ANALYSIS OF A HIERARCHICAL STRUCTURED QUEUING SYSTEM WITH THREE SERVICE CHANNELS, AND CHANCES OF REPETITION OF SERVICE WITH FEEDBACK

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Abstract: The paper deals with the analysis of a queuing system with three service channels wherein a customer after getting the service from first server may leaves the system or may moves to the second server according as he/she is satisfied with service of first customer or not. However, the customer after getting the service from second or third server may moves to any server or may leaves the system. Repetition of service by a server may also be there but allowed only once. The steady-state equations have been derived for finding mean queue length using generating function technique.

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Key Words: queuing system, reversibility, feedback, mean queue length

1. Introduction

Feedback queues are those queues in which a customer served once, may not be satisfied with the service and hence stays with the system to get service again and again till satisfied. Many real life situations could be modeled as a feedback queues in manufacturing concerns, office management, etc. A large number of

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researchers have done a lot of work on queuing models considering various aspects such as bulk service, impatient customers, cyclic queues, batch arrivals, reneging, blocking. Study on queuing models considering the concept of feedback has also been done by a large number of researchers including Kumar (1990), Garg and Kumari (1998), Garg and Srivastava (2006), Kusum (2010), Huang et al. (2015), Raheja et al. (2016). Kusum et al. (2010) discussed queues with feedback wherein three servers have been considered wherein a customer may go back to his/her preceding service channel or moving forward to immediate next service channel, except from the third server. She did not impose any condition on the frequency of such movements. However, in practical situations, there may be the possibility of moving back/forward to the preceding/succeeding channel and from the third server and the number of services done by any server for a customer may be limited, usually at most once. Also, the servers may have hierarchical order which have not been given due consideration.

Keeping this in view, we, in the present paper, considers a queuing system with three service channels wherein a customer after getting the service from first server may leave the system or may move to the second higher level server according as he/she is satisfied with service of first customer or not. However, the customer after getting the service from second server may move to the next higher level or back to the first server with feedback or may exit the system. From the third server, the customer may go to first or second server or may exit the system. Repetition of service by a server may also be there for a customer but allowed only once with different probability. Such a situation can be experienced in the administrative offices, hospitals or the organizations having hierarchical structures.

The steady-state equations have been derived for the queuing system studied in the present paper and mean queue length is obtained using generating function technique.

2. Formulation of Problem

We consider a system of three service channels/servers. Customer arrives in Poisson Process at first server, and then goes through 2nd and 3rd server for required service. Let \( \lambda \) be the Poisson mean arrival rate at first server and \( \mu_1, \mu_2, \mu_3 \) denotes exponential service parameters at first, 2nd and third channels respectively. Attempt has been made to explain the situation through various states as shown in Figure 1 for more clarity:
1. Possible states leaving $S_1$

![Transition diagram for $S_1$](image1)

2. Possible States leaving $S_2$ Server

![Transition diagram for $S_2$](image2)

3. Possible States leaving $S_3$

![Transition diagram for $S_3$](image3)

**Figure 1.** Transition diagram showing various states of the system
Customer after service at first server first time either leaves the system with probability \( p_1 \) or goes through 2nd server for further service with probability \( p_{12} \). Thus we have \( p_1 + p_{12} = 1 \). After getting service at 2nd server first time either customer leaves the system with probability \( p_2 \) or goes through \( S_3 \) for further service with Prob. \( p_{23} \) or moves back to first server with probability \( p_{21} \), so \( p_2 + p_{23} + p_{21} = 1 \). If the customer moves back to first server then after service 2nd time, either he leaves system with probability \( p_1 \) or moves server 2nd for further service with probability \( p_{12} \). So \( p_1 + p_{12} = 1 \). If customer moves from 2nd to third server first time for further service then either he leaves system with probability \( p_3 \) or moves back either to the first server or to 2nd server with probabilities \( p_{31} \) or \( p_{32} \) respectively. So \( p_3 + p_{31} + p_{32} = 1 \).

When customer reaches at 2nd server second time then either he leaves the system after satisfactory service with probability \( p_2' \) or moves further for service at 3rd server with probability \( p_{23}' \). So \( p_2' + p_{23}' = 1 \). So after reaching at third server 2nd time with probability \( p_{23} \) it is assumed that the customer is satisfied with service and leaves the system with probability \( p_3' = 1 \).

Let us assume that

\[
a = \text{is the probability of customer on first server (i.e. } S_1 \text{) first time.}
\]

\[
a' = \text{is the probability of customer on first server (i.e. } S_1' \text{) 2nd time.}
\]

\[
b = \text{is the probability that customer is on second server (i.e. } S_2 \text{) first time.}
\]

\[
b' = \text{is the probability of customer is on second server (i.e. } S_2' \text{) 2nd time.}
\]

\[
c = \text{is the probability that customer is on third server (i.e. } S_3 \text{) first time.}
\]

\[
c' = \text{is the probability of customer is on third server (i.e. } S_3' \text{) 2nd time.}
\]

Thus

\[
a + a' = 1, \quad b + b' = 1, \quad c + c' = 1
\]

and

\[
\begin{align*}
ap_1 + ap_{12} + a' p_1 + a' p_{12} &= 1; \\
b p_2 + b p_{23} + b p_{21} + b' p_2 + b' p_{23} &= 1; \\
c p_3 + c p_{31} + c p_{32} + c' p_3 &= 1;
\end{align*}
\]

(\(^{**}\))

The steady state equation for \( n_1, n_2 \) and \( n_3 > 0 \), is:

\[
(\lambda + \mu_1 + \mu_2 + \mu_3)P_{n_1,n_2,n_3}(t) = \lambda \cdot P_{n_1-1,n_2,n_3}(t) + \mu_1(ap_{12} + a' p_{12})P_{n_1+1,n_2-1,n_3}(t) + \\
\mu_1(ap_1 + a' p_1)P_{n_1+1,n_2,n_3}(t) + \mu_2(bp_2 + b' p_2)P_{n_1,n_2+1,n_3}(t) + \\
(bp_{23} + b' p_{23})\mu_2 \cdot P_{n_1,n_2+1,n_3-1}(t) + \mu_2 bp_{21} P_{n_1-1,n_2+1,n_3}
\]
Steady state equation for $n_1 = 0$ and $n_2, n_3 > 0$, is:

\[
\begin{align*}
(\lambda + \mu_2 + \mu_3) \cdot P_{0,n_2,n_3} &= (ap_{12} + a \dot{p}_{12}) \mu_1 P_{1,n_2-1,n_3} + (ap_1 + a \dot{p}_1) \mu_1 P_{1,n_2,n_3} \\
&+ \mu_2(b p_2 + b \dot{p}_2) P_{0,n_2+1,n_3} + \mu_2(b p_{23} + b \dot{p}_{23}) P_{0,n_2+1,n_3-1} \\
&+ (c p_3 + c \dot{p}_3) \mu_3 P_{0,n_2,n_3+1} \\
&+ c p_{32} \cdot \mu_3 P_{0,n_2-1,n_3+1}
\end{align*}
\]  

(2)

Steady state equation for $n_2 = 0$; $n_1, n_3 > 0$, is:

\[
\begin{align*}
[\lambda + \mu_1 + \mu_3] P_{n_1,0,n_3}(t) &= \lambda \cdot P_{n_1-1,0,n_3}(t) + (ap_{12} + a \dot{p}_{12}) \mu_1 P_{n_1+1,0,n_3} \\
&+ (b p_2 + b \dot{p}_2) \mu_2 P_{n_1,1,n_3} + (b p_{23} + b \dot{p}_{23}) \mu_2 P_{n_1,1,n_3-1} \\
&+ b p_{21} \cdot P_{n_1-1,1,n_3} \cdot \mu_2 + (c p_3 + c \dot{p}_3) \mu_3 P_{n_1,0,n_3+1} \\
&+ c p_{31} \cdot \mu_3 P_{n_1-1,0,n_3+1}
\end{align*}
\]  

(3)

Steady state equation for $n_3 = 0$, $n_1, n_2 > 0$ is:

\[
\begin{align*}
[\lambda + \mu_1 + \mu_2] P_{n_1,n_2,0}(t) &= \lambda P_{n_1-1,n_2,0}(t) + (ap_{12} + a \dot{p}_{12}) P_{n_1+1,n_2-1,0} \mu_1 \\
&+ (ap_1 + a \dot{p}_1) \mu_1 P_{n_1+1,n_2,0}(t) + (b p_2 + b \dot{p}_2) P_{n_1,n_2+1,0} \\
&+ (b p_{21}) \mu_2 P_{n_1-1,n_2+1,0}(t) + (c p_3 + c \dot{p}_3) \mu_3 P_{n_1,n_2,1}(t) \\
&+ c p_{31} \mu_3 P_{n_1-1,n_2,1}(t) + c p_{32} P_{n_1,n_2-1,1}(t)
\end{align*}
\]  

(4)

Steady-state equation for $n_1, n_2 = 0$ and $n_3 > 0$, is:

\[
\begin{align*}
[\lambda + \mu_3] P_{0,0,n_3}(t) &= \mu_1 (ap_1 + a \dot{p}_1) P_{1,0,n_3} + \mu_2 (b p_2 + b \dot{p}_2) P_{0,1,n_3} \\
&+ \mu_2(b p_{23} + b \dot{p}_{23}) P_{0,1,n_3-1} + \mu_3(c p_3 + c \dot{p}_3) P_{0,0,n_3+1}
\end{align*}
\]  

(5)

Steady-state equation for $n_2, n_3 = 0$ and $n_1 > 0$ is:

\[
\begin{align*}
[\lambda + \mu_1] P_{n_1,0,0} &= \lambda \cdot P_{n_1-1,0,0} + \mu_1 (ap_1 + a \dot{p}_1) P_{n_1+1,0,0} \\
&+ \mu_2(b p_2 + b \dot{p}_2) P_{n_1,1,0} + b p_{21} \mu_2 P_{n_1-1,1,0}
\end{align*}
\]
In order to solve the expression, we use the following partia l generating functions:

\[ F_{n_2,n_3}(x) = \sum_{n_1=0}^{\infty} P_{n_1,n_2,n_3}(t)x^{n_1} \]

\[ G_{n_1,n_3}(y) = \sum_{n_2=0}^{\infty} P_{n_1,n_2,n_3}(t)y^{n_2} \]

\[ H_{n_1,n_2}(z) = \sum_{n_3=0}^{\infty} P_{n_1,n_2,n_3}(t)z^{n_3} \]

\[ I_{n_3}(x,y) = \sum_{n_2=0}^{\infty} F_{n_2,n_3}(x)y^{n_2} = \sum_{n_1=0}^{\infty} F_{n_1,n_3}(y)x^{n_1} \]

\[ J_{n_1}(y,z) = \sum_{n_3=0}^{\infty} G_{n_1,n_3}(y)z^{n_3} = \sum_{n_2=0}^{\infty} H_{n_1,n_2}(z)y^{n_2} \]

\[ K_{n_2}(x,z) = \sum_{n_3=0}^{\infty} F_{n_2,n_3}(x)z^{n_3} = \sum_{n_1=0}^{\infty} H_{n_1,n_2}(z)x^{n_1} \]

The steady-state equation for \( n_1, n_3 = 0 \) and \( n_2 > 0 \) is:

\[
[\lambda + \mu_2]P_{0,n_2,0}(t) = \mu_1 (a p_{12} + a p_{1}') P_{1,n_2-1,0}(t) + \mu_1 (a p_{1} + a p_{1}') P_{1,n_2,0}(t) + \mu_2 (b p_{2} + b p_{2}') P_{0,n_2+1,0} + \mu_3 (c p_{3} + c p_{3}') P_{0,n_2,1} + \mu_3 c p_{32} P_{0,n_2-1,1}
\]  (7)

Steady state equation for \( n_1, n_2, n_3 = 0 \) is:

\[
\lambda P_{0,0,0} = \mu_1 (a p_{1} + a p_{1}') P_{1,0,0} + \mu_2 (b p_{2} + b p_{2}') P_{0,1,0} + \mu_3 (c p_{3} + c p_{3}') P_{0,0,1}(t)
\]  (8)

To Find Steady-State Solution of the Model

Let us define g.f. to solve the Steady-State equations from (1) to (8)

\[
F(x,y,z) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} P_{n_1,n_2,n_3}(t)x^{n_1}y^{n_2}z^{n_3}
\]  (9)

In order to solve the expression, we use the following partial generating functions:
Multiplying (1) by \( x^{n_1} \) and taking sum over \( n_1 \) from 0 to \( \infty \) and using (2) and definition of generating function defined in (9), we have:

\[
(\lambda + \mu_2 + \mu_3) F_{n_2,n_3}(x) + \mu_1 F_{n_2,n_3}(t) - \mu_1 P_{0,n_2,n_3}(t) \\
= x \cdot \lambda F_{n_2,n_3}(x) + \mu_1 \left( \frac{a p_{12} + a' p_{12}'}{x} \right) [F_{n_2-1,n_3}(x) - P_{0,n_2-1,n_3}] \\
+ \mu_1 \left( \frac{a p_1 + a' p_1'}{x} \right) [F_{n_2,n_3} - P_{0,n_2,n_3}] + \mu_2 (b p_2 + b' p_2') F_{n_2+1,n_3} \\
+ \mu_2 (b p_{23} + b' p_{23}') F_{n_2+1,n_3-1} + x b p_{21} \cdot \mu_2 \cdot F_{n_2+1,n_3}(x) \\
+ \mu_3 (c p_3 + c' p_3') F_{n_2,n_3+1} + x \mu_3 c p_3 F_{n_2,n_3+1} \\
+ \mu_3 c p_{32} F_{n_2-1,n_3+1}(t) \tag{12}
\]

Multiplying (3) by \( x^{n_1} \), taking sum over \( n_1 \) from 0 to \( \infty \) and using (5) and definition of generating function in (9) we have:

\[
(\lambda + \mu_3) F_{0,n_3}(x) + \mu_1 F_{0,n_3}(t) - \mu_1 P_{0,0,n_3}(t) \\
= \lambda x F_{0,n_3}(x) + (b p_2 + b' p_2') \mu_2 F_{1,n_3-1} \\
+ \mu_1 \left( \frac{a p_1 + a' p_1'}{x} \right) [F_{0,n_3}(x) - P_{0,0,n_3}] + (b p_{23} + b' p_{23}') \mu_2 F_{1,n_3-1} \\
+ (b p_{2} \mu_2 x) F_{1,n_3}(x) + (c p_3 + c' p_3') \mu_3 F_{0,n_3+1} + c p_3 \mu_3 x F_{0,n_3+1}(x) \tag{13}
\]

for \( n_1, n_3 \) and \( n_2 = 0 \).

Multiplying (4) by \( x^{n_1} \) taking sum over \( n_1 \) from 0 to \( \infty \), using (7) and definition of g.f. in (9), we have:

\[
(\lambda + \mu_1 + \mu_2) F_{n_2,0}(x) - \mu_1 P_{0,n_2,0} \\
= \lambda x F_{n_2,0}(x) + \mu_1 \left( \frac{a p_{12} + a' p_{12}'}{x} \right) [F_{n_2-1,0}(x)] \\
- \mu_1 \left( \frac{a p_{12} + a' p_{12}'}{x} \right) P_{0,n_2-1,0}(t) + \left( \frac{a p_1 + a' p_1'}{x} \right) F_{n_2,0}(x) \\
- \left( \frac{a p_1 + a' p_1'}{x} \right) \mu_1 P_{0,n_2,0}(t) + (b p_2 + b' p_2') \mu_2 F_{n_2+1,0}(x) \\
+ (b p_{21} \mu_2 x) F_{n_2+1,0}(x) + (c p_3 + c' p_3') \mu_3 F_{n_2,1}(x) \\
+ c p_3 \mu_3 x F_{n_2,1}(x) + c p_{32} \cdot \mu_3 \cdot F_{n_2-1}(x) \tag{14}
\]

Multiplying (6) by \( x^{n_1} \), taking sum over \( n_1 \) from 0 to \( \infty \), and definition of generating function and using (8) we have:

\[
(\lambda + \mu_1) F_{0,0}(x) - \mu_1 P_{0,0,0}(t)
\]
\[
\begin{align*}
&= \lambda x F_{0,0}(x) + \frac{\mu_1(ap_1 + a'p_1)}{x} F_{0,0}(x) + \mu_2(bp_2 + b'p_2) F_{1,0} \\
&\quad - \frac{\mu_1(ap_1 + a'p_1)}{x} P_{0,0,0} + bp_{21} \mu_2 x F_{1,0}(x) + \mu_3(cp_3 + c'p_3) F_{0,1}(x) \\
&\quad + \mu_3 cp_3 x F_{0,1}(x) \quad \text{for } n_1, n_3 = 0 \text{ and } n_1 \geq 0 
\end{align*}
\]
Multiplying (12) by \(y^{n_2}\), taking sum over \(n_2\) from 0 to \(\infty\), using (13) and definition of generating function, we have:
\[
(\lambda + \mu_1 + \mu_3 + \mu_2) I_{n_3}(x, y) - \mu_2 F_{0,n_3} - \mu_1 G_{0,n_3} \\
= x \lambda I_{n_3}(x, y) + \frac{\mu_1(ap_{12} + a'p_{12})y}{x} [I_{n_3}(x, y) - G_{0,n_3}(y)] \\
\quad + \frac{\mu_1(ap_1 + a'p_1)}{x} [I_{n_3}(x, y) - G_{0,n_3}(x)] \\
\quad + \frac{\mu_2(bp_2 + b'p_2)}{y} [I_{n_3}(x, y) - F_{0,n_3}(x)] \\
\quad + \frac{\mu_2(bp_{23} + b'p_{23})}{y} [I_{n_3-1} - F_{0,n_3-1}] \\
\quad + \frac{xbp_{21} \mu_2}{y} [I_{n_3}(x, y) - F_{0,n_3}(x)] + \mu_3(cp_3 + c'p_3) I_{n_3+1}(x, y) \\
\quad + x \mu_3 cp_3 I_{n_3+1}(x, y) + \mu_3 cp_{32} y I_{n_3+1}(x, y) 
\]  
(16)

Multiplying (14) by \(y^{n_2}\), taking sum over \(n_2\) from 0 to \(\infty\), using (15) and definition of g.f., we have:
\[
(\lambda + \mu + \mu_2) I_0(x, y) - \mu_2 F_{0,0}(x) - \mu_1 G_{0,0}(y) \\
= \lambda x I_0(x, y) + \frac{\mu_1(ap_{12} + a'p_{12})y}{x} [I_0(x, y) - G_{0,0}(y)] \\
\quad + \frac{\mu_1(ap_1 + a'p_1)}{x} [I_0(x, y) - G_{0,0}(x)] \\
\quad + \frac{(bp_2 + b'p_2)}{y} \mu_2 [I_0(x, y) - F_{0,0}(x)] \\
\quad + (cp_3 + c'p_3) \mu_3 I_1(x, y) + \frac{bp_{21} \mu_2 x}{y} [I_0(x, y) - F_{0,0}] \\
\quad + (cp_{31} \mu_3 x) I_1(x, y) + cp_{32} \mu_3 y I_1(x, y) 
\]  
(17)

Multiplying (16) by \(z^{n_3}\), taking sum over \(n_3\) from 0 to \(\infty\), using (17) and definition of g.f. we have
\[
(\lambda + \mu_1 + \mu_2 + \mu_3) F(x, y, z) - \mu_3 I_0(x, y) - \mu_2 k_0(x, z) - \mu_1 J_0(y, z) 
\]
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\[
F(x, y, z) = x\lambda F(x, y, z) + \frac{\mu_1 y (ap_{12} + a'_{12})}{x} [F(x, y, z) - J_0(y, z)] \\
+ \frac{\mu_1 (ap_1 + a'_{12})}{x} [F(x, y, z) - J_0(y, z)] \\
+ \frac{\mu_2 (bp_2 + b'_{23})}{y} [F(x, y, z) - K_0(x, z)] \\
+ \frac{\mu_2 (bp_{23} + b'_{23})}{y} [F(x, y, z) - K_0(x, z)] \\
+ \frac{\mu_2 x bp_{21}}{y} [F(x, y, z) - K_0(x, z)] \\
+ \mu_3 (cp_3 + c'_{23}) [F(x, y, z) - I_0(x, y)] \\
+ \frac{\mu_3 x cp_{31}}{z} [F(x, y, z) - I_0(x, y)] \\
+ \frac{\mu_3 cp_{32} y}{z} [F(x, y, z) - I]
\]

Now collecting terms of \( F(x, y, z) \) one side, we have

\[
F(x, y, z) = \left( \left( \frac{\mu_3}{x} \left[ 1 - \frac{1}{z} \{ (cp_3 + c^'_{23}) + xcp_{31} + cp_{32} y \} \right] J_0(x, y) \\
+ \mu_2 \left[ 1 - \frac{1}{y} \{ (bp_2 + b^'_{23}) + (bp_{23} + b^'_{23}) z + xbp_{21} \} \right] \right) k_0(x, z) \\
+ \left( \frac{\lambda (1 - x) + \mu_1}{x} \left[ 1 - \frac{1}{y} \{ (ap_{12} + a^'_{12}) y + (ap_1 + a^'_{12}) \} \right] \right) J_0(y, z) \\
+ \mu_2 \left[ 1 - \frac{1}{y} \{ (bp_2 + b^'_{23}) + (bp_{23} + b^'_{23}) z + xbp_{21} \} \right] \right) J_0(y, z) \\
+ \mu_3 \left[ 1 - \frac{1}{y} \{ (cp_3 + c^'_{23}) + xcp_{31} + cp_{32} y \} \right]
\]

For \( x = y = z = 1 \) and using \((\times \times)\), (18) reduces to indeterminate form \( (0 \ 0) \).

Now taking \( y = z = 1 \) and taking limit \( x \to 1 \) in (18), we have:

\[
- \mu_3 cp_{31} I_0(1, 1) - \mu_2 bp_{21} K_0(1, 1) + \mu_1 J_0(1, 1) \\
= -\lambda + \mu_1 - \mu_2 bp_{21} - \mu_3 cp_{31} \tag{19}
\]

Let \( x = z = 1 \) and taking \( y \to 1 \), in (18) we have

\[
- \mu_3 cp_{32} I_0(1, 1) - \mu_1 (ap_{12} + a'_{12}) J_0(1, 1) + \mu_2 K_0(1, 1) \\
= -\mu_3 cp_{32} + \mu_2 - \mu_1 (ap_{12} + a'_{12}) \tag{20}
\]

Now taking \( x = y = 1 \) and \( z \to 1 \) in (18), we have:

\[
\mu_3 I_0(1, 1) - \mu_2 (bp_{23} + b'_{23}) K_0(1, 1) = -\mu_2 (bp_{23} + b'_{23}) + \mu_3 \tag{21}
\]
From (19), (20) and (21), we have:

\[ K_0(1, 1) = 1 + \frac{\lambda (ap_{12} + a' p'_{12})}{\mu_2 A} \]  

(22)

where

\[ A = [-1 + bp_{21}(ap_{12} + a' p_{12}) + c(bp_{23} + b' p'_{23})\{p_{32} + p_{31}(ap_{12} + a' p_{12})\}] \]

\[ J_0(1, 1) = 1 + \frac{\lambda}{\mu_1 A}[1 - cp_{32}(bp_{23} + b' p'_{23})] \]  

(23)

and

\[ I_0(1, 1) = 1 + \frac{\lambda}{\mu_3 A}[(ap_{12} + a' p_{12})(bp_{23} + b' p'_{23})] \]  

(24)

Let we consider

\[ F(x, y, z) = \frac{f(x, y, z)}{g(x, y, z)} \]

where:

\[ f(x, y, z) \]

\[ = \mu_3 \left[ 1 - \frac{1}{z}\{(cp_3 + c' p'_3) + xcp_{31} + cp_{32}y\} \right] I_0(x, y) \]

\[ + \mu_2 \left[ 1 - \frac{1}{y}\{(bp_2 + b' p'_2) + (bp_{23} + b' p'_{23})z + xbp_{21}\} \right] K_0(x, z) \]

\[ + \mu_1 \left[ 1 - \frac{1}{x}\{(ap_{12} + a' p_{12})y + (ap_1 + a' p_1)\} \right] J_0(y, z) \]  

(25)

and

\[ g(x, y, z) \]

\[ = \lambda(1 - x) + \mu_1 \left[ 1 - \frac{1}{x}\{(ap_{12} + a' p_{12})y + (ap_1 + a' p_1)\} \right] \]

\[ + \mu_2 \left[ 1 - \frac{1}{y}\{(bp_{12} + b' p'_{12}) + (bp_{23} + b' p'_{23})z + xbp_{21}\} \right] \]

\[ + \mu_3 \left[ 1 - \frac{1}{z}\{(cp_3 + c' p'_3) + xcp_{31} + cp_{32}y\} \right] \]  

(26)

Now the partial derivatives at \( x = y = z = 1 \) gives

\[ \left( \frac{\partial f}{\partial x} \right)_{(1,1,1)} = -\mu_3 cp_{31} - \mu_2 bp_{21} + \mu_1 - \lambda \]  

(27)
\[
\left( \frac{\partial^2 f}{\partial x^2} \right)_{(1,1,1)} = -2\mu_1 + \frac{2\lambda}{A} \left\{ 1 - c p_{32} \left( b p_{23} + b' p'_{23} \right) \right\} \quad (28)
\]

\[
\left( \frac{\partial f}{\partial x} \right)_{(1,1,1)} = -c p_{32} \mu_3 + \mu_2 - \left( a p_{12} + a' p'_{12} \right) \mu_1 \quad (29)
\]

\[
\left( \frac{\partial^2 f}{\partial y^2} \right)_{(1,1,1)} = -2\mu_2 - \frac{2\lambda}{A} \left( a p_{12} + a' p'_{12} \right) \quad (30)
\]

\[
\left( \frac{\partial f}{\partial y} \right)_{(1,1,1)} = \mu_3 - \mu_2 \left( b p_{23} + b' p'_{23} \right) \quad (31)
\]

\[
\left( \frac{\partial^2 f}{\partial z^2} \right)_{(1,1,1)} = -2\mu_3 - \frac{2\lambda}{A} \left( a p_{12} + a' p'_{12} \right) \left( b p_{23} + b' p'_{23} \right) \quad (32)
\]

\[
\left( \frac{\partial g}{\partial x} \right)_{(1,1,1)} = -\lambda + \mu_1 - b p_{21} \mu_2 - c p_{31} \mu_3 \quad (33)
\]

\[
\left( \frac{\partial^2 g}{\partial x^2} \right)_{(1,1,1)} = -2\mu_1 \quad (34)
\]

\[
\left( \frac{\partial g}{\partial y} \right)_{(1,1,1)} = -\mu_1 \left( a p_{12} + a' p'_{12} \right) + \mu_2 - \mu_3 c p_{32} \quad (35)
\]

\[
\left( \frac{\partial^2 g}{\partial y^2} \right)_{(1,1,1)} = -2\mu_2 \quad (36)
\]

\[
\left( \frac{\partial g}{\partial z} \right)_{(1,1,1)} = \mu_3 - \mu_2 \left( b p_{23} + b' p'_{23} \right) \quad (37)
\]

\[
\left( \frac{\partial^2 g}{\partial z^2} \right)_{(1,1,1)} = -2\mu_3 \quad (38)
\]

The marginal mean queue length on first server is given by

\[
L_{q_1} = \frac{-\left( \frac{\partial f}{\partial x} \right)_{(1,1,1)} \left( \frac{\partial^2 g}{\partial x^2} \right)_{(1,1,1)} + \left( \frac{\partial^2 f}{\partial x^2} \right)_{(1,1,1)} \left( \frac{\partial g}{\partial x} \right)_{(1,1,1)}}{2 \left[ \left( \frac{\partial g}{\partial x} \right)_{(1,1,1)} \right]^2} \quad (39)
\]

\[
L_{q_1} = \frac{\lambda}{A} \left\{ 1 - c p_{32} \left( b p_{23} + b' p'_{23} \right) \right\} \quad (39)
\]

\[
\lambda = \mu_1 - b p_{21} \mu_2 + c p_{31} \mu_3
\]
The marginal mean queue length before second server is given by

\[
L_{q2} = \frac{\left( -\frac{\partial f}{\partial y} \right)_{(1, 1, 1)} \left( \frac{\partial^2 g}{\partial y^2} \right)_{(1, 1, 1)} + \left( \frac{\partial g}{\partial y} \right)_{(1, 1, 1)} \left( \frac{\partial^2 f}{\partial y^2} \right)_{(1, 1, 1)}}{2 \left[ \left( \frac{\partial g}{\partial y} \right)_{(1, 1, 1)} \right]^2}
\]

\[
L_{q2} = \frac{\lambda}{A} \frac{(ap_{12} + a' p'_{12})}{\mu_1 (ap_{12} + a p_{12}) - \mu_2 + \mu_3 c p_{32}}
\]

(40)

The marginal mean queue length on third server is given by:

\[
L_{q3} = \frac{\left( -\frac{\partial f}{\partial z} \right)_{(1, 1, 1)} \left( \frac{\partial^2 g}{\partial z^2} \right)_{(1, 1, 1)} + \left( \frac{\partial g}{\partial z} \right)_{(1, 1, 1)} \left( \frac{\partial^2 f}{\partial z^2} \right)_{(1, 1, 1)}}{2 \left[ \left( \frac{\partial g}{\partial z} \right)_{(1, 1, 1)} \right]^2}
\]

\[
L_{q3} = \frac{\lambda}{A} \frac{(ap_{12} + a' p'_{12}) (bp_{23} + b' p'_{23})}{\mu_2 (bp_{23} + b p_{23}) - \mu_3}
\]

(41)

\[L_q = \text{Mean Queue Length of the System} = L_{q1} + L_{q2} + L_{q3}\]

\[
L_q = \frac{\lambda}{A} \frac{\{1 - cp_{32} (bp_{23} + b' p'_{23})\}}{\lambda - \mu_1 + bp_{21} \mu_2 + cp_{31} \mu_3}
\]

\[+ \frac{\lambda}{A} \frac{(ap_{12} + a' p'_{12})}{\mu_1 (ap_{12} + a p_{12}) - \mu_2 + \mu_3 c p_{32}}
\]

\[+ \frac{\lambda}{A} \frac{(ap_{12} + a' p'_{12}) (bp_{23} + b' p'_{23})}{\mu_2 (bp_{23} + b p_{23}) - \mu_3}
\]

**Numerical Results**

1. Behaviour of mean queue length of the system with respect to arrival rate \(\lambda\) is depicted in Table 1. The probability \((ap_{1})\) has also been varied whereas the values of other parameters have been kept fixed.
ANALYSIS OF A HIERARCHICAL STRUCTURED QUEUING...

Table 1

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$Lq_1$</th>
<th>$Lq_2$</th>
<th>$Lq_3$</th>
<th>$Lq$</th>
<th>$Lq_1$</th>
<th>$Lq_2$</th>
<th>$Lq_3$</th>
<th>$Lq$</th>
<th>$Lq_1$</th>
<th>$Lq_2$</th>
<th>$Lq_3$</th>
<th>$Lq$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.179</td>
<td>0.316</td>
<td>0.083</td>
<td>0.577</td>
<td>0.177</td>
<td>0.263</td>
<td>0.078</td>
<td>0.518</td>
<td>0.176</td>
<td>0.220</td>
<td>0.073</td>
<td>0.469</td>
</tr>
<tr>
<td>1.2</td>
<td>0.221</td>
<td>0.379</td>
<td>0.100</td>
<td>0.700</td>
<td>0.219</td>
<td>0.315</td>
<td>0.094</td>
<td>0.628</td>
<td>0.218</td>
<td>0.264</td>
<td>0.088</td>
<td>0.570</td>
</tr>
<tr>
<td>1.4</td>
<td>0.267</td>
<td>0.442</td>
<td>0.116</td>
<td>0.825</td>
<td>0.265</td>
<td>0.368</td>
<td>0.109</td>
<td>0.741</td>
<td>0.262</td>
<td>0.308</td>
<td>0.103</td>
<td>0.673</td>
</tr>
<tr>
<td>1.6</td>
<td>0.315</td>
<td>0.505</td>
<td>0.133</td>
<td>0.953</td>
<td>0.313</td>
<td>0.420</td>
<td>0.125</td>
<td>0.858</td>
<td>0.310</td>
<td>0.352</td>
<td>0.117</td>
<td>0.779</td>
</tr>
<tr>
<td>1.8</td>
<td>0.368</td>
<td>0.568</td>
<td>0.149</td>
<td>1.085</td>
<td>0.364</td>
<td>0.473</td>
<td>0.141</td>
<td>0.978</td>
<td>0.361</td>
<td>0.396</td>
<td>0.132</td>
<td>0.889</td>
</tr>
<tr>
<td>2.0</td>
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<td>0.631</td>
<td>0.166</td>
<td>1.221</td>
<td>0.420</td>
<td>0.525</td>
<td>0.156</td>
<td>1.101</td>
<td>0.416</td>
<td>0.440</td>
<td>0.146</td>
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</tr>
<tr>
<td>2.2</td>
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<td>0.694</td>
<td>0.183</td>
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<td>0.480</td>
<td>0.578</td>
<td>0.172</td>
<td>1.229</td>
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<td>0.484</td>
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</tr>
<tr>
<td>2.4</td>
<td>0.549</td>
<td>0.758</td>
<td>0.199</td>
<td>1.505</td>
<td>0.544</td>
<td>0.630</td>
<td>0.187</td>
<td>1.362</td>
<td>0.540</td>
<td>0.528</td>
<td>0.176</td>
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<tr>
<td>2.6</td>
<td>0.619</td>
<td>0.821</td>
<td>0.216</td>
<td>1.656</td>
<td>0.614</td>
<td>0.683</td>
<td>0.203</td>
<td>1.500</td>
<td>0.609</td>
<td>0.572</td>
<td>0.190</td>
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</tr>
<tr>
<td>2.8</td>
<td>0.696</td>
<td>0.884</td>
<td>0.232</td>
<td>1.812</td>
<td>0.690</td>
<td>0.735</td>
<td>0.219</td>
<td>1.644</td>
<td>0.684</td>
<td>0.616</td>
<td>0.205</td>
<td>1.506</td>
</tr>
<tr>
<td>3.0</td>
<td>0.780</td>
<td>0.947</td>
<td>0.249</td>
<td>1.975</td>
<td>0.773</td>
<td>0.788</td>
<td>0.234</td>
<td>1.795</td>
<td>0.767</td>
<td>0.660</td>
<td>0.220</td>
<td>1.647</td>
</tr>
</tbody>
</table>

Following can be interpreted from Table 1.

Mean queue length of the system increases with the increase in mean arrival rate ($\lambda$) and has lower values for higher values of probability ($a p_1$). However there is no significant difference between mean queue lengths with respect to ($a p_1$).

2. Behavior of marginal mean queue length at first server ($Lq_1$) with respect to service rate of first server ($\mu_1$) is depicted in Table 2. The service rate of second server ($\mu_2$) has also been varied whereas the values of other parameters have been kept fixed.
Table 2

\[\mu_3 = 9, \lambda = 1, a = 0.3, a' = 0.7, b = 0.7,\]

\[b' = 0.3, c = 0.8, c' = 0.2, p_1 = 0.7, p_{12} = 0.3,\]

\[p_1' = 0.4, p_{12}' = 0.6, p_2 = 0.3, p_{23} = 0.5, p_{21} = 0.2,\]

\[p_2' = 0.8, p_{23}' = 0.2, p_3 = 0.3, p_{31} = 0.2, A = -0.73114,\]

\[p_{32} = 0.5, p_3' = 1\]

<table>
<thead>
<tr>
<th>(\mu_1)</th>
<th>(Lq_1)</th>
<th>(Lq_2)</th>
<th>(Lq_3)</th>
<th>(Lq)</th>
<th>(\mu_2 = 7)</th>
<th>(\mu_2 = 8)</th>
<th>(\mu_2 = 9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>1.9714</td>
<td>0.5129</td>
<td>0.0467</td>
<td>2.5310</td>
<td>2.5987</td>
<td>0.2956</td>
<td>0.0500</td>
</tr>
<tr>
<td>4.2</td>
<td>1.4659</td>
<td>0.5545</td>
<td>0.0467</td>
<td>2.0671</td>
<td>1.7866</td>
<td>0.3089</td>
<td>0.0500</td>
</tr>
<tr>
<td>4.4</td>
<td>1.1667</td>
<td>0.6034</td>
<td>0.0467</td>
<td>1.8168</td>
<td>1.3612</td>
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<td>0.0500</td>
</tr>
<tr>
<td>4.6</td>
<td>0.9690</td>
<td>0.6618</td>
<td>0.0467</td>
<td>1.6774</td>
<td>1.0994</td>
<td>0.3396</td>
<td>0.0500</td>
</tr>
<tr>
<td>4.8</td>
<td>0.8286</td>
<td>0.7327</td>
<td>0.0467</td>
<td>1.6079</td>
<td>0.9221</td>
<td>0.3573</td>
<td>0.0500</td>
</tr>
<tr>
<td>5.0</td>
<td>0.7237</td>
<td>0.8206</td>
<td>0.0467</td>
<td>1.5910</td>
<td>0.7940</td>
<td>0.3770</td>
<td>0.0500</td>
</tr>
<tr>
<td>5.2</td>
<td>0.6424</td>
<td>0.9325</td>
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<td>5.4</td>
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<td>0.0500</td>
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<td>0.5605</td>
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<td>1.5781</td>
<td>0.0467</td>
<td>2.1052</td>
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<tr>
<td>6.0</td>
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<td>0.0467</td>
<td>2.5414</td>
<td>0.4686</td>
<td>0.5206</td>
<td>0.0500</td>
</tr>
</tbody>
</table>

Following can be interpreted from Table 2.

1. Marginal mean queue length at first server \((Lq_1)\) decreases with the increase in service rate of first server \((\mu_1)\) but has higher values for higher values of service rate of second server \((\mu_2)\) when \(\mu_1 \leq 4.8\). However, there is no significant difference between marginal mean queue lengths with respect to \(\mu_2\) when \(\mu_1 > 4.8\).

2. Marginal mean queue length at second server \((Lq_2)\) increases with the increase in service rate of first server \((\mu_1)\) but has lower values for higher values of service rate of second server \((\mu_2)\).

3. Marginal mean queue length at third server \((Lq_3)\) remains same irrespective of the value of service rate of first server \((\mu_1)\). However, it has higher values for higher values of service rate of second server \((\mu_2)\).
4. Graph for Mean Queue Length of the System \((L_q)\) has been plotted with respect to service rate of first server \((\mu_1)\) for different values of \(\mu_2\) as shown in the Figure 2 and following can be interpreted from the above graph:

(i) Mean Queue length of the system \((L_q)\) decreases with the increase in service rate of first server \((\mu_1)\) for \(\mu_2 = 8\) and \(\mu_2 = 9\). However, for \(\mu_2 = 7\) it decreases when \(\mu_1 = 5\) and increases thereafter.

(ii) Let \(L_{qi}\) denote the mean queue length of the system at \(\mu_2 = i\), for \(i = 7, 8,\) or 9. Then mean queue length for different values of \(\mu_2\) may be compared as:

<table>
<thead>
<tr>
<th>Value of Service Rate ((\mu_1))</th>
<th>Mean Queue Length ((L_{qi}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu_1 &lt; 4.29769)</td>
<td>(L_{q7} &lt; L_{q8} &lt; L_{q9})</td>
</tr>
<tr>
<td>(4.29769 &lt; \mu_1 &lt; 4.48464)</td>
<td>(L_{q8} &lt; L_{q7} &lt; L_{q9})</td>
</tr>
<tr>
<td>(4.48464 &lt; \mu_1 &lt; 4.800465)</td>
<td>(L_{q8} &lt; L_{q9} &lt; L_{q7})</td>
</tr>
<tr>
<td>(\mu_1 &gt; 4.800465)</td>
<td>(L_{q9} &lt; L_{q8} &lt; L_{q7})</td>
</tr>
</tbody>
</table>

**Figure 2.** Mean Queue Length vs Service Rate of first server \((\mu_1)\)

**Conclusion**

A queuing system having three service channels is discussed. Mean queue length at the counter of each of these servers is obtained for particular cases as dis-
cussed in the previous section of the paper. This model can be fitted to many real life situations. Such situations may be observed in hierarchical systems. The user of such systems can take various important decisions by putting the values of arrival, service and other parameters existing therein in the equations established in this paper.

References


