International Journal of Pure and Applied Mathematics
Volume 115 No. 2 2017, 383-398
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v115i2.15

ANALYSIS OF A HIERARCHICAL STRUCTURED QUEUING SYSTEM WITH THREE SERVICE CHANNELS, AND CHANCES OF REPETITION OF SERVICE WITH FEEDBACK

Surender Kumar[1], Gulshan Taneja[2 §]
[1,2]Department of Mathematics
Maharshi Dayanand University
Rohtak, INDIA

Abstract: The paper deals with the analysis of a queuing system with three service channels wherein a customer after getting the service from first server may leaves the system or may moves to the second server according as he/she is satisfied with service of first customer or not. However, the customer after getting the service from second or third server may moves to any server or may leaves the system. Repetition of service by a server may also be there but allowed only once. The steady-state equations have been derived for finding mean queue length using generating function technique.

AMS Subject Classification: 90B22
Key Words: queuing system, reversibility, feedback, mean queue length

1. Introduction

Feedback queues are those queues in which a customer served once, may not be satisfied with the service and hence stays with the system to get service again and again till satisfied. Many real life situations could be modeled as a feedback queues in manufacturing concerns, office management, etc. A large number of

Received: March 25, 2017
Revised: May 24, 2017
Published: July 14, 2017

§Correspondence author

researchers have done a lot of work on queuing models considering various aspects such as bulk service, impatient customers, cyclic queues, batch arrivals, reneging, blocking. Study on queuing models considering the concept of feedback has also been done by a large number of researchers including Kumar (1990), Garg and Kumari (1998), Garg and Srivastava (2006), Kusum (2010), Huang et al. (2015), Raheja et al. (2016). Kusum et al. (2010) discussed queues with feedback wherein three servers have been considered wherein a customer may go back to his/her preceding service channel or moving forward to immediate next service channel, except from the third server. She did not impose any condition on the frequency of such movements. However, in practical situations, there may be the possibility of moving back/forward to the preceding/succeeding channel and from the third server and the number of services done by any server for a customer may be limited, usually at most once. Also, the servers may have hierarchical order which have not been given due consideration.

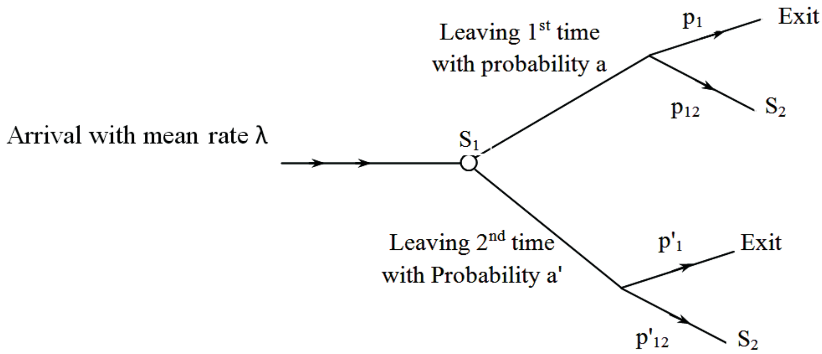
Keeping this in view, we, in the present paper, considers a queuing system with three service channels wherein a customer after getting the service from first server may leave the system or may move to the second higher level server according as he/she is satisfied with service of first customer or not. However, the customer after getting the service from second server may move to the next higher level or back to the first server with feedback or may exit the system. From the third server, the customer may go to first or second server or may exit the system. Repetition of service by a server may also be there for a customer but allowed only once with different probability. Such a situation can be experienced in the administrative offices, hospitals or the organizations having hierarchical structures.

The steady-state equations have been derived for the queuing system studied in the present paper and mean queue length is obtained using generating function technique.

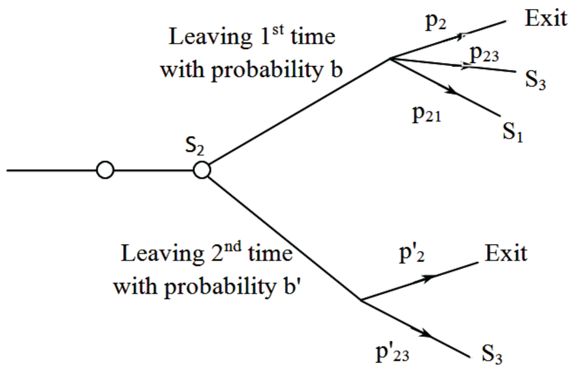
2. Formulation of Problem

We consider a system of three service channels/ servers. Customer arrives in Poisson Process at first server, and then goes through 2nd and 3rd server for required service. Let λ be the Poisson mean arrival rate at first server and μ_1, μ_2, μ_3 denotes exponential service parameters at first, 2nd and third channels respectively. Attempt has been made to explain the situation through various states as shown in Figure 1 for more clarity:

1. Possible states leaving S_1



2. Possible States leaving S_2 Server



3. Possible States leaving S_3

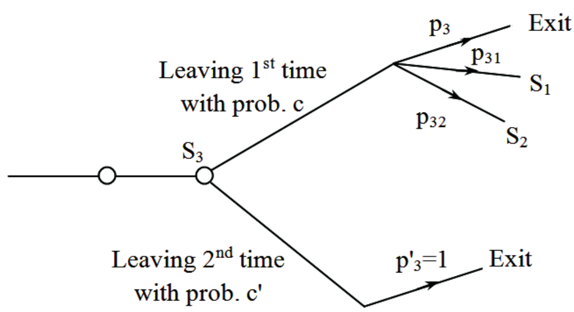


Figure 1. Transition diagram showing various states of the system

Customer after service at first server first time either leaves the system with probability p_1 or goes through 2nd server for further service with probability p_{12}. Thus we have $p_1 + p_{12} = 1$. After getting service at 2nd server first time either customer leaves the system with probability p_2 or goes through S_3 for further service with Prob. p_{23} or moves back to first server with probability p_{21}, so $p_2+p_{23}+p_{21} = 1$. If the customer moves back to first server then after service 2nd time, either he leaves system with probability p_1' or moves server 2nd for further service with probability p_{12}'. So $p_1' + p_{12}' = 1$. If customer moves from 2nd to third server first time for further service then either he leaves system with probability p_3 or moves back either to the first server or to 2nd server with probabilities p_{31} or p_{32} respectively. So $p_3 + p_{31} + p_{32} = 1$.

When customer reaches at 2nd server second time then either he leaves the system after satisfactory service with probability p_2' or moves further for service at 3rd server with probability p_{23}'. So $p_2' + p_{23}' = 1$. So after reaching at third server 2nd time with probability p_{23}' it is assumed that the customer is satisfied with service and leaves the system with probability $p_3' = 1$.
Let us assume that
$a =$ is the probability of customer on first server (i.e. S_1) first time.
$a' =$ is the probability of customer on first server (i.e. S_1') 2nd time.
$b =$ is the probability that customer is on second server (i.e. S_2) first time.
$b' =$ is the probability of customer is on second server (i.e. S_2') 2nd time.
$c =$ is the probability that customer is on third server (i.e. S_3) first time.
$c' =$ is the probability of customer is on third server (i.e. S_3') 2nd time.
Thus

$$a + a' = 1, \quad b + b' = 1, \quad c + c' = 1$$

and

$$\left.\begin{aligned} &ap_1 + ap_{12} + a'p_1' + a'p_{12}' = 1;\\ &bp_2 + bp_{23} + bp_{21} + b'p_2' + b'p_{23}' = 1;\\ &cp_3 + cp_{31} + cp_{32} + c'p_3' = 1; \end{aligned}\right\} \tag{**}$$

The steady state equation for n_1, n_2 and $n_3 > 0$, is:

$$\begin{aligned} &(\lambda + \mu_1 + \mu_2 + \mu_3)P_{n_1,n_2,n_3}(t)\\ &= \lambda \cdot P_{n_1-1,n_2,n_3}(t) + \mu_1(ap_{12} + a'p_{12}')P_{n_1+1,n_2-1,n_3}(t)\\ &\quad + \mu_1(ap_1 + a'p_1')P_{n_1+1,n_2,n_3}(t) + \mu_2(bp_2 + b'p_2')P_{n_1,n_2+1,n_3}(t)\\ &\quad + (bp_{23} + b'p_{23}')\mu_2 \cdot P_{n_1,n_2+1,n_3-1}(t) + \mu_2 bp_{21}P_{n_1-1,n_2+1,n_3} \end{aligned}$$

$$
\begin{aligned}
&+\mu_3(cp_3+c'p'_3)P_{n_1,n_2,n_3+1}(t)+\mu_3 cp_{31}P_{n_1-1,n_2,n_3+1}\\
&+\mu_3 cp_{32}P_{n_1,n_2-1,n_3+1}
\end{aligned}
\tag{1}
$$

Steady state equation for $n_1=0$ and $n_2,\ n_3>0$, is:

$$
\begin{aligned}
&(\lambda+\mu_2+\mu_3)\cdot P_{0,n_2,n_3}\\
&=(ap_{12}+a'p'_{12})\mu_1 P_{1,n_2-1,n_3}+(ap_1+a'p'_1)\mu_1 P_{1,n_2,n_3}\\
&\quad+\mu_2(bp_2+b'p'_2)P_{0,n_2+1,n_3}+\mu_2(bp_{23}+b'p'_{23})\\
&\quad\cdot P_{0,n_2+1,n_3-1}+(cp_3+c'p'_3)\mu_3 P_{0,n_2,n_{3+1}}\\
&\quad+cp_{32}\cdot\mu_3 P_{0,n_2-1,n_3+1}
\end{aligned}
\tag{2}
$$

Steady state equation for $n_2=0$; $n_1,n_3>0$, is:

$$
\begin{aligned}
&[\lambda+\mu_1+\mu_3]P_{n_1,0,n_3}(t)\\
&=\lambda\cdot P_{n_1-1,0,n_3}(t)+(ap_1+a'p'_1)\mu_1 P_{n_1+1,0,n_3}\\
&\quad+(bp_2+b'p'_2)\mu_2 P_{n_1,1,n_3}+(b'p'_{23}+bp_{23})\mu_2\cdot P_{n_1,1,n_3-1}\\
&\quad+bp_{21}\cdot P_{n_1-1,1,n_3}\cdot\mu_2+(cp_3+c'p'_3)\mu_3 P_{n_1,0,n_3+1}\\
&\quad+cp_{31}\cdot\mu_3 P_{n_1-1,0,n_3+1}
\end{aligned}
\tag{3}
$$

Steady state equation for $n_3=0$, $n_1,n_2>0$ is:

$$
\begin{aligned}
&[\lambda+\mu_1+\mu_2]P_{n_1,n_2,0}(t)\\
&=\lambda P_{n_1-1,n_2,0}(t)+(ap_{12}+a'p'_{12})P_{n_1+1,n_2-1,0}\mu_1\\
&\quad+(ap_1+a'p'_1)\mu_1 P_{n_1+1,n_2,0}(t)+(bp_2+b'p'_2)\mu_2 P_{n_1,n_2+1,0}\\
&\quad+(bp_{21})\mu_2 P_{n_1-1,n_2+1,0}(t)+(cp_3+c'p'_3)\mu_3 P_{n_1,n_2,1}(t)\\
&\quad+cp_{31}\mu_3 P_{n_1-1,n_2,1}(t)+cp_{32}\mu_3 P_{n_1,n_2-1,1}(t)
\end{aligned}
\tag{4}
$$

Steady-state equation for $n_1,n_2=0$ and $n_3>0$, is:

$$
\begin{aligned}
&[\lambda+\mu_3]P_{0,0,n_3}(t)\\
&=\mu_1(ap_1+a'p'_1)P_{1,0,n_3}+\mu_2(bp_2+b'p'_2)P_{0,1,n_3}\\
&\quad+\mu_2(bp_{23}+b'p'_{23})P_{0,1,n_3-1}+\mu_3(cp_3+c'p'_3)P_{0,0,n_3+1}
\end{aligned}
\tag{5}
$$

Steady-state equation for $n_2,n_3=0$ and $n_1>0$ is:

$$
\begin{aligned}
&[\lambda+\mu_1]P_{n_1,0,0}\\
&=\lambda\cdot P_{n_1-1,0,0}+\mu_1(ap_1+a'p'_1)P_{n_1+1,0,0}\\
&\quad+\mu_2(bp_2+b'p'_2)P_{n_1,1,0}+bp_{21}\mu_2 P_{n_1-1,1,0}
\end{aligned}
$$

$$+ \mu_3(cp_3 + c'p_3')P_{n_1,0,1} + \mu_3 cp_{31}P_{n_1-1,0,1} \tag{6}$$

The steady-state equation for $n_1, n_3 = 0$ and $n_2' > 0$ is:

$$\begin{aligned}
&[\lambda + \mu_2]P_{0,n_2,0}(t) \\
&= \mu_1(ap_{12} + a'p_{12}')P_{1,n_2-1,0}(t) + \mu_1(ap_1 + a'p_1')P_{1,n_2,0}(t) \\
&\quad + \mu_2(bp_2 + b'p_2')P_{0,n_2+1,0} + \mu_3(cp_3 + c'p_3')P_{0,n_2,1} \\
&\quad + \mu_3 cp_{32}P_{0,n_2-1,1}
\end{aligned} \tag{7}$$

Steady state equation for $n_1, n_2, n_3 = 0$ is:

$$\begin{aligned}
\lambda P_{0,0,0} &= \mu_1(ap_1 + a'p_1')P_{1,0,0} + \mu_2(bp_2 + b'p_2')P_{0,1,0} \\
&\quad + \mu_3(cp_3 + c'p_3')P_{0,0,1}(t)
\end{aligned} \tag{8}$$

To Find Steady-State Solution of the Model

Let us define g.f. to solve the Steady-State equations from (1) to (8)

$$F(x,y,z) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} P_{n_1,n_2,n_3}(t)x^{n_1}y^{n_2}z^{n_3} \tag{9}$$

In order to solve the expression, we use the following partial generating functions:

$$\left.\begin{aligned}
F_{n_2,n_3}(x) &= \sum_{n_1=0}^{\infty} P_{n_1,n_2,n_3}(t).x^{n_1} \\
G_{n_1,n_3}(y) &= \sum_{n_2=0}^{\infty} P_{n_1,n_2,n_3}(t).y^{n_2} \\
H_{n_1,n_2}(z) &= \sum_{n_3=0}^{\infty} P_{n_1,n_2,n_3}(t).z^{n_3}
\end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned}
I_{n_3}(x,y) &= \sum_{n_2=0}^{\infty} F_{n_2,n_3}(x).y^{n_2} = \sum_{n_1=0}^{\infty} F_{n_1,n_3}(y).x^{n_1} \\
J_{n_1}(y,z) &= \sum_{n_3=0}^{\infty} G_{n_1,n_3}(y).z^{n_3} = \sum_{n_2=0}^{\infty} H_{n_1,n_2}(z).y^{n_2} \\
K_{n_2}(x,z) &= \sum_{n_3=0}^{\infty} F_{n_2,n_3}(x).z^{n_3} = \sum_{n_1=0}^{\infty} H_{n_1,n_2}(z).x^{n_1}
\end{aligned}\right\} \tag{11}$$

Multiplying (1) by x^{n_1} and taking sum over n_1 from 0 to ∞ and using (2) and definition of generating function defined in (9), we have

$$\begin{aligned}
&(\lambda+\mu_2+\mu_3)F_{n_2,n_3}(x)+\mu_1 F_{n_2,n_3}(t)-\mu_1 P_{0,n_2,n_3}(t)\\
&= x\cdot\lambda.F_{n_2,n_3}(x)+\mu_1\frac{(ap_{12}+a'p'_{12})}{x}[F_{n_2-1,n_3}(x)-P_{0,n_2-1,n_3}]\\
&\quad+\mu_1\frac{(ap_1+a'p'_1)}{x}[(F_{n_2,n_3}-P_{0,n_2,n_3}]+\mu_2(bp_2+b'p'_2)F_{n_2+1,n_3}\\
&\quad+\mu_2(bp_{23}+b'p'_{23})F_{n_2+1,n_3-1}+xbp_{21}\cdot\mu_2\cdot F_{n_2+1,n_3}(x)\\
&\quad+\mu_3(cp_3+c'p'_3)F_{n_2,n_3+1}+x\mu_3 cp_{31}F_{n_2n_3+1}\\
&\quad+\mu_3 cp_{32}F_{n_2-1,n_3+1}(t) \qquad (12)
\end{aligned}$$

Multiplying (3) by x^{n_1}, taking sum over n_1 from 0 to ∞ and using (5) and using definition of generating function in (9) we have:

$$\begin{aligned}
&(\lambda+\mu_3)F_{0,n_3}(x)+\mu_1 F_{0,n_3}(x)-\mu_1 P_{0,0,n_3}(t)\\
&=\lambda x F_{0,n_3}(x)+(bp_2+b'p_2)\mu_2 F_{1,n_3-1}\\
&\quad+\frac{\mu_1(ap_1+a'p'_1)}{x}[F_{0,n_3}(x)-P_{0,0,n_3}]+(bp_{23}+b'p'_{23})\mu_2 F_{1,n_3-1}\\
&\quad+(bp_2\mu_2 x)F_{1,n_3}(x)+(cp_3+c'p'_3)\mu_3 F_{0,n_3+1}+cp_{31}\mu_3 x F_{0,n_3+1}(x) \qquad (13)
\end{aligned}$$

for n_1, n_3 and $n_2=0$.

Multiplying (4) by x^{n_1} taking sum over n_1 from 0 to ∞, using (7) and definition of g.f. in (9), we have:

$$\begin{aligned}
&(\lambda+\mu_1+\mu_2)F_{n_2,0}(x)-\mu_1 P_{0,n_2,0}\\
&=\lambda x F_{n_2,0}(x)+\frac{\mu_1(ap_{12}+a'p'_{12})}{x}[F_{n_2-1,0}(x)]\\
&\quad-\frac{\mu_1(ap_{12}+a'p'_{12})}{x}P_{0,n_2-1,0}(t)+\frac{(ap_1+a'p'_1)}{x}F_{n_2,0}(x)\\
&\quad-\frac{(ap_1+a'p'_1)}{x}\mu_1 P_{0,n_2,0}(t)+(bp_2+b'p'_2)\mu_2 F_{n_2+1,0}(x)\\
&\quad+(bp_{21}\mu_2 x)F_{n_2+1,0}(x)+(cp_3+c'p'_3)\mu_3 F_{n_2,1}(x)\\
&\quad+cp_{31}\mu_3 x F_{n_2,1}(x)+cp_{32}.\mu_3.F_{n_2-1}(x) \qquad (14)
\end{aligned}$$

Multiplying (6) by x^{n_1}, taking sum over n_1 from 0 to ∞, and definition of generating function and using (8) we have:

$$(\lambda+\mu_1)F_{0,0}(x)-\mu_1 P_{0,0,0}(t)$$

$$
\begin{aligned}
&= \lambda x F_{0,0}(x) + \frac{\mu_1(ap_1 + a'p_1')}{x} F_{0,0}(x) + \mu_2(bp_2 + b'p_2')F_{1,0} \\
&\quad - \frac{\mu_1(ap_1 + a'p_1')}{x} P_{0,0,0} + bp_{21}\mu_2 x F_{1,0}(x) + \mu_3(cp_3 + c'p_3')F_{0,1}(x) \\
&\quad + \mu_3 cp_{31} x F_{0,1}(x) \quad \text{for } n_1, n_3 = 0 \text{ and } n_1 \geq 0 \qquad (15)
\end{aligned}
$$

Multiplying (12) by y^{n_2}, taking sum over n_2 from 0 to ∞, using (13) and definition of generating function, we have:

$$
\begin{aligned}
&(\lambda + \mu_1 + \mu_3 + \mu_2) I_{n_3}(x,y) - \mu_2 F_{0,n_3} - \mu_1 G_{0,n_3} \\
&= x\lambda I_{n_3}(x,y) + \frac{\mu_1(ap_{12} + a'p_{12}')y}{x}[I_{n_3}(x,y) - G_{0,n_3}(y)] \\
&\quad + \frac{\mu_1(ap_1 + a'p_1')}{x}[I_{n_3}(x,y) - G_{0,n_3}(x)] \\
&\quad + \frac{\mu_2(bp_2 + b'p_2')}{y}[I_{n_3}(x,y) - F_{0,n_3}(x)] \\
&\quad + \frac{\mu_2(bp_{23} + b'p_{23}')}{y}[I_{n_3 - 1} - F_{0,n_3 - 1}] \\
&\quad + \frac{xbp_{21}\mu_2}{y}[I_{n_3}(x,y) - F_{0,n_3}(x)] + \mu_3(cp_3 + c'p_3')I_{n_3+1}(x,y) \\
&\quad + x\mu_3 cp_{31} I_{n_3+1}(x,y) + \mu_3 cp_{32} y.I_{n_3+1}(x,y) \qquad (16)
\end{aligned}
$$

Multiplying (14) by y^{n_2}, taking sum over n_2 from 0 to ∞, using (15) and definition of g.f., we have:

$$
\begin{aligned}
&(\lambda + \mu + \mu_2) I_0(x,y) - \mu_2 F_{0,0}(x) - \mu_1 G_{0,0}(y) \\
&= \lambda x I_0(x,y) + \frac{\mu_1(ap_{12} + a'p_{12}')y}{x}[I_0(x,y) - G_{0,0}(y)] \\
&\quad + \frac{\mu_1(ap_1 + a'p_1')}{x}[I_0(x,y) - G_{0,0}(x)] \\
&\quad + \frac{(bp_2 + b'p_2')}{y}\mu_2[I_0(x,y) - F_{0,0}(x)] \\
&\quad + (cp_3 + c'p_3')\mu_3 I_1(x,y) + \frac{bp_{21}\mu_2 x}{y}[I_0(x,y) - F_{0,0}] \\
&\quad + (cp_{31}\mu_3 x) I_1(x,y) + cp_{32}\mu_3 y.I_1(x,y) \qquad (17)
\end{aligned}
$$

Multiplying (16) by z^{n_3}, taking sum over n_3 from 0 to ∞, using (17) and definition of g.f. we have

$$
(\lambda + \mu_1 + \mu_2 + \mu_3) F(x,y,z) - \mu_3 I_0(x,y) - \mu_2 k_0(x,z) - \mu_1 J_0(y,z)
$$

$$
\begin{aligned}
&= x\lambda F(x,y,z) + \frac{\mu_1 y(ap_{12} + a^{\prime}p^{\prime}_{12})}{x}[F(x,y,z) - J_0(y,z)] \\
&\quad + \frac{\mu_1(ap_1 + a^{\prime}p^{\prime}_1)}{x}[F(x,y,z) - J_0(y,z)] \\
&\quad + \frac{\mu_2(bp_2 + b^{\prime}p_2)}{y}[F(x,y,z) - K_0(x,z)] \\
&\quad + \frac{\mu_2(bp_{23} + b^{\prime}p^{\prime}_{23})z}{y}[F(x,y,z) - K_0(x,z)] \\
&\quad + \frac{\mu_2 xbp_{21}}{y}[F(x,y,z) - K_0(x,z)] \\
&\quad + \mu_3(cp_3 + c^{\prime}p^{\prime}_3)[F(x,y,z) - I_0(x,y)] \\
&\quad + \frac{\mu_3 x.cp_{31}}{z}[F(x,y,z) - I_0(x,y)] \\
&\quad + \frac{\mu_3 cp_{32} y}{z}[F(x,y,z) - I)]
\end{aligned}
$$

Now collecting terms of $F(x,y,z)$ one side, we have

$$
F(x,y,z) = \frac{\begin{pmatrix} \mu_3\left[1 - \frac{1}{z}\{(cp_3 + c^{\prime}p^{\prime}_3) + xcp_{31} + cp_{32}y\right] I_0(x,y) \\ + \mu_2\left[1 - \frac{1}{y}\{(bp_2 + b^{\prime}p^{\prime}_2) + (bp_{23} + b^{\prime}p^{\prime}_{23})z + xbp_{21}\}\right] k_0(x,z) \\ +\mu_1\left[1 - \frac{1}{x}\{ap_{12} + a^{\prime}p^{\prime}_{12})y + (ap_1 + a^{\prime}p^{\prime}_1)\}\right] J_0(y,z) \end{pmatrix}}{\begin{pmatrix} \lambda(1-x) + \mu_1\left[1 - \frac{1}{x}\{(ap_{12} + a^{\prime}p^{\prime}_{12})y + (ap_1 + a^{\prime}p^{\prime}_1)\}\right] \\ +\mu_2\left[1 - \frac{1}{y}\{(b\ p_2 + b^{\prime}p^{\prime}_2) + (bp_{23} + b^{\prime}p^{\prime}_{23})z + xbp_{21}\}\right] \\ +\mu_3\left[1 - \frac{1}{z}\{(cp_3 + c^{\prime}p^{\prime}_3) + xcp_{31} + cp_{32}y\}\right] \end{pmatrix}} \tag{18}
$$

For $x = y = z = 1$ and using $(\times\times)$, (18) reduces to indeterminate form $\left(\frac{0}{0}\right)$.

Now taking $y = z = 1$ and taking limit $x \to 1$ in (18), we have:

$$
\begin{aligned}
&- \mu_3 cp_{31} I_0(1,1) - \mu_2 bp_{21} K_0(1,1) + \mu_1 J_0(1,1) \\
&\quad = -\lambda + \mu_1 - \mu_2 bp_{21} - \mu_3 cp_{31}
\end{aligned} \tag{19}
$$

Let $x = z = 1$ and taking $y \to 1$, in (18) we have

$$
\begin{aligned}
&- \mu_3 cp_{32} I_0(1,1) - \mu_1(ap_{12} + a^{\prime}p^{\prime}_{12}) J_0(1,1) + \mu_2 K_0(1,1) \\
&\quad = -\mu_3 cp_{32} + \mu_2 - \mu_1(ap_{12} + a^{\prime}p^{\prime}_{12})
\end{aligned} \tag{20}
$$

Now taking $x = y = 1$ and $z \to 1$ in (18), we have:

$$
\mu_3 I_0(1,1) - \mu_2(bp_{23} + b^{\prime}p^{\prime}_{23}) K_0(1,1) = -\mu_2(bp_{23} + b^{\prime}p^{\prime}_{23}) + \mu_3 \tag{21}
$$

From (19), (20) and (21), we have:

$$K_0(1,1) = 1 + \frac{\lambda(ap_{12} + a'p'_{12})}{\mu_2 A} \tag{22}$$

where

$$A = [-1 + bp_{21}(ap_{12} + a'p'_{12}) + c(bp_{23} + b'p'_{23})\{p_{32} + p_{31}(ap_{12} + a'p'_{12})\}]$$

$$J_0(1,1) = 1 + \frac{\lambda}{\mu_1 A}[1 - cp_{32}(bp_{23} + b'p'_{23})] \tag{23}$$

and

$$I_0(1,1) = 1 + \frac{\lambda}{\mu_3 A}[(ap_{12} + a'p'_{12})(bp_{23} + b'p'_{23})] \tag{24}$$

Let we consider

$$F(x,y,z) = \frac{f(x,y,z)}{g(x,y,z)}$$

where:

$$\begin{aligned} f&(x,y,z) \\ &= \mu_3 \left[1 - \frac{1}{z}\{(cp_3 + c'p'_3) + xcp_{31} + cp_{32}y\}\right] I_0(x,y) \\ &\quad + \mu_2 \left[1 - \frac{1}{y}\{(bp_2 + b'p'_2) \ +(bp_{23} + b'p'_{23})z + xbp_{21}\}\right] K_0(x,z) \\ &\quad + \mu_1 \left[1 - \frac{1}{x}\{(ap_{12} + a'p'_{12})y + (ap_1 + a'p'_1)\}\right] J_0(y,z) \end{aligned} \tag{25}$$

and

$$\begin{aligned} g&(x,y,z) \\ &= \lambda(1-x) + \mu_1 \left[1 - \frac{1}{x}\{(ap_{12} + a'p'_{12})y + (ap_1 + a'p'_1)\}\right] \\ &\quad + \mu_2 \left[1 - \frac{1}{y}\{(bp_{12} + b'p'_2) + (bp_{23} + b'p'_{23})z + xbp_{21}\}\right] \\ &\quad + \mu_3 \left[1 - \frac{1}{z}\{(cp_3 + c'p'_3) + xcp_{31} + cp_{32}y\}\right] \end{aligned} \tag{26}$$

Now the partial derivatives at $x = y = z = 1$ gives

$$\left(\frac{\partial f}{\partial x}\right)_{(1,1,1)} = -\mu_3 cp_{31} - \mu_2 bp_{21} + \mu_1 - \lambda \tag{27}$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(1,1,1)} = -2\mu_1 + \frac{2\lambda}{A}\left\{1 - cp_{32}\left(bp_{23} + b'p'_{23}\right)\right\} \tag{28}$$

$$\left(\frac{\partial f}{\partial x}\right)_{(1,1,1)} = -cp_{32}\mu_3 + \mu_2 - \left(ap_{12} + a'p'_{12}\right)\mu_1 \tag{29}$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(1,1,1)} = -2\mu_2 - \frac{2\lambda}{A}\left(ap_{12} + a'p'_{12}\right) \tag{30}$$

$$\left(\frac{\partial f}{\partial z}\right)_{(1,1,1)} = \mu_3 - \mu_2\left(bp_{23} + b'p'_{23}\right) \tag{31}$$

$$\left(\frac{\partial^2 f}{\partial z^2}\right)_{(1,1,1)} = -2\mu_3 - \frac{2\lambda}{A}\left(ap_{12} + a'p'_{12}\right)\left(bp_{23} + b'p'_{23}\right) \tag{32}$$

$$\left(\frac{\partial g}{\partial x}\right)_{(1,1,1)} = -\lambda + \mu_1 - bp_{21}\mu_2 - cp_{31}\mu_3 \tag{33}$$

$$\left(\frac{\partial^2 g}{\partial x^2}\right)_{(1,1,1)} = -2\mu_1 \tag{34}$$

$$\left(\frac{\partial g}{\partial y}\right)_{(1,1,1)} = -\mu_1(ap_{12} + a'p'_{12}) + \mu_2 - \mu_3 cp_{32} \tag{35}$$

$$\left(\frac{\partial^2 g}{\partial y^2}\right)_{(1,1,1)} = -2\mu_2 \tag{36}$$

$$\left(\frac{\partial g}{\partial z}\right)_{(1,1,1)} = \mu_3 - \mu_2(bp_{23} + b'p'_{23}) \tag{37}$$

$$\left(\frac{\partial^2 g}{\partial z^2}\right)_{(1,1,1)} = -2\mu_3 \tag{38}$$

The marginal mean queue length on first server is given by

$$Lq_1 = \frac{\left(-\frac{\partial f}{\partial x}\right)_{(1,1,1)}\left(\frac{\partial^2 g}{\partial x^2}\right)_{(1,1,1)} + \left(\frac{\partial g}{\partial x}\right)_{(1,1,1)}\left(\frac{\partial^2 f}{\partial x^2}\right)_{(1,1,1)}}{2\left[\left(\frac{\partial g}{\partial x}\right)_{(1,1,1)}\right]^2}$$

$$L_{q_1} = \frac{\frac{\lambda}{A}\left\{1 - cp_{32}\left(bp_{23} + b'p'_{23}\right)\right\}}{\lambda - \mu_1 + bp_{21}\mu_2 + cp_{31}\mu_3} \tag{39}$$

The marginal mean queue length before second server is given by

$$L_{q_2} = \frac{\left(-\frac{\partial f}{\partial y}\right)_{(1,1,1)}\left(\frac{\partial^2 g}{\partial y^2}\right)_{(1,1,1)} + \left(\frac{\partial g}{\partial y}\right)_{(1,1,1)}\left(\frac{\partial^2 f}{\partial y^2}\right)_{(1,1,1)}}{2\left[\left(\frac{\partial g}{\partial y}\right)_{(1,1,1)}\right]^2}$$

$$L_{q_2} = \frac{\lambda}{A}\frac{(ap_{12} + a'p'_{12})}{[\mu_1(ap_{12} + a'p'_{12}) - \mu_2 + \mu_3 cp_{32}]} \tag{40}$$

The marginal mean queue length on third server is given by:

$$L_{q_3} = \frac{\left(-\frac{\partial f}{\partial z}\right)_{(1,1,1)}\left(\frac{\partial^2 g}{\partial z^2}\right)_{(1,1,1)} + \left(\frac{\partial g}{\partial z}\right)_{(1,1,1)}\left(\frac{\partial^2 f}{\partial z^2}\right)_{(1,1,1)}}{2\left[\left(\frac{\partial g}{\partial z}\right)_{(1,1,1)}\right]^2}$$

$$L_{q_3} = \frac{\lambda}{A}\frac{(ap_{12} + a'p'_{12})\,(bp_{23} + b'p'_{23})}{[\mu_2(bp_{23} + b'p'_{23}) - \mu_3]} \tag{41}$$

Lq = Mean Queue Length of the System $= L_{q_1} + L_{q_2} + L_{q_3}$

$$\begin{aligned} Lq = {} & \frac{\frac{\lambda}{A}\{1 - cp_{32}\,(bp_{23} + b'p'_{23})\}}{\lambda - \mu_1 + bp_{21}\mu_2 + cp_{31}\mu_3} \\ & + \frac{\lambda}{A}\frac{(ap_{12} + a'p'_{12})}{[\mu_1(ap_{12} + a'p'_{12}) - \mu_2 + \mu_3 cp_{32}]} \\ & + \frac{\lambda}{A}\frac{(ap_{12} + a'p'_{12})\,(bp_{23} + b'p'_{23})}{[\mu_2(bp_{23} + b'p'_{23}) - \mu_3]} \end{aligned}$$

Numerical Results

1. Behaviour of mean queue length of the system with respect to arrival rate (λ) is depicted in Table 1. The probability (ap_1) has also been varied whereas the values of other parameters have been kept fixed.

Table 1

$\mu_1 = 9$, $\mu_2 = 8$, $\mu_3 = 3$, $b = 0.7$,$b' = 0.3$, $c = 0.8$, $c' = 0.2$, $p_1 = 0.7$, $p_{12} = 0.3$,$p'_1 = 0.4$, $p'_{12} = 0.6$, $p_2 = 0.3$, $p_{23} = 0.5$,$p_{21} = 0.2$, $p'_2 = 0.8$, $p'_{23} = 0.2$, $p_3 = 0.3$, $p_{31} = 0.2$, $p_{32} = 0.5$, $p'_3 = 1$												
	$a = 0.3$, $a' = 0.7$, $A = -0.73114$, $a^*p_1 = 0.21$				$a = 0.4$, $a' = 0.6$, $A = -0.73731$, $a^*p_1 = 0.28$				$a = 0.5$, $a' = 0.5$, $A = -0.74348$, $a^*p_1 = 0.35$			
λ	Lq_1	Lq_2	Lq_3	Lq	Lq_1	Lq_2	Lq_3	Lq	Lq_1	Lq_2	Lq_3	Lq
1.0	0.179	0.316	0.083	0.577	0.177	0.263	0.078	0.518	0.176	0.220	0.073	0.469
1.2	0.221	0.379	0.100	0.700	0.219	0.315	0.094	0.628	0.218	0.264	0.088	0.570
1.4	0.267	0.442	0.116	0.825	0.265	0.368	0.109	0.741	0.262	0.308	0.103	0.673
1.6	0.315	0.505	0.133	0.953	0.313	0.420	0.125	0.858	0.310	0.352	0.117	0.779
1.8	0.368	0.568	0.149	1.085	0.364	0.473	0.141	0.978	0.361	0.396	0.132	0.889
2.0	0.423	0.631	0.166	1.221	0.420	0.525	0.156	1.101	0.416	0.440	0.146	1.003
2.2	0.484	0.694	0.183	1.361	0.480	0.578	0.172	1.229	0.476	0.484	0.161	1.121
2.4	0.549	0.758	0.199	1.505	0.544	0.630	0.187	1.362	0.540	0.528	0.176	1.244
2.6	0.619	0.821	0.216	1.656	0.614	0.683	0.203	1.500	0.609	0.572	0.190	1.372
2.8	0.696	0.884	0.232	1.812	0.690	0.735	0.219	1.644	0.684	0.616	0.205	1.506
3.0	0.780	0.947	0.249	1.975	0.773	0.788	0.234	1.795	0.767	0.660	0.220	1.647

Following can be interpreted from Table 1.

Mean queue length of the system increases with the increase in mean arrival rate (λ) and has lower values for higher values of probability (ap_1). However there is no significant difference between mean queue lengths with respect to (ap_1).

2. Behavior of marginal mean queue length at first server (Lq_1) with respect to service rate of first server (μ_1) is depicted in Table 2. The service rate of second server (μ_2) has also been varied whereas the values of other parameters have been kept fixed.

Table 2

$\mu_3 = 9,\ \lambda = 1,\ a = 0.3,\ a' = 0.7,\ b = 0.7,$

$b' = 0.3,\ c = 0.8,\ c' = 0.2,\ p_1 = 0.7,\ p_{12} = 0.3,$

$p_1' = 0.4,\ p_{12}' = 0.6,\ p_2 = 0.3,\ p_{23} = 0.5,\ p_{21} = 0.2,$

$p_2' = 0.8,\ p_{23}' = 0.2,\ p_3 = 0.3,\ p_{31} = 0.2,\ A = -0.73114,$

$p_{32} = 0.5,\ p_3' = 1$

	$\mu_2 = 7$				$\mu_2 = 8$				$\mu_2 = 9$			
μ_1	Lq_1	Lq_2	Lq_3	Lq	Lq_1	Lq_2	Lq_3	Lq	Lq_1	Lq_2	Lq_3	Lq
4.0	1.9714	0.5129	0.0467	2.5310	2.5987	0.2956	0.0500	2.9442	3.8114	0.2076	0.0539	4.0728
4.2	1.4659	0.5545	0.0467	2.0671	1.7866	0.3089	0.0500	2.1455	2.2868	0.2141	0.0539	2.5548
4.4	1.1667	0.6034	0.0467	1.8168	1.3612	0.3235	0.0500	1.7347	1.6334	0.2210	0.0539	1.9083
4.6	0.9690	0.6618	0.0467	1.6774	1.0994	0.3396	0.0500	1.4890	1.2705	0.2284	0.0539	1.5527
4.8	0.8286	0.7327	0.0467	1.6079	0.9221	0.3573	0.0500	1.3295	1.0395	0.2363	0.0539	1.3296
5.0	0.7237	0.8206	0.0467	1.5910	0.7940	0.3770	0.0500	1.2211	0.8795	0.2447	0.0539	1.1782
5.2	0.6424	0.9325	0.0467	1.6216	0.6972	0.3990	0.0500	1.1463	0.7623	0.2538	0.0539	1.0700
5.4	0.5775	1.0798	0.0467	1.7039	0.6214	0.4238	0.0500	1.0952	0.6726	0.2636	0.0539	0.9901
5.6	0.5245	1.2822	0.0467	1.8534	0.5605	0.4518	0.0500	1.0623	0.6018	0.2742	0.0539	0.9298
5.8	0.4804	1.5781	0.0467	2.1052	0.5105	0.4837	0.0500	1.0442	0.5445	0.2856	0.0539	0.8840
6.0	0.4432	2.0516	0.0467	2.5414	0.4686	0.5206	0.0500	1.0392	0.4971	0.2981	0.0539	0.8491

Following can be interpreted from Table 2.

1. Marginal mean queue length at first server (Lq_1) decreases with the increase in service rate of first server (μ_1) but has higher values for higher values of service rate of second server (μ_2) when $\mu_1 \leq 4.8$. However, there is no significant difference between marginal mean queue lengths with respect to μ_2 when $\mu_1 > 4.8$.

2. Marginal mean queue length at second server (Lq_2) increases with the increase in service rate of first server (μ_1) but has lower values for higher values of service rate of second server (μ_2).

3. Marginal mean queue length at third server (Lq_3) remains same irrespective of the value of service rate of first server (μ_1). However, it has higher values for higher values of service rate of second server (μ_2).

4. Graph for Mean Queue Length of the System (Lq) has been plotted with respect to service rate of first server (μ_1) for different values of μ_2 as shown in the Figure 2 and following can be interpreted from the above graph:

(i) Mean Queue length of the system (Lq) decreases with the increase in service rate of first server (μ_1) for $\mu_2 = 8$ and $\mu_2 = 9$. However, for $\mu_2 = 7$ it decreases when $\mu_1 = 5$ and increases thereafter.

(ii) Let Lq_i denote the mean queue length of the system at $\mu_2 = i$, for $i = 7, 8$, or 9. Then mean queue length for different values of μ_2 may be compared as:

Value of Service Rate (μ_1)	Mean Queue Length (Lq_i)
$\mu_1 < 4.29769$	$Lq_7 < Lq_8 < Lq_9$
$4.29769 < \mu_1 < 4.48464$	$Lq_8 < Lq_7 < Lq_9$
$4.48464 < \mu_1 < 4.800465$	$Lq_8 < Lq_9 < Lq_7$
$\mu_1 > 4.800465$	$Lq_9 < Lq_8 < Lq_7$

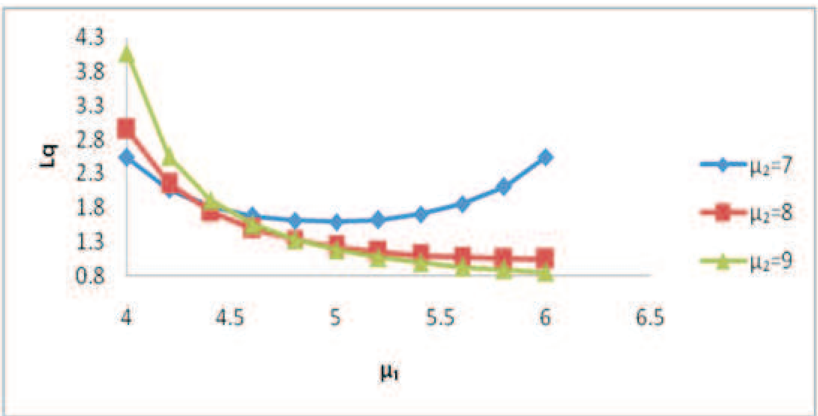


Figure 2. Mean Queue Length vs Service Rate of first server (μ_1)

Conclusion

A queuing system having three service channels is discussed. Mean queue length at the counter of each of these servers is obtained for particular cases as discussed in the previous section of the paper. This model can be fitted to many real life situations. Such situations may be observed in hierarchical systems. The user of such systems can take various important decisions by putting the values of arrival, service and other parameters existing therein in the equations established in this paper.

References

[1] A. Kumar (1990), On feed back in the network queues with service rates proportional to their queue numbers, *JISSOR*, **11**, 29-42.

[2] P.C. Garg and S. Kumari (1998), A note on time dependent solution for a bulk queue with feedback, *IJIMS*, **9** (3), 59-67.

[3] Garg and Srivastava (2006), A measure for time dependent queueing system with feedback and service in batches of variable size, *IJMS*, **5** (1), 109-119.

[4] Kusum et al. (2010), Feedback queue model assumed service rate proportional to queue numbers, *AJMI*, **2** (1), 103-110.

[5] Huang et al. (2015), Control of patient flow in emergency departments or multiclass queues with deadlines and feedback, *Operations Research: Informs*, **63** (4).

[6] Raheja et al. (2016), Designing a vague logic based multi-level feedback queue scheduler, *Egyptian Informatics*, **17** (1), 125-137.