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SOME PROPERTIES OF COVERED Γ-IDEALS IN PO-Γ-SEMIGROUPS

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Abstract: The concept of covered ideal in semigroups has been introduced by I. Fabrici[1]. In this paper, we introduce covered Γ-ideal in po-Γ-semigroups. We study some results based on covered Γ-ideals in po-Γ-semigroup.

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1. Preliminaries

To start with, we need the following definition

Definition 1.1 [1]. An ideal M of a semigroup S is called covered ideal if $M \subset S(S \setminus M)S$.

We study some properties of po-Γ-semigroups containing covered Γ-ideals. In fact the class of covered Γ-ideals in po-Γ-semigroups are a generalization of the class of covered ideals in semigroups. Consequently, as an application of results of this paper, the corresponding results for Γ-semigroups and semigroup(without order) can be obtained.

The concept of po-Γ-semigroup was introduced by Y. I. Kwon and S. K. Lee[4]. A po-Γ-semigroup is an ordered set $(S, \leq)$ at the same time a Γ-semigroup $(S, \Gamma, \cdot)$ such that $a \leq b \Rightarrow a \cdot \alpha \cdot x \leq b \cdot \alpha \cdot x$ and $x \cdot \beta \cdot a \leq x \cdot \beta \cdot b$ for all $a, b, x \in S$ and $\alpha, \beta \in \Gamma$.

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For subsets A, B of a po-Γ-semigroup S, the product set $A \cdot B$ of the pair (A, B) relative to S is defined as $A \cdot \Gamma \cdot B = \{a \cdot \gamma \cdot b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$ and for $A \subseteq S$, the product set $A \cdot A$ relative to S is defined as $A^2 = A \cdot A = A \cdot \Gamma \cdot A$. For $M \subseteq S$, $(M] = \{s \in S \mid s \leq m \text{ for some } m \in M\}$. Also, we write $(s]$ instead of $(\{s\}]$ for $s \in S$.

For $s \in S$, the principal Γ-ideal generated by s is of the form $I(s) = (s \cup S\Gamma s \cup s\Gamma S \cup S\Gamma s\Gamma S]$. We shall denote po-$\Gamma$-semigroup $(S, \Gamma, \cdot, \leq)$ by S. Green's relation $\mathfrak{T}$ is defined on S by, for any $a, b \in S$,

$$a\mathfrak{T}b \text{ if and only if I(a) = I(b).}$$

A $\mathfrak{T}$-class containing an element a of S will be denoted by J_a. The $\mathfrak{T}$-classes of S is a quasi-ordered set where the quasi-order $\preceq$ is defined as follows: For any $a, b \in S$,

$$J_a \preceq J_b \text{ if and only if I(a)} \subseteq \text{I(b).}$$

The symbol $J_a \prec J_b$ means $J_a \preceq J_b$, but $J_a \neq J_b$. Throughout the paper, for the sake of clarity, we denote $a \cdot \gamma \cdot b$ by $a\gamma b$.

Example 1.1.[3] Let S be the set of all $m \times n$ matrices and Γ be the set of $n \times m$ matrices, where m, n are positive integers. Furthermore, define $P \leq Q \Leftrightarrow P \subseteq Q$ for all $P, Q \subseteq S$, then S is a po-Γ-semigroup under the usual matrix multiplication.

Example 1.2.[3] Let $P(S)$ be the power set of any nonempty set S and Γ a topology on S. If we define $LMN = L \cap M \cap N$ and $L \leq N \Leftrightarrow L \subseteq N$ for all $L, N \in P(S)$ and $M \in \Gamma$, then $P(S)$ is a po-Γ-semigroup.

For further properties of po-Γ-semigroups and ideal-theoretic results, we refer [2], [3].

2. Main Results

Lemma 2.1. *Let s be any element of a po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$. If $I(s)$ is not a proper subset of any principal ideal of S, then J_s is maximal.*

Proof. This is obvious. □

Lemma 2.2. *Let J be a subset of a po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$. Then J is a maximal $\mathfrak{T}$-class of S if and only if $S \setminus J$ is a maximal Γ-ideal of S.*

Proof. Let J be a maximal $\mathfrak{T}$-class of S. Then $J = J_s$ for some s in S. We obtain $S\Gamma(S \setminus J_s) \subseteq S \setminus J_s$ and $(S \setminus J_s)\Gamma S \subseteq S \setminus J_s$. Let $a \in S \setminus J_s$ and $b \in S$

be such that $b \leq a$. Then $J_b \preceq Ja$. If $b \in J_s$, then J_s is a maximal $\mathfrak{T}$-class of S, and therefore $J_s = J_a$. This is a contradiction. Hence $b \in S \setminus J_s$. This implies that $S \setminus J_s$ is an ideal of S. To prove that $S \setminus J_s$ is a maximal ideal of S, we prove that I is an ideal of S such that $(S \setminus J_s) \subset I$. Then there exists $z \in I \setminus (S \setminus J_s)$, and so $z \in J_s$. If $b \in J_s$, then

$$I(b) = I(s) = I(z) \subseteq I,$$

and therefore $J_s \subseteq I$. Hence $S = I$.

Conversely, let $S \setminus J$ is a maximal ideal of S. Set $s \in S \setminus (S \setminus J)$. If $a \in J_s$, then $I(a) = I(s)$; hence $a \in J$. So $J_s \subseteq J$. As $S \setminus J \subset (S \setminus J) \cup I(s)$, it follows by assumption that $(S \setminus J) \cup I(s) = S$. It follows that $I(a) = I(b)$ for all $a, b \in J$. Therefore $a \in J$ implies $a \in J_s$. Then $J \subseteq J_s$. Hence $J = J_s$. If J_s is not maximal, then there exists $b \in S$ such that $J_s \prec J_b$. It implies that $I(s) \subset I(b)$. It further implies $I(b) \subseteq S \setminus J$. So $s \in S \setminus J$. This is a contradiction. □

We now define covered Γ-ideals of po-Γ-semigroup.

Definition 2.1 A proper ideal M of po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$ is called covered Γ-ideal of S if $M \subseteq (S\Gamma(S \setminus M)\Gamma S]$.

Proposition 2.1 *If M_1 and M_2 are different proper ideals of po-Γ-semigroup S such that $M_1 \cup M_2 = S$, then both M_1 and M_2 are covered ideals of S.*

Proof. As $M_1 \cup M_2 = S$, it implies that $S \setminus M_1 \subseteq M_2$ and $S \setminus M_2 \subseteq M_1$. If M_1 is a covered Γ-ideal of S, then

$$M_1 \subseteq (S\Gamma(S \setminus M_1)\Gamma S] \subseteq (S\Gamma M_2 \Gamma S] \subseteq M_2.$$

Hence $S = M_2$. This is impossible. Hence the Proposition is established. □

Next corollary is a consequence of Proposition 2.1

Corollary 2.1. *If a po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$ contains more than one maximal ideal, then none of them is a covered Γ-ideal of S.*

Proposition 2.2. *Suppose S is a po-Γ-semigroup. If M_1 and M_2 are covered Γ-ideals of S, then $M_1 \cup M_2$ is a covered Γ-ideal of S.*

Proof. Let M_1 and M_2 be covered Γ-ideals of S. Then $M_1 \subseteq (S\Gamma(S \setminus M_1)\Gamma S]$ and $M_2 \subseteq (S\Gamma(S \setminus M_2)\Gamma S]$. Let $x \in M_1 \cup M_2$. If $x \in M_1$, then $x \in (S\Gamma a \Gamma S]$ for some $a \in S \setminus M_1$. If $a \in S \setminus (M_1 \cup M_2)$, then $x \in (S\Gamma(S \setminus (M_1 \cup M_2))\Gamma S]$. If $a \in M_1 \cup M_2$, then $a \in M_2$. Hence $a \in (S\Gamma b \Gamma S]$ for some $b \in S \setminus M_2$. We have

$$x \in (S\Gamma s \Gamma S] \subseteq (S\Gamma(S\Gamma b \Gamma S]\Gamma S] = (S\Gamma S \Gamma b \Gamma S \Gamma S] \subseteq (S\Gamma b \Gamma S].$$

If $b \in M_1$, then $a \in M_1$. This is a contradiction. Therefore $b \in S \setminus (M_1 \cup M_2)$, and therefore $x \in (S\Gamma(S \setminus (M_1 \cup M_2))\Gamma S]$. In a similar fashion, $x \in M_2$ implies $x \in (S\Gamma(S \setminus (M_1 \cup M_2))\Gamma S]$. This shows that $M_1 \cup M_2$ is a covered Γ-ideal of S. □

Proposition 2.3. *Suppose M is an ideal of po-Γ-semigroup S. If M_1 is a covered Γ-ideal of S, then $M_1 \cap M$ is a covered Γ-ideal of S.*

Proof. If M_1 is a covered Γ-ideal of S, then $M_1 \subseteq (S\Gamma(S \setminus M_1)\Gamma S]$. Hence

$$M_1 \cap M \subseteq M_1 \subseteq (S\Gamma(S \setminus M_1)\Gamma S] \subseteq (S\Gamma(S \setminus (M_1 \cap M))\Gamma S].$$

Hence $M_1 \cap M$ is a covered Γ-ideal of S. □

Corollary 2.2. *Suppose $(S, \Gamma, \cdot, \leq)$ is a po-Γ-semigroup. If M_1 and M_2 are covered Γ-ideals of S, then $M_1 \cap M_2$ is a covered Γ-ideal of S.*

Using Proposition 2.2 and Corollary 2.1, we obtain the following:

Theorem 2.1. *The set of all covered Γ-ideals of a po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$ is a sublattice of the lattice of all ideals of S.*

Theorem 2.2. *Suppose $(S, \Gamma, \cdot, \leq)$ is a po-Γ-semigroup. If S is not simple, then S contains a covered Γ-ideal.*

Proof. As S is not simple, S contains a proper Γ-ideal T. Since $T \cap (S\Gamma(S \setminus T)\Gamma S]$ is a proper Γ-ideal of S and

$$T \cap (S\Gamma(S \setminus T)\Gamma S] \subseteq (S\Gamma(S \setminus T)\Gamma S] \subseteq (S\Gamma(S \setminus (S \setminus (T \cap (S\Gamma(S \setminus T)\Gamma S]))\Gamma S].$$

It follows that $T \cap (S\Gamma(S \setminus T)\Gamma S]$ is a covered Γ-ideal of S.

Definition 2.2. A subset A of a po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$ is called a two-sided base of S if it satisfies the following:

(i) $S = (A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S]$;

(ii) If B is a subset of A such that $S = (B \cup B\Gamma S \cup S\Gamma B \cup S\Gamma B\Gamma S]$, then $B = A$.

A covered Γ-ideal M of a po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$ is called the greatest covered Γ-ideal of S if it contains every covered Γ-ideal of S. If a po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$ contains the greatest covered Γ-ideal, we identify it by M^g. To give a necessary condition so that a po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$ contains a two-sided base, we need the following lemma.

Lemma 2.3. *Suppose $(S, \Gamma, \cdot, \leq)$ is a po-Γ-semigroup containing the greatest covered Γ-ideal M^g. If $M^g \subset (S^3]$, then the following assertions hold:*

(i) Every $\mathfrak{T}$*-class in* $(S^3] \setminus M^g$ *is maximal;*
(ii) $I(s) = (S\Gamma s\Gamma S]$ *for all* s *in* $(S^3] \setminus M^g$*.*

Proof. (ii) Suppose that $M^g \subset (S^3]$. Then $(S^3] \setminus M^g$ is nonempty. Suppose $s \in (S^3] \setminus M^g$. As M^g is a Γ-ideal of S, it follows that $J_s \subseteq (S^3] \setminus M^g$. Then $s \in (S\Gamma b\Gamma S]$ for some $b \in S$, and therefore $(S\Gamma s\Gamma S] \subseteq (S\Gamma b\Gamma S]$. As $(S\Gamma b\Gamma S] \subseteq I(b)$, we obtain $I(s) \subseteq I(b)$. Let b be not contained in J_s; so $J_s \neq J_b$. If $b \in I(s)$, then $I(s) = I(b)$. So $J_s = J_b$. This is impossible. Then $b \in S \setminus I(s)$. It follows that $I(s) \subseteq (S\Gamma(S \setminus I(s))\Gamma S]$, and $I(s)$ is a covered Γ-ideal of S. By Proposition 2.2, $M^g \cup I(s)$ is a covered Γ-ideal of S. As s is not contained in M^g, therefore $M^g \subset M^g \cup I(s)$. This is a contradiction. Therefore $b \in J_s$. Hence

$$I(s) \subseteq (S\Gamma b\Gamma S] \subseteq I(b) = I(s).$$

Then

$$I(s) = (S\Gamma b\Gamma S] = I(b).$$

Obviously, $(S\Gamma s\Gamma S] \subseteq I(s)$. If $b \leq s$, then $I(s) = (S\Gamma b\Gamma S] \subseteq (S\Gamma s\Gamma S]$. Thus $I(s) \subseteq (S\Gamma s\Gamma S]$. If $b \leq s$ is false, then $b \in (S\Gamma s \cup s\Gamma S \cup S\Gamma s\Gamma S]$, so

$$S\Gamma b\Gamma S \subseteq S\Gamma(S\Gamma s]\Gamma S \subseteq (S\Gamma(S\Gamma s]\Gamma S] \subseteq (S\Gamma S\Gamma s\Gamma S] \subseteq (S\Gamma s\Gamma S].$$

In a similar fashion, if $b \in (s\Gamma S]$ or $b \in (S\Gamma s\Gamma S]$, then $(S\Gamma b\Gamma S] \subseteq (S\Gamma s\Gamma S]$. So,

$$I(s) = I(b) = (S\Gamma b\Gamma S] \subseteq (S\Gamma s\Gamma S].$$

(i) Let J_s be a $\mathfrak{T}$-class in $(S^3] \setminus M^g$. Let J_s be not maximal. By Lemma 1.1, we have $I(s) \subset I(c)$ for some c in S. Then $s \in I(c)$. So $s \in (c]$ or $s \in (S\Gamma c]$ or $s \in (c\Gamma S]$ or $s \in (S\Gamma c\Gamma S]$. Each of the cases implies $(S\Gamma s\Gamma S] \subseteq (S\Gamma c\Gamma S]$, and therefore $I(s) \subseteq (S\Gamma c\Gamma S]$. As $c \in S \setminus I(s)$, it shows that $I(s)$ is a covered Γ-ideal of S. Therefore $M^g \subset M^g \cup I(s)$. This is a contradiction. Thus any $\mathfrak{T}$-class in $(S^3] \setminus M^g$ is maximal. □

Theorem 2.3. *Suppose* $(S, \Gamma, \cdot, \leq)$ *is a po-*Γ*-semigroup containing the greatest covered* Γ*-ideal* M^g*. Then* S *contains a two-sided base if it satisfies the following:*
(i) $M^g \subset (S^3]$*;*
(ii) For any two elements $a, b \in S \setminus (S^2]$*, neither* $J_a \preceq J_b$ *nor* $J_b \preceq J_a$*.*

Proof. Let $M^g \subset (S^3]$ and any two elements $a, b \in S \setminus (S^2]$ are incomparable. By

$$M^g \subseteq (S\Gamma(S \setminus M^g)\Gamma S] \subseteq (S^3] \subseteq (S^2] \subseteq S,$$

there are three families of $\mathfrak{T}$-classes to consider: $C_1 = \{J_a \mid a \in S \setminus (S^2]\}$, $C_2 = \{J_a \mid a \in (S^2] \setminus (S^3]\}$, and $C_3 = \{J_a \mid a \in (S^3] \setminus M^g\}$. Consider one element from each $\mathfrak{T}$-class from C_1 and C_3. Suppose A is the set of all elements we take, we claim that A is a two-sided base of S. Furthermore, suppose $I(A) = (A \cup S\Gamma A \cup A\Gamma S \cup S\Gamma A\Gamma S]$. To prove that $S = I(A)$, it is sufficient to prove that M^g, $(S^3] \setminus M^g$, $(S^2] \setminus (S^3]$, and $S \setminus (S^2]$ are subsets of $I(A)$.

(a) Let $x \in M^g$. Then $x \in (S\Gamma(S \setminus M^g)\Gamma S]$, or equivalently $x \in (S\Gamma b\Gamma S]$ for some $b \in S \setminus M^g$. We obtain $b \in J_a$ for some $a \in S \setminus (S^2]$ or $a \in (S^2] \setminus (S^3]$ or $a \in (S^3] \setminus M^g$. If $a \in S \setminus (S^2]$ or $a \in (S^3] \setminus M^g$, then by constructing A, we obtain $b \in I(A)$. Hence $x \in I(A)$. Let $a \in (S^2] \setminus (S^3]$. Then $a \leq c\gamma d$ for some $c, d \in S$ and $\gamma \in \Gamma$. As a is not contained in $(S^3]$, it shows that $c, d \in S \setminus (S^2]$. It follows that $a \in I(A)$, and so $b \in I(A)$. Therefore $x \in I(A)$.

(b) If $x \in (S^3] \setminus M^g$, then there exists $a_1 \in A$ such that $x \in I(a_1)$. Therefore $x \in I(A)$.

(c) If $x \in (S^2] \setminus (S^3]$, then the proof is similar to (a).

(d) If $x \in S \setminus (S^2]$, then there exists $a_2 \in A$ such that $x \in I(a_2) \subseteq I(A)$.

Finally, we prove that A is the minimal subset of S such that $S = I(A)$. By Lemma 2.1, it follows that $J_a \in C_3$ is maximal. Moreover, every $J_a \in C_1$ is also maximal as for any elements $a, b \in S \setminus (S^2]$, neither $J_a \preceq J_b$ nor $J_b \preceq J_a$. Now suppose B is a proper subset of A such that $S = (B \cup S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S]$. Suppose $x \in A \setminus B$. Then $x \leq y$ for some $y \in B \cup S\Gamma B \cup B\Gamma S \cup S\Gamma B\Gamma S$. As $y \in I(b)$ for some $b \in B$, it implies that $I(x) \subset I(b)$. This contradicts to the construction of A. □

Suppose $(S, \Gamma, \cdot, \leq)$ is a po-Γ-semigroup. An ideal M of S is called the greatest ideal of S if it contains every proper Γ-ideal of S. If a po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$ contains the greatest Γ-ideal, we denote it by $M^{\square}$.

Theorem 2.3. *Suppose $(S, \Gamma, \cdot, \leq)$ is a po-Γ-semigroup containing only one maximal ideal M. If M is a covered Γ-ideal of S, then M is the greatest Γ-ideal of S.*

Proof. This is clear to see since if T is a proper Γ-ideal of S, then $T \subseteq M$. Hence $M = M^{\square}$ by Proposition 2.1. □

Theorem 2.4. *Suppose $(S, \Gamma, \cdot, \leq)$ is a po-Γ-semigroup with the property that every proper Γ-ideal of S is a covered Γ-ideal of S. Then either one of the following statements hold:*

(1) S contains $M^{\square}$;

(2) $S = (S^2]$ and for any proper Γ-ideal M and for every Γ-ideal $I(a) \subseteq M$, there is b in $S \setminus M$ such that $I(a) \subset I(b) \subset S$.

Proof. Let J_x and J_y be maximal $\mathfrak{T}$-classes of S such that $J_x \neq J_y$. Then by Lemma 2.1, $M_x = S \setminus J_x$ and $M_y = S \setminus J_y$ are maximal proper Γ-ideals of S such that none of them is a covered Γ-ideal of S. This is a contradiction. Then S contains no different maximal $\mathfrak{T}$-class. Therefore S contains one maximal $\mathfrak{T}$-class or S does not contain maximal $\mathfrak{T}$-class. If S contains one maximal $\mathfrak{T}$-class J_x. Then $M_x = S \setminus J_x$ is a maximal proper Γ-ideal of S. By hypothesis, M_x is a covered Γ-ideal of S. By Theorem 2.3, $M_x = M^{\square}$. Suppose S does not contain maximal $\mathfrak{T}$-class. To prove that $S = (S^2]$. Assume $(S^2] \subset S$. Then there exists s in $S \setminus (S^2]$. If $I(s) = S$, then S contains a maximal $\mathfrak{T}$-class. This is impossible. Then $I(s) \subset S$, and so $I(s) \subseteq (S\Gamma(S \setminus I(s))\Gamma S]$. Then $s \in (S^3] \subseteq (S^2]$. This is a contradiction. Let M be a proper Γ-ideal of S, and suppose $I(a) \subseteq M$. As $M \subseteq (S\Gamma(S \setminus M)\Gamma S]$, there exists $b \in S \setminus M$ so that $a \in (S\Gamma b\Gamma S]$, and hence $I(a) \subseteq I(b) \subseteq S$. As $b \in S \setminus M$, so $I(a) \subset I(b)$. By the hypothesis, $I(b) \subset S$. □

Theorem 2.5. *Suppose that a po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$ satisfies one of the following:*

(1) S contains $M^{\square}$ which is a covered Γ-ideal of S;

(2) $S = (S^2]$, and for any proper Γ-ideal M and for every Γ-ideal $I(a) \subseteq M$, there is b in $S \setminus M$ such that $I(a) \subseteq I(b)$.

Then every proper Γ-ideal of S is a covered Γ-ideal of S.

Proof. Suppose M is a proper Γ-ideal of S. Suppose S satisfies (1). Then $M \subseteq M^{\square}$. As $S \setminus M^{\square} \subseteq S \setminus M$, it implies that

$$M \subseteq M^{\square} \subseteq (S\Gamma(S \setminus M^{\square})\Gamma S] \subseteq (S\Gamma(S \setminus M)\Gamma S].$$

Then M is a covered Γ-ideal of S.

Now suppose that the condition (2) holds. Let $x \in M$; so $I(x) \subseteq M$. We have $I(x) \subset I(b)$. As $S = (S^2]$, therefore $S = (S^3]$. Hence $b \in (S\Gamma d\Gamma S]$ for some $d \in S$. As $b \in S \setminus M$, therefore $d \in S \setminus M$. Hence $x \in (S\Gamma d\Gamma S] \subseteq (S\Gamma(S \setminus M)\Gamma S]$. It implies that $M \subseteq (S\Gamma(S \setminus M)\Gamma S]$.

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