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# FIXED POINTS OF CHATTERJEE AND
# CIRIC CONTRACTIONS ON AN $S$-METRIC SPACE

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**Abstract:**   Unique fixed points are obtained for Chatterjee and Ciric contractions on an $S$-metric space, which are then shown to be $S$-contractive fixed points.

## 1. Introduction

Let $X$ be a nonempty set. Sedghi et al [7] introduced an $S$-metric $S : X \times X \times X \to [0, \infty)$ on $X$ satisfying the following conditions:

(S1)  $S(x, y, z) = 0$ if and only if $x, y, z \in X$ are such that $x = y = z$,

(S2)  $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.

The pair $(X, S)$ is called an $S$-metric space. We obtain from Axiom (S2) that

$$S(x, x, y) = S(y, y, x) \text{ for all } x, y \in X. \tag{1.1}$$

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**Definition 1.1.** A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a $S$-metric space $(X, S)$ is said to be $S$-convergent, if there exists a point $x$ in $X$ such that $S(x_n, x_n, x) \to 0$ as $n \to \infty$.

**Definition 1.2.** A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a $S$-metric space $(X, S)$ is said to be $S$-Cauchy if $\lim_{n,m \to \infty} S(x_n, x_n, x_m) = 0$.

**Definition 1.3.** The space $(X, S)$ is said to be $S$-complete, if every $S$-Cauchy sequence in $X$ converges in it.

The well-known infimum property of real numbers states that a nonempty and bounded set of real numbers has an infimum in $\mathbb{R}$. In particular,

**Lemma 1.1.** *If $S$ is a nonempty subset of nonnegative real numbers, then $\alpha = \inf S \geq 0$ and $\lim_{n \to \infty} p_n = \alpha$ for some sequence $\langle p_n \rangle_{n=1}^{\infty}$ in $S$.*

As an elementary application of Lemma 1.1, unique fixed points of a Chatterjee and Ciric- contractions are obtained in $S$-metric space. Further, the unique fixed points for these two contractions are shown to be $S$-contractive fixed points.

## 2. Main Results

Our first result is

**Theorem 2.1.** *Let $f$ be a Chatterjee contraction on a complete $S$-metric space $(X, S)$ with the choice*

$$S(fx, fx, fy) \leq \alpha \max\{S(fx, fx, y), S(fy, fy, y)\} \ \text{for all} \ x, y \in X, \quad (2.1)$$

*where $0 \leq \alpha < 1/3$. Then $f$ has a unique fixed point.*

*Proof.* We divide the proof into various steps:

### Step 1 – Existence of the infimum

Define $A = \{S(fx, fx, x) : x \in X\}$. Then by Lemma 1.1, the infimum infimum $a$ of $A$ exists and is nonnegative.

### Step 2 – Vanishing infimum

If $a > 0$, writing $y = fx$ in (2.1) and using (1.1), we get

$$S(fx, fx, f^2x) \leq \alpha \max\{S(f^2x, fx, fx), S(f^2x, f^2x, x)\}$$

$$\leq \alpha[S(f^2x, f^2x, fx) + S(f^2x, f^2x, fx) + S(x, x, fx)]$$
$$= \alpha[2S(fx, fx, f^2x) + S(fx, fx, x)]$$

or

$$(1 - 2\alpha)S(fx, fx, f^2x) \leq \alpha S(fx, fx, x)$$
$$S(fx, fx, f^2x) \leq \left(\frac{\alpha}{1-2\alpha}\right)S(fx, fx, x)$$

This implies that $0 < a \leq \alpha a < a$, which is again a contradiction. Hence $a = 0$.

## Step 3 – Existence of a sequence

Hence, there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $X$ such that

$$S(fx_n, fx_n, x_n) \in A \text{ for all } n = 1, 2, 3, ... \text{ and } \lim_{n \to \infty} S(fx_n, fx_n, x_n) = 0. \quad (2.2)$$

**Step 4 $- \langle x_n \rangle_{n=1}^{\infty}$ is $S$-Cauchy**
In fact, by $(S2)$ and $(1.1)$, we have

$$S(x_n, x_n, x_m) \leq S(x_n, x_n, fx_n) + S(x_n, x_n, fx_n) + S(x_m, x_m, fx_n)$$
$$= 2S(x_n, x_n, fx_n) + S(x_m, x_m, fx_n)$$
$$\leq 2S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m) + S(x_m, x_m, fx_m)$$
$$+ S(fx_n, fx_n, fx_m)$$
$$= 2[S(fx_n, fx_n, x_n) + S(fx_m, fx_m, x_m)]$$
$$+ S(fx_n, fx_n, fx_m). \quad (2.3)$$

Now, with $x = x_n$ and $y = x_m$, $(2.1)$ gives,

$$S(fx_n, fx_n, fx_m) \leq \alpha \max\{S(fx_n, fx_n, x_m), S(fx_m, fx_m, x_n)\}$$
$$\leq \alpha \max\{[2S(fx_n, fx_n, x_n) + S(x_m, x_m, x_n)],$$
$$[2S(fx_m, fx_m, x_m) + S(x_n, x_n, x_m)]\}$$
$$= \alpha[2S(fx_n, fx_n, x_n) + 2S(fx_m, fx_m, x_m) + 2S(x_n, x_n, x_m)]$$

Inserting this in $(2.3)$, we get

$$S(x_n, x_n, x_m) \leq (2\alpha + 1)[S(fx_n, fx_n, x_n) + S(fx_m, fx_m, x_m)]$$
$$+ 2\alpha S(x_n, x_n, x_m)$$

or

$$(1 - 2\alpha)S(x_n, x_n, x_m) \leq (2\alpha + 1)[S(fx_n, fx_n, x_n) + S(fx_m, fx_m, x_m)]$$

so that

$$S(x_n, x_n, x_m) \leq \left(\tfrac{2\alpha+1}{1-2\alpha}\right)[S(x_n, x_n, fx_n) + S(x_m, x_m, fx_m)].$$

Applying the limit as $m, n \to \infty$ in this and using (2.2) we obtain that $\langle x_n \rangle_{n=1}^{\infty}$ is a $S$-Cauchy sequence in $X$.

**Step 5 – $S$-convergence**

Since, $X$ is $S$-complete, we find the point $p$ in $X$ such that

$$\lim_{n \to \infty} x_n = p. \tag{2.4}$$

**Step 6 – $S$-convergent limit as a fixed point**

Again repeatedly using $(S2)$,

$$
\begin{aligned}
S(fp, fp, p) &\leq S(fp, fp, fx_n) + S(fp, fp, fx_n) + S(p, p, fx_n). \\
&= 2S(fp, fp, fx_n) + S(p, p, fx_n) \\
&= 2S(fx_n, fx_n, fp) + S(fx_n, fx_n, p) \tag{2.5}
\end{aligned}
$$

Now, from (2.1) with $x = x_n$ and $y = p$, it follows that

$$
\begin{aligned}
S(fx_n, fx_n, fp) &\leq \alpha \max\left\{S(fx_n, fx_n, p), S(fp, fp, x_n)\right\} \\
&\leq \alpha[S(fx_n, fx_n, p) + S(fp, fp, x_n)] \tag{2.6}
\end{aligned}
$$

Substituting (2.6) in (2.5), we get

$$
\begin{aligned}
S(fp, fp, p) &\leq 2\alpha[S(fx_n, fx_n, p) + S(fp, fp, x_n)] + S(fx_n, fx_n, p) \\
&= (2\alpha + 1)S(fx_n, fx_n, p) + 2\alpha S(fp, fp, x_n)
\end{aligned}
$$

In the limiting case as $n \to \infty$, this in view of (2.2) and (2.4) implies $S(fp, fp, p) = 0$ or $fp = p$. Thus $p$ is a fixed point.

**Step 7 – Uniqueness of the fixed point**

Let $q$ be another fixed point of $f$. Then, (2.1) with $x = p$ and $y = q$ gives

$$
\begin{aligned}
S(p, p, q) &= S(fp, fp, fq) \\
&\leq \alpha \max\left\{S(fp, fp, q), S(fq, fq, p)\right\}
\end{aligned}
$$

$$= \alpha \max \big\{ S(p,p,q), S(q,q,p) \big\}$$
$$= \alpha S(p,p,q)$$

or

$$(1 - \alpha)S(p,p,q) \leq 0$$

so that $p = q$. That is, $p$ is the unique fixed point of $f$.  □

Our next result is:

**Theorem 2.2.** *Let $f$ be a Ciric-type contraction on a complete $S$-metric space $(X, S)$ such that*

$$S(fx, fx, fy) \leq \alpha \max \big\{ S(x,x,y), S(fx,fx,x), S(fx,fx,y),$$
$$S(fy,fy,x), S(fy,fy,y) \big\} \qquad (2.7)$$

*for all $x, y \in X$, where $0 \leq \alpha < 1/3$. Then $f$ has a unique fixed point.*

A unique fixed point $p$ for (2.7) is obtained, similar to the previous proof and is omitted here.

## 3. $S$-Contractive Fixed Point

The notion of a $G$-metric space was introduced by Mustafa and Sims in [1], as a generalization of a metric space. In this setting, contractive fixed points were introduced in [2]. For further study on this idea, one can refer to [3, 4, 5, 6].

Now, we have

**Definition 3.1.** Let $f$ be a self-map on an $S$-metric space $(X, S)$. A fixed point $p$ of $f$ is a contractive fixed point, if for every $x_0 \in X$, the $f$-orbit $O_f(x_0) = \langle x_0, fx_0, ..., f^n x_0, ... \rangle$ converges to $p$.

We now show that the unique fixed point $p$ of Chatterjee contraction (2.1) is an $S$-contractive fixed point.

*Proof.* Writing $x = f^{n-1}x_0, y = p$ in (2.1), we get

$$S(f^n x, f^n x, p) = S(f^n x, f^n x, fp)$$
$$\leq \alpha \max \big\{ S(f^n x, f^n x, p), S(fp, fp, f^{n-1}x) \big\}$$
$$= \alpha \max \big\{ S(f^n x, f^n x, p), S(f^{n-1}x, f^{n-1}x, fp) \big\}$$

$$\leq \max \left\{ S(f^n x, f^n x, p), [2S(f^{n-1}x, f^{n-1}x, f^n x) + S(fp, fp, f^n x)] \right\}$$
$$= \max \left\{ S(f^n x, f^n x, p), [2S(f^{n-1}x, f^{n-1}x, f^n x) + S(f^n x, f^n x, p)] \right\}$$
$$\leq \alpha[2S(f^{n-1}x, f^{n-1}x, f^n x) + S(f^n x, f^n x, p)]$$

or

$$(1-\alpha)S(f^n x, f^n x, p) \leq 2\alpha S(f^{n-1}x, f^{n-1}x, f^n x)$$
$$S(f^n x, f^n x, p) \leq \left( \tfrac{2\alpha}{1-\alpha} \right) S(f^{n-1}x, f^{n-1}x, f^n x). \qquad (3.1)$$

Proceeding the limit as $n \to \infty$ in (3.1) , we get $S(f^n x, f^n x, p) \to 0$. Thus $f^n x_0 \to p$ for each $x_0 \in X$. Thus $p$ is a $S$-contractive fixed point of $f$. □

We finally show that the unique fixed point $p$ of Ciric contraction (2.7) is an $S$-contractive fixed point as follows:

*Proof.* Writing $x = f^{n-1}x_0, y = p$ in (2.7), and then using (1.1), we get

$$S(f^n x, f^n x, fp) = S(f^n x, f^n x, p)$$
$$\leq \alpha \max \left\{ S(f^{n-1}x, f^{n-1}x, p), S(f^n x, f^n x, f^{n-1}x), S(f^n x, f^n x, p), \right.$$
$$\left. S(fp, fp, f^{n-1}x), S(fp, fp, p) \right\}$$
$$\leq \alpha \max \left\{ S(f^{n-1}x, f^{n-1}x, p), S(f^n x, f^n x, f^{n-1}x), S(f^n x, f^n x, p) \right\}$$
$$\leq \alpha \max \left\{ [2S(f^{n-1}x, f^{n-1}x, p) + S(p, p, f^n x)], \right.$$
$$\left. S(f^n x, f^n x, f^{n-1}x), S(f^n x, f^n x, p) \right\}$$
$$\leq \alpha[2S(f^{n-1}x, f^{n-1}x, p) + S(f^n x, f^n x, p)]$$

or

$$(1-\alpha)S(f^n x, f^n x, p) \leq 2\alpha S(f^{n-1}x, f^{n-1}x, f^n x)$$
$$S(f^n x, f^n x, p) \leq \left( \tfrac{2\alpha}{1-\alpha} \right) S(f^{n-1}x, f^{n-1}x, f^n x). \qquad (3.2)$$

As $n \to \infty$ in (3.1) , we see that

$$S(f^n x, f^n x, p) \to 0.$$

Thus $f^n x_0 \to p$ for each $x_0 \in X$. Thus $p$ is a $S$-contractive fixed point of $f$. □

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