SOME PROPERTIES OF COVERED Γ-IDEALS 
IN PO-Γ-SEMIGROUPS

Abul Basar¹§, M.Y. Abbasi², Sabahat Ali Khan³
¹,²,³Department of Mathematics
Jamia Millia Islamia
New Delhi, 110 025, INDIA

Abstract: The concept of covered ideal in semigroups has been introduced by I. Fabrici[1]. In this paper, we introduce covered Γ-ideal in po-Γ-semigroups. We study some results based on covered Γ-ideals in po-Γ-semigroup.

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1. Preliminaries

To start with, we need the following definition

Definition 1.1 [1]. An ideal \( M \) of a semigroup \( S \) is called covered ideal if \( M \subset S(S \setminus M)S \).

We study some properties of po-Γ-semigroups containing covered Γ-ideals. In fact the class of covered Γ-ideals in po-Γ-semigroups are a generalization of the class of covered ideals in semigroups. Consequently, as an application of results of this paper, the corresponding results for Γ-semigroups and semigroup(without order) can be obtained.

The concept of po-Γ-semigroup was introduced by Y. I. Kwon and S. K. Lee[4]. A po-Γ-semigroup is an ordered set \( (S, \leq) \) at the same time a Γ-semigroup \( (S, \Gamma, \cdot) \) such that \( a \leq b \Rightarrow a \cdot \alpha \cdot x \leq b \cdot \alpha \cdot x \) and \( x \cdot \beta \cdot a \leq x \cdot \beta \cdot b \) for all \( a, b, x \in S \) and \( \alpha, \beta \in \Gamma \).
For subsets $A$, $B$ of a po-$\Gamma$-semigroup $S$, the product set $A \cdot B$ of the pair $(A, B)$ relative to $S$ is defined as $A \cdot \Gamma \cdot B = \{a \cdot \gamma \cdot b \mid a \in A, b \in B$ and $\gamma \in \Gamma\}$ and for $A \subseteq S$, the product set $A \cdot A$ relative to $S$ is defined as $A^2 = A \cdot A = A \cdot \Gamma \cdot A$. For $M \subseteq S$, $(M) = \{s \in S \mid s \leq m$ for some $m \in M\}$. Also, we write $(s)$ instead of $(\{s\})$ for $s \in S$.

For $s \in S$, the principal $\Gamma$-ideal generated by $s$ is of the form $I(s) = (s \cup s\Gamma \cup s\Gamma S \cup s\Gamma s\Gamma S)$. We shall denote po-$\Gamma$-semigroup $(S, \Gamma, \cdot, \leq)$ by $S$.

Green’s relation $T$ is defined on $S$ by, for any $a, b \in S$,

$$aTb$$

if and only if $I(a) = I(b)$.

A $T$-class containing an element $a$ of $S$ will be denoted by $J_a$. The $T$-classes of $S$ is a quasi-ordered set where the quasi-order $\preceq$ is defined as follows: For any $a, b \in S$,

$$J_a \preceq J_b$$

if and only if $I(a) \subseteq I(b)$.

The symbol $J_a \prec J_b$ means $J_a \preceq J_b$, but $J_a \neq J_b$. Throughout the paper, for the sake of clarity, we denote $a \cdot \gamma \cdot b$ by $a\gamma b$.

**Example 1.1.**[3] Let $S$ be the set of all $m \times n$ matrices and $\Gamma$ be the set of $n \times m$ matrices, where $m, n$ are positive integers. Furthermore, define $P \leq Q \iff P \subseteq Q$ for all $P, Q \subseteq S$, then $S$ is a po-$\Gamma$-semigroup under the usual matrix multiplication.

**Example 1.2.**[3] Let $P(S)$ be the power set of any nonempty set $S$ and $\Gamma$ a topology on $S$. If we define $LMN = L \cap M \cap N$ and $L \leq N \iff L \subseteq N$ for all $L, N \in P(S)$ and $M \in \Gamma$, then $P(S)$ is a po-$\Gamma$-semigroup.

For further properties of po-$\Gamma$-semigroups and ideal-theoretic results, we refer [2], [3].

2. Main Results

**Lemma 2.1.** Let $s$ be any element of a po-$\Gamma$-semigroup $(S, \Gamma, \cdot, \leq)$. If $I(s)$ is not a proper subset of any principal ideal of $S$, then $J_s$ is maximal.

**Proof.** This is obvious. $\square$

**Lemma 2.2.** Let $J$ be a subset of a po-$\Gamma$-semigroup $(S, \Gamma, \cdot, \leq)$. Then $J$ is a maximal $\exists$-class of $S$ if and only if $S \setminus J$ is a maximal $\Gamma$-ideal of $S$.

**Proof.** Let $J$ be a maximal $\exists$-class of $S$. Then $J = J_s$ for some $s \in S$. We obtain $S\Gamma (S \setminus J_s) \subseteq S \setminus J_s$ and $(S \setminus J_s)\Gamma \subseteq S \setminus J_s$. Let $a \in S \setminus J_s$ and $b \in S$
be such that \( b \leq a \). Then \( J_b \preceq Ja \). If \( b \in J_s \), then \( J_s \) is a maximal \( \mathcal{F} \)-class of \( S \), and therefore \( J_s = J_a \). This is a contradiction. Hence \( b \in S \setminus J_s \). This implies that \( S \setminus J_s \) is an ideal of \( S \). To prove that \( S \setminus J_s \) is a maximal ideal of \( S \), we prove that \( I \) is an ideal of \( S \) such that \( (S \setminus J_s) \subset I \). Then there exists \( z \in I \setminus (S \setminus J_s) \), and so \( z \in J_s \). If \( b \in J_s \), then

\[
I(b) = I(s) = I(z) \subseteq I,
\]

and therefore \( J_s \subseteq I \). Hence \( S = I \).

Conversely, let \( S \setminus J \) is a maximal ideal of \( S \). Set \( s \in S \setminus (S \setminus J) \). If \( a \in J_s \), then \( I(a) = I(s) \); hence \( a \in J \). So \( J_s \subseteq J \). As \( S \setminus J \subseteq (S \setminus J) \cup I(s) \), it follows by assumption that \( (S \setminus J) \cup I(s) = S \). It follows that \( I(a) = I(b) \) for all \( a, b \in J \). Therefore \( a \in J \) implies \( a \in J_s \). Then \( J \subseteq J_s \). Hence \( J = J_s \). If \( J_s \) is not maximal, then there exists \( b \in S \) such that \( J_s \preceq J_b \). It implies that \( I(s) \subseteq I(b) \). It further implies \( I(b) \subseteq S \setminus J \). So \( s \in S \setminus J \). This is a contradiction. \( \square \)

We now define covered \( \Gamma \)-ideals of po-\( \Gamma \)-semigroup.

**Definition 2.1** A proper ideal \( M \) of po-\( \Gamma \)-semigroup \( (S, \Gamma, \cdot, \preceq) \) is called covered \( \Gamma \)-ideal of \( S \) if \( M \subseteq (ST(S \setminus M)\Gamma S) \).

**Proposition 2.1** If \( M_1 \) and \( M_2 \) are different proper ideals of po-\( \Gamma \)-semigroup \( S \) such that \( M_1 \cup M_2 = S \), then both \( M_1 \) and \( M_2 \) are covered ideals of \( S \).

**Proof.** As \( M_1 \cup M_2 = S \), it implies that \( S \setminus M_1 \subseteq M_2 \) and \( S \setminus M_2 \subseteq M_1 \). If \( M_1 \) is a covered \( \Gamma \)-ideal of \( S \), then

\[
M_1 \subseteq (ST(S \setminus M_1)\Gamma S) \subseteq (STM_2\Gamma S) \subseteq M_2.
\]

Hence \( S = M_2 \). This is impossible. Hence the Proposition is established. \( \square \)

Next corollary is a consequence of Proposition 2.1

**Corollary 2.1.** If a po-\( \Gamma \)-semigroup \( (S, \Gamma, \cdot, \preceq) \) contains more than one maximal ideal, then none of them is a covered \( \Gamma \)-ideal of \( S \).

**Proposition 2.2.** Suppose \( S \) is a po-\( \Gamma \)-semigroup. If \( M_1 \) and \( M_2 \) are covered \( \Gamma \)-ideals of \( S \), then \( M_1 \cup M_2 \) is a covered \( \Gamma \)-ideal of \( S \).

**Proof.** Let \( M_1 \) and \( M_2 \) be covered \( \Gamma \)-ideals of \( S \). Then \( M_1 \subseteq (ST(S \setminus M_1)\Gamma S) \) and \( M_2 \subseteq (ST(S \setminus M_2)\Gamma S) \). Let \( x \in M_1 \cup M_2 \). If \( x \in M_1 \), then \( x \in (STa\Gamma S) \) for some \( a \in S \setminus M_1 \). If \( a \in S \setminus (M_1 \cup M_2) \), then \( x \in (ST(S \setminus (M_1 \cup M_2))\Gamma S) \). If \( a \in M_1 \cup M_2 \), then \( a \in M_2 \). Hence \( a \in (STb\Gamma S) \) for some \( b \in S \setminus M_2 \). We have

\[
x \in (STs\Gamma S) \subseteq (ST(STb\Gamma S)\Gamma S) = (STSTb\Gamma STS) \subseteq (STb\Gamma S).
\]
If $b \in M_1$, then $a \in M_1$. This is a contradiction. Therefore $b \in S \setminus (M_1 \cup M_2)$, and therefore $x \in (ST(S \setminus (M_1 \cup M_2))\Gamma S]$. In a similar fashion, $x \in M_2$ implies $x \in (ST(S \setminus (M_1 \cup M_2))\Gamma S]$. This shows that $M_1 \cup M_2$ is a covered $\Gamma$-ideal of $S$. \qed

**Proposition 2.3.** Suppose $M$ is an ideal of po-$\Gamma$-semigroup $S$. If $M_1$ is a covered $\Gamma$-ideal of $S$, then $M_1 \cap M$ is a covered $\Gamma$-ideal of $S$.

**Proof.** If $M_1$ is a covered $\Gamma$-ideal of $S$, then $M_1 \subseteq (ST(S \setminus M_1)\Gamma S]$. Hence $M_1 \cap M \subseteq M_1 \subseteq (ST(S \setminus M_1)\Gamma S] \subseteq (ST(S \setminus (M_1 \cap M))\Gamma S]$. Hence $M_1 \cap M$ is a covered $\Gamma$-ideal of $S$. \qed

**Corollary 2.2.** Suppose $(S, \Gamma, \cdot, \leq)$ is a po-$\Gamma$-semigroup. If $M_1$ and $M_2$ are covered $\Gamma$-ideals of $S$, then $M_1 \cap M_2$ is a covered $\Gamma$-ideal of $S$.

Using Proposition 2.2 and Corollary 2.1, we obtain the following:

**Theorem 2.1.** The set of all covered $\Gamma$-ideals of a po-$\Gamma$-semigroup $(S, \Gamma, \cdot, \leq)$ is a sublattice of the lattice of all ideals of $S$.

**Theorem 2.2.** Suppose $(S, \Gamma, \cdot, \leq)$ is a po-$\Gamma$-semigroup. If $S$ is not simple, then $S$ contains a covered $\Gamma$-ideal.

**Proof.** As $S$ is not simple, $S$ contains a proper $\Gamma$-ideal $T$. Since $T \cap (ST(S \setminus T)\Gamma S]$ is a proper $\Gamma$-ideal of $S$ and $T \cap (ST(S \setminus T)\Gamma S] \subseteq (ST(S \setminus T)\Gamma S] \subseteq (ST(S \setminus (T \cap (ST(S \setminus T)\Gamma S]))\Gamma S]$. It follows that $T \cap (ST(S \setminus T)\Gamma S]$ is a covered $\Gamma$-ideal of $S$.

**Definition 2.2.** A subset $A$ of a po-$\Gamma$-semigroup $(S, \Gamma, \cdot, \leq)$ is called a two-sided base of $S$ if it satisfies the following: 

(i) $S = (A \cup A\Gamma S \cup STA \cup ST A\Gamma S]$;

(ii) If $B$ is a subset of $A$ such that $S = (B \cup B\Gamma S \cup STB \cup ST B\Gamma S]$, then $B = A$.

A covered $\Gamma$-ideal $M$ of a po-$\Gamma$-semigroup $(S, \Gamma, \cdot, \leq)$ is called the greatest covered $\Gamma$-ideal of $S$ if it contains every covered $\Gamma$-ideal of $S$. If a po-$\Gamma$-semigroup $(S, \Gamma, \cdot, \leq)$ contains the greatest covered $\Gamma$-ideal, we identify it by $M^g$. To give a necessary condition so that a po-$\Gamma$-semigroup $(S, \Gamma, \cdot, \leq)$ contains a two-sided base, we need the following lemma.

**Lemma 2.3.** Suppose $(S, \Gamma, \cdot, \leq)$ is a po-$\Gamma$-semigroup containing the greatest covered $\Gamma$-ideal $M^g$. If $M^g \subset (S^3]$, then the following assertions hold:
(i) Every $\mathcal{G}$-class in $(S^3) \setminus M^g$ is maximal;
(ii) $I(s) = (S\Gamma s\Gamma S]$ for all $s$ in $(S^3) \setminus M^g$.

Proof. (ii) Suppose that $M^g \subset (S^3)$. Then $(S^3) \setminus M^g$ is nonempty. Suppose $s \in (S^3) \setminus M^g$. As $M^g$ is a $\Gamma$-ideal of $S$, it follows that $J_s \subset (S^3) \setminus M^g$. Then $s \in (S\Gamma b\Gamma S]$ for some $b \in S$, and therefore $(S\Gamma s\Gamma S] \subset (S\Gamma b\Gamma S]$. As $(S\Gamma b\Gamma S] \subset I(b)$, we obtain $I(s) \subset I(b)$. Let $b$ be not contained in $J_s$; so $J_s \neq J_b$. If $b \in I(s)$, then $I(s) = I(b)$. So $J_s = J_b$. This is impossible. Then $b \in S \setminus I(s)$. It follows that $I(s) \subset (S\Gamma(S \setminus I(s))\Gamma S]$, and $I(s)$ is a covered $\Gamma$-ideal of $S$. By Proposition 2.2, $M^g \cup I(s)$ is a covered $\Gamma$-ideal of $S$. As $s$ is not contained in $M^g$, therefore $M^g \subset M^g \cup I(s)$. This is a contradiction. Therefore $b \in J_s$. Hence

$$I(s) \subset (S\Gamma b\Gamma S] \subset I(b) = I(s).$$

Then

$$I(s) = (S\Gamma b\Gamma S] = I(b).$$

Obviously, $(S\Gamma s\Gamma S] \subset I(s)$. If $b \leq s$, then $I(s) = (S\Gamma b\Gamma S] \subset (S\Gamma s\Gamma S]$. Thus $I(s) \subset (S\Gamma s\Gamma S]$. If $b \leq s$ is false, then $b \in (S\Gamma s \cup s\Gamma S \cup S\Gamma s\Gamma S]$, so

$$S\Gamma b\Gamma S \subset S\Gamma(S\Gamma s\Gamma S] \subset (S\Gamma(S\Gamma s\Gamma S]) \subset (S\Gamma S\Gamma S\Gamma S \subset (S\Gamma s\Gamma S].$$

In a similar fashion, if $b \in (s\Gamma S]$ or $b \in (S\Gamma s\Gamma S]$, then $(S\Gamma b\Gamma S] \subset (S\Gamma s\Gamma S]$. So,

$$I(s) = I(b) = (S\Gamma b\Gamma S] \subset (S\Gamma s\Gamma S].$$

(i) Let $J_s$ be a $\mathcal{G}$-class in $(S^3) \setminus M^g$. Let $J_s$ be not maximal. By Lemma 1.1, we have $I(s) \subset I(c)$ for some $c$ in $S$. Then $s \in I(c)$. So $s \in (c\Gamma S]$ or $s \in (S\Gamma c\Gamma S]$. Each of the cases implies $(S\Gamma s\Gamma S] \subset (S\Gamma c\Gamma S]$, and therefore $I(s) \subset (S\Gamma c\Gamma S]$. As $c \in S \setminus I(s)$, it shows that $I(s)$ is a covered $\Gamma$-ideal of $S$. Therefore $M^g \subset M^g \cup I(s)$. This is a contradiction. Thus any $\mathcal{G}$-class in $(S^3) \setminus M^g$ is maximal. \qed

Theorem 2.3. Suppose $(S, \Gamma, \cdot, \leq)$ is a po-$\Gamma$-semigroup containing the greatest covered $\Gamma$-ideal $M^g$. Then $S$ contains a two-sided base if it satisfies the following:

(i) $M^g \subset (S^3]$;
(ii) For any two elements $a, b \in S \setminus (S^2)$, neither $J_a \leq J_b$ nor $J_b \leq J_a$.

Proof. Let $M^g \subset (S^3]$ and any two elements $a, b \in S \setminus (S^2)$ are incomparable. By

$$M^g \subset (S\Gamma(S \setminus M^g)\Gamma S] \subset (S^3] \subset (S^2] \subset S,$$
there are three families of $\mathcal{I}$-classes to consider: $C_1 = \{J_a \mid a \in S \setminus (S^2)\}$, $C_2 = \{J_a \mid a \in (S^2) \setminus (S^3)\}$, and $C_3 = \{J_a \mid a \in (S^3) \setminus M^g\}$. Consider one element from each $\mathcal{I}$-class from $C_1$ and $C_3$. Suppose $A$ is the set of all elements we take, we claim that $A$ is a two-sided base of $S$. Furthermore, suppose $I(A) = (A \cup \Sigma A \cup A \Gamma S \cup \Sigma \Gamma A S)$. To prove that $S = I(A)$, it is sufficient to prove that $M^g$, $(S^3) \setminus M^g$, $(S^2) \setminus (S^3)$, and $S \setminus (S^2)$ are subsets of $I(A)$.

(a) Let $x \in M^g$. Then $x \in (\Sigma (S \setminus M^g) \Gamma S)$, or equivalently $x \in (\Sigma b \Gamma S)$ for some $b \in S \setminus M^g$. We obtain $b \in J_a$ for some $a \in S \setminus (S^2)$ or $a \in (S^2) \setminus (S^3)$ or $a \in (S^3) \setminus M^g$. If $a \in S \setminus (S^2)$ or $a \in (S^3) \setminus M^g$, then by constructing $A$, we obtain $b \in I(A)$. Hence $x \in I(A)$. Let $a \in (S^2) \setminus (S^3)$. Then $a \leq c \gamma d$ for some $c, d \in S$ and $\gamma \in \Gamma$. As $a$ is not contained in $(S^3)$, it shows that $c, d \in S \setminus (S^2)$. It follows that $a \in I(A)$, and so $b \in I(A)$. Therefore $x \in I(A)$.

(b) If $x \in (S^3) \setminus M^g$, then there exists $a_1 \in A$ such that $x \in I(a_1)$. Therefore $x \in I(A)$.

(c) If $x \in (S^2) \setminus (S^3)$, then the proof is similar to (a).

(d) If $x \in S \setminus (S^2)$, then there exists $a_2 \in A$ such that $x \in I(a_2) \subseteq I(A)$.

Finally, we prove that $A$ is the minimal subset of $S$ such that $S = I(A)$. By Lemma 2.1, it follows that $J_a \in C_3$ is maximal. Moreover, every $J_a \in C_1$ is also maximal as for any elements $a, b \in S \setminus (S^2)$, neither $J_a \succeq J_b$ nor $J_b \succeq J_a$. Now suppose $B$ is a proper subset of $A$ such that $S = (B \cup \Sigma B \Gamma B \cup B \Gamma S \cup \Sigma B \Gamma S)$. Suppose $x \in A \setminus B$. Then $x \leq y$ for some $y \in B \cup \Sigma B \Gamma B \cup B \Gamma S \cup \Sigma B \Gamma S$. As $y \in I(b)$ for some $b \in B$, it implies that $I(x) \subset I(b)$. This contradicts to the construction of $A$.

Suppose $(S, \Gamma, \cdot, \leq)$ is a po-$\Gamma$-semigroup. An ideal $M$ of $S$ is called the greatest ideal of $S$ if it contains every proper $\Gamma$-ideal of $S$. If a po-$\Gamma$-semigroup $(S, \Gamma, \cdot, \leq)$ contains the greatest $\Gamma$-ideal, we denote it by $M^\ast$.

**Theorem 2.3.** Suppose $(S, \Gamma, \cdot, \leq)$ is a po-$\Gamma$-semigroup containing only one maximal ideal $M$. If $M$ is a covered $\Gamma$-ideal of $S$, then $M$ is the greatest $\Gamma$-ideal of $S$.

Proof. This is clear to see since if $T$ is a proper $\Gamma$-ideal of $S$, then $T \subseteq M$. Hence $M = M^\ast$ by Proposition 2.1. $\square$

**Theorem 2.4.** Suppose $(S, \Gamma, \cdot, \leq)$ is a po-$\Gamma$-semigroup with the property that every proper $\Gamma$-ideal of $S$ is a covered $\Gamma$-ideal of $S$. Then either one of the following statements hold:

1. $S$ contains $M^\ast$.
2. $S = (S^2)$ and for any proper $\Gamma$-ideal $M$ and for every $\Gamma$-ideal $I(a) \subseteq M$, there is $b$ in $S \setminus M$ such that $I(a) \subset I(b) \subset S$. 

Proof. This is clear to see. $\square$
Proof. Let $J_x$ and $J_y$ be maximal $\mathfrak{T}$-classes of $S$ such that $J_x \neq J_y$. Then by Lemma 2.1, $M_x = S \setminus J_x$ and $M_y = S \setminus J_y$ are maximal proper $\Gamma$-ideals of $S$ such that none of them is a covered $\Gamma$-ideal of $S$. This is a contradiction. Then $S$ contains no different maximal $\mathfrak{T}$-class. Therefore $S$ contains one maximal $\mathfrak{T}$-class or $S$ does not contain maximal $\mathfrak{T}$-class. If $S$ contains one maximal $\mathfrak{T}$-class $J_x$. Then $M_x = S \setminus J_x$ is a maximal proper $\Gamma$-ideal of $S$. By hypothesis, $M_x$ is a covered $\Gamma$-ideal of $S$. By Theorem 2.3, $M_x = M^*$. Suppose $S$ does not contain maximal $\mathfrak{T}$-class. To prove that $S = (S^2)$. Assume $(S^2) \subset S$. Then there exists $s$ in $S \setminus (S^2)$. If $I(s) = S$, then $S$ contains a maximal $\mathfrak{T}$-class. This is impossible. Then $I(s) \subset S$, and so $I(s) \subseteq (ST(S \setminus I(s)) \Gamma S)$. Then $s \in (S^3) \subseteq (S^2)$. This is a contradiction. Let $M$ be a proper $\Gamma$-ideal of $S$, and suppose $I(a) \subseteq M$. As $M \subseteq (ST(S \setminus M) \Gamma S)$, there exists $b \in S \setminus M$ so that $a \in (STb \Gamma S)$, and hence $I(a) \subseteq I(b) \subseteq S$. As $b \in S \setminus M$, so $I(a) \subset I(b)$. By the hypothesis, $I(b) \subset S$. □

Theorem 2.5. Suppose that a po-$\Gamma$-semigroup $(S, \Gamma, \cdot, \leq)$ satisfies one of the following:

1. $S$ contains $M^*$ which is a covered $\Gamma$-ideal of $S$;
2. $S = (S^2)$, and for any proper $\Gamma$-ideal $M$ and for every $\Gamma$-ideal $I(a) \subseteq M$, there is $b$ in $S \setminus M$ such that $I(a) \subseteq I(b)$.

Then every proper $\Gamma$-ideal of $S$ is a covered $\Gamma$-ideal of $S$.

Proof. Suppose $M$ is a proper $\Gamma$-ideal of $S$. Suppose $S$ satisfies (1). Then $M \subseteq M^*$. As $S \setminus M^* \subseteq S \setminus M$, it implies that

$$M \subseteq M^* \subseteq (ST(S \setminus M^*) \Gamma S) \subseteq (ST(S \setminus M) \Gamma S).$$

Then $M$ is a covered $\Gamma$-ideal of $S$.

Now suppose that the condition (2) holds. Let $x \in M$; so $I(x) \subseteq M$. We have $I(x) \subseteq I(b)$. As $S = (S^2)$, therefore $S = (S^3)$. Hence $b \in (STd \Gamma S)$ for some $d \in S$. As $b \in S \setminus M$, therefore $d \in S \setminus M$. Hence $x \in (STd \Gamma S) \subseteq (ST(S \setminus M) \Gamma S)$. It implies that $M \subseteq (ST(S \setminus M) \Gamma S)$.

References