)$ containing y such that $f(U) \cap \beta(V) = \phi$.*

Definition 6.3. An operation $\rho\colon (\tau \times \sigma)_g \to P(X \times Y)$ is said to be P_S-associated with γ and β if $\rho(U \times V) = \gamma(U) \times \beta(V)$ holds for each $U \in P_SO(X)$ and $V \in P_SO(Y)$.

Definition 6.4. The operation $\rho\colon (\tau \times \sigma)_g \to P(X \times Y)$ is said to be P_S-regular with respect to γ and β if for each $(x,y) \in X \times Y$ and each P_S-open set W containing (x,y), there exist P_S-open sets U in X and V in Y such that $x \in U$, $y \in V$ and $\gamma(U) \times \beta(V) \subseteq \rho(W)$.

Theorem 6.5. *Let $\rho\colon (\tau\times\tau)_g \to P(X\times X)$ be an P_S-associated operation with γ and γ. If $f\colon (X,\tau) \to (Y,\sigma)$ is an P_S-(γ,β)-continuous mapping and (Y,σ) is an P_S^β-T_2' space, then the set $A = \{(x,y) \in X \times X : f(x) = f(y)\}$ is an P_S^ρ-closed set of $(X \times X, \tau \times \tau)$.*

Proof. We want to prove that P_S-$Cl_\rho(A) \subseteq A$. Let $(x,y) \in (X \times X)\backslash A$. Since (Y,σ) is P_S^β-T_2'. Then there exist two P_S-open sets U and V in (Y,σ) such that $f(x) \in U$, $f(y) \in V$ and $\beta(U) \cap \beta(V) = \phi$. Moreover, for U and V there exist P_S-open sets R and S in (X,τ) such that $x \in R$, $y \in S$ and $f(\gamma(R)) \subseteq \beta(U)$ and $f(\gamma(S)) \subseteq \beta(V)$ since f is P_S-(γ,β)-continuous. Therefore we have $(x,y) \in \gamma(R) \times \gamma(S) = \rho(R \times S) \cap A = \phi$ because $R \times S \in (\tau \times \tau)_g$. This shows that $(x,y) \notin P_SCl_\rho(A)$. □

Corollary 6.6. *Suppose $\rho\colon (\tau \times \tau)_g \to P(X \times X)$ is P_S-associated operation with γ and γ, and it is P_S-regular with γ and γ. A space (X,τ) is P_S^γ-T_2' if and only if the diagonal set $\triangle = \{(x,x) : x \in X\}$ is P_S^ρ-closed of $(X \times X, \tau \times \tau)$.*

Theorem 6.7. *Let* $\rho\colon (\tau\times\sigma)_g \to P(X\times Y)$ *be an* P_S*-associated operation with* γ *and* β*. If* $f\colon (X,\tau) \to (Y,\sigma)$ *is* P_S*-*(γ,β)*-continuous and* (Y,σ) *is* P_S^β*-*T_2'*, then the graph of* f*,* $G(f) = \{(x, f(x)) \in X \times Y\}$ *is an* P_S^ρ*-closed set of* $(X \times Y, \tau \times \sigma)$*.*

Proof. The proof is similar to Theorem 6.5. □

Definition 6.8. Let (X,τ) be a topological space and γ be an operation on $P_SO(X)$. A subset S of X is said to be P_S^γ-compact if for every P_S-open cover $\{U_i, i \in \mathbb{N}\}$ of S, there exists a finite subfamily $\{U_1, U_2, ..., U_n\}$ such that $S \subseteq \gamma(U_1) \cup \gamma(U_2) \cup ... \cup \gamma(U_n)$.

Theorem 6.9. *Suppose that* γ *is* P_S*-regular and* $\rho\colon (\tau\times\sigma)_g \to P(X\times Y)$ *is* P_S*-regular with respect to* γ *and* β*. Let* $f\colon (X,\tau) \to (Y,\sigma)$ *be a mapping whose graph* $G(f)$ *is* P_S^ρ*-closed in* $(X \times Y, \tau \times \sigma)$*. If a subset* S *is* P_S^β*-compact in* (Y,σ)*, then* $f^{-1}(S)$ *is* P_S^γ*-closed in* (X,τ)*.*

Proof. Suppose that $f^{-1}(S)$ is not P_S^γ-closed, then there exist a point x such that $x \in P_SCl_\gamma(f^{-1}(S))$ and $x \notin f^{-1}(S)$. Since $(x,s) \notin G(f)$ and each $s \in S$ and P_S-$Cl_\rho(G(f)) \subseteq G(f)$, there exists an P_S-open set W of $(X\times Y, \tau\times\sigma)$ such that $(x,s) \in W$ and $\beta(W) \cap G(f) = \phi$. By P_S-regularity of ρ, for each $s \in S$ we can take two P_S-open sets $U(s)$ and $V(s)$ in (Y,σ) such that $x \in U(s)$, $s \in V(s)$ and $\gamma(U(s)) \times \beta(V(s)) \subseteq \rho(W)$. Then we have $f(\gamma(U(s))) \cap \beta(V(s)) = \phi$. Since $\{V(s) : s \in S\}$ is P_S-open cover of S, then by P_S^γ-compactness there exists a finite number $s_1, s_2, ..., s_n \in S$ such that $S \subseteq \beta(V(s_1)) \cup \beta(V(s_2)) \cup ... \cup \beta(V(s_n))$. By the P_S-regularity of γ, there exist an P_S-open set U such that $x \in U$, $\gamma(U) \subseteq \gamma(U(s_1)) \cap \gamma(U(s_2)) \cap ... \cap \gamma(U(s_n))$. Therefore, we have $\gamma(U) \cap f^{-1}(S) \subseteq U(s_i) \cap f^{-1}(\beta(V(s_i))) = \phi$. This shows that $x \notin P_SCl_\gamma(f^{-1}(S))$. This is a contradiction. Therefore, $f^{-1}(S)$ is P_S^γ-closed. □

Theorem 6.10. *Suppose that the following condition hold:*

1. $\gamma\colon P_SO(X) \to P(X)$ *is* P_S*-open*
2. $\beta\colon P_SO(Y) \to P(Y)$ *is* P_S*-regular, and*
3. $\rho\colon (\tau \times \sigma)_g \to P(X \times Y)$ *is associated with* γ *and* β*, and* ρ *is* P_S*-regular with respect to* γ *and* β*.*

Let $f\colon (X,\tau) \to (Y,\sigma)$ *be a mapping whose graph* $G(f)$ *is* P_S^ρ*-closed in* $(X \times Y, \tau \times \sigma)$*. If every cover of* A *by* P_S^γ*-open sets of* (X,τ) *has finite sub cover, then* $f(A)$ *is* P_S^β*-closed in* (Y,σ)*.*

Proof. Similar to Theorem 6.9. □

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