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INTEGRAL OPERATORS OF HARMONIC ANALYSIS IN SPACES DEFINED IN TERMS OF LOCAL CHARACTERISTICS OF FUNCTIONS II

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Abstract: In the paper we consider boundedness of subadditive operators from a rather wide class containing in particular singular integral operators, the Riesz and Bessel potentials maximal functions, fractional maximal functions, the Poisson integrals associated with Laplace-Bessel operator in new scales of Banach spaces, introduced in terms of integral characteristics of type Ω_p^*.

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1. Introduction

Study of integral operators in terms of characteristics of type Ω_p goes back to the papers [1], [2], [3], where the operators of classic Fourier harmonic analysis are considered.

In the present paper similar investigations are carried out for the operators of Fourier-Bessel harmonic analysis. The powerful working apparatus of Fourier-Bessel's harmonic analysis (associated with Fourier-Bessel transform) undoubtedly are singular integral operators (SIO), maximal function, Riesz and Bessel potentials and others. Here we consider convolution structures generated

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not by ordinary but by some special shift T^y (the so called generalized shift [4]) adapted to Fourier-Bessel transform in some coordinates of a point.

Note that in some issues of mathematics, for example when studying boundary values of B-harmonic functions associated with Laplace-Bessel's differential equation

$$\Delta_{B_{m+k,k}}(x) = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2} + \sum_{j=m+1}^{m+k} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\gamma_j}{x_j} \frac{\partial}{\partial x_j} \right),$$

$$x \in R^+_{m+k,k}, \ \ \gamma_{m+1} > 0, ..., \gamma_{m+k} > 0$$

there arises a necessity for studying potentials and SIO generated by the generalized shift $T^y_{\gamma_{n,k}}$ in one-dimensional case by B.M. Levitan [4], [5] and called the generalized or Bessel shift operator (see the papers of I.A. Kiprinov [6] and I.A. Kiprianov, M.I. Klyuchanchev [7]).

In the paper of I.A. Kiprianov and L.A. Ivanov [8] it was proved that the integral operator as cylindrical potential

$$u(x) \equiv I^\alpha_B(f)(x) = \int\limits_{R^+_{m+k,k}} |y|^{\alpha - n - |\gamma_{k,m+k}|} T^y_{y_{n,k}}(f(x))\, y^{\gamma_{k,m+k}} dy,$$

$$0 < \alpha < m + k + |\gamma_{k,m+k}|,$$

(when $\alpha = 2$) called the Riesz generalized potential, is the solution of the equation $\Delta_{B_{m+k,k}} u(x) = f(x)$, and the problem of obtaining a priori estimations in fact was reduced to estimation of these generalized Riesz potentials and their corresponding derivatives (singular integrals). In this connection, they introduced the spaces $L_{p,\gamma_{m+k,k}}$.

Establishment of weight $L_{p,\gamma_{m+k,k}}$ estimations is one of the significant directions of studying integral operators of Fourier-Bessel's harmonic analysis. For SIO generated by the generalized shift $T^y_{\gamma_{n,k}}$ (in the case $k = 1$), this problem was first studied in the paper of I.A. Aliyev and A.Dj. Gadjiev [9] where only the case of radial weights was considered. The case of nonradial weights was first considered in the papers of S.K. Abdullayev and N.R. Karamaliyev [10], S.K. Abdullayev, A.K. Akbarov and M.K. Kerimov [11] when weight function depends only on one coordinate of space variable. The case of certain class f general weights is considered in the paper of S.K. Abdullayev, E.A. Gadjieva and F.A. Isayev [12].

In the present paper, in particular, these results are taken to the case of weights dependent on arbitrary set of $s \in \{1, ..., m + k\}$ coordinates.

To this end, following [3], we introduce the class of operators $\overline{K}_{\gamma_{n,k}}(p,q)$ and integral characteristics

$$\Omega^{*(x)}_{p,\mu_{n,k}}(u,\xi)=\left\{\int\limits_{\left\{x\in R^+_{m+k,k}:|_s x|\le\xi\right\}}|u(x)|^p\,d\mu_{\gamma_{n,k}}(x)\right\}^{1/p}$$

of locally summable functions $\upsilon(\xi)$, $\xi\in P^+_{\mu+k,k}$ and prove the estimations associating this characteristics of the image Au with the same characteristics of the preimage u of the operator $A\in\overline{K}_{\gamma_{n,k}}(p,q)$. In the paper [13] in the same statement this problem was completely studied in terms of characteristics

$$\Omega^{(_s x)}_{p,\mu_{n,k}}(u,\xi)=\left\{\int\limits_{\left\{x\in R^+_{m+k,k}:|_s x|\ge\xi\right\}}|u(x)|^p\,d\mu_{\gamma_{n,k}}(x)\right\}^{1/p}.$$

The obtained estimations becomes a starting point when studying these operators in different scales of Banach spaces determined in terms of the introduced characteristics, one of which is the scale of weight $L_{p,\gamma_{m+k,k}}$ spaces. Note that the considered weight $L_{p,\gamma_{m+k,k}}$ spaces with monotone increasing (decreasing) weights are expressed in terms of characteristics $\Omega^{(_s x)}_{p,\mu_{n,k}}\left(\Omega^{*(_s x)}_{p,\mu_{n,k}}\right)$.

The classes as $\overline{K}_{\gamma_{n,k}}(p,q)$ subadditive operators containing in particular singular integral operators (SIO), maximal and fractional-maximal functions, the Riesz and Bessel and other operators majorized by convolution type integral operators with generalized shift $T^y_{\gamma_{n,k}}$, were introduced for studying a wider class of harmonic analysis operators from a unified position. We also note that owing to generality of the approach, the results obtained by us contain the case of ordinary shift in all coordinates, more exactly, the case of operators of Fourier harmonic analysis.

2. Some Designations and Preliminary Information

Let R^l be Euclidean space of dimension l, and m, $k\ge 0$ be integers,

$$n=m+k\ge 1, R^+_{m+k,k}=$$

$$\left\{(x_1,...,x_{m+k})\in R^{m+k}:x_{m+i}>0,\ i=1,...,k\right\},\ \ R^+_{m+0,0}\equiv R^m.$$

$$-T^{y}_{\gamma_{n,k}}\left(u\left(x\right)\right)=c\int\limits_{0}^{\pi}\cdots\int\limits_{0}^{\pi}u\left(x'-y',\left(x_{m+1},y_{m+1}\right)_{\alpha_1},...,\left(x_{m+k},y_{m+k}\right)\alpha_k\right)\times$$

$$\times\prod_{i=1}^{m+k}\sin^{\gamma_{m+i}-1}\alpha_i d\alpha_i$$

be a generalized shift operator (GSO) generated by the Laplace-Bessel operator

$$\Delta_{B_{m+k,k}}\left(x\right)=\sum_{j=1}^{m}\frac{\partial^2}{\partial x_j^2}+\sum_{j=m+1}^{m+k}\left(\frac{\partial^2}{\partial x_j^2}+\frac{\gamma_j}{x_j}\frac{\partial}{\partial x_j}\right),$$

$$x\in R^{+}_{m+k,k}\ \ \gamma_{m+1}>0,...,\gamma_{m+k}>0\ \ x',y'\in R^m,$$

$$x=\left(x',x_{m+1},...,x_{m+k}\right),\quad y=\left(y',y_{m+1},...,y_{m+k}\right),$$

$$\left(x_{m+i},y_{m+i}\right)_{\alpha_i}=\sqrt{x^2_{m+i}-2x_{m+i}y_{m+i}\cos\alpha_i+y^2_{m+i}}$$

$i=1,...,k, C_v$ be a normalizing multiplier.

Further, we assume

$$\gamma_{n,k}=\left(0,...,0,\gamma_{m+1},...,\gamma_{m+k}\right)\in R^{+}_{m+k,k},\quad\left|\gamma_{n,k}\right|=\sum_{i=1}^{k}\gamma_{m+i},$$

$$y^{\gamma_{n,k}}=\prod_{i=1}^{m+k}y_i^{\gamma_i}=y_{m+1}^{\gamma_{m+1}}\cdots y_{m+k}^{\gamma_{m+k}},$$

$$d\mu_{n,k}\left(y\right)=y^{\gamma_{n,k}}dy,\qquad\text{if}\quad y\in R^{+}_{m+k,k}.$$

In designation $\gamma_{n,k}$, n indicates the dimensionality of this vector, k the amount of its positive coordinates.

Remark 2.1. If $k=0$, we will assume $\gamma_{n,k}=(0,...,0)\in R^m$, $T^{y}_{\gamma_{n,k}}f\left(x\right)=f\left(y-x\right)$ is an ordinary shift, and $d\mu_{n,k}\left(y\right)=dy$.

When $n=m+k\geq 2$ and $s\in\{1,...,m+k-1\}$ we divide the space $R^{+}_{m+k,k}$ of the points $x=(x_1,...,x_{m+k})$ into the direct sum of space R^{+}_{s,k_s} of the points ${}_sx=(x_{n_1},....,x_{n_s})$ by the coordinates $x_{n_1},....,x_{n_s}$, where $k_s=rang\left(\{n_1,...,n_s\}\cap\{m+1,...,m+k\}\right)$,
$1\leq n_1<...<n_s\leq m+k$ and the space $R^{+}_{n-s,(k-k_s)}$ of the points ${}_sx'$ so that $x=\uparrow({}_sx,{}_sx')\in R^{+}_{n,k}$ (for designations see [14]).

Let $m_s=rang\left(\{n_1,...,n_s\}\cap\{1,...,m\}\right)$, then m_s,k_s ....$0\leq m_s\leq m$, $0\leq k_s\leq k$ and $m_s+k_s=s$.

In these designations we also assume

$$m'_s = m - m_s, \quad k'_s = k - k_s, \quad R^+_{s,k_s} \equiv R^+_{m_s+k_s,k_s},$$

$$R^+_{n-s,k-k_s} \equiv R^+_{m'_s+k'_s,k'_s} \equiv R^+_{n-s,k'_s} = R^+_{s',k'_s}$$

In what follows, ${}_sy', \gamma_{n-s,k's}, ({}_sy')^{\gamma_{n-s,k'_s}}$ and $d\mu_{n-s,k'_s}({}_sy')$ are determined from the equalities

$$y = \uparrow \left({}_sy, {}_sy'\right) \gamma_{n,k} = \uparrow \left(\gamma_{s,k_s}, \gamma_{n-s,k'_s}\right),$$

$$y^{\gamma_{n,k}} \equiv y_{m+1}^{\gamma_{m+1}} \cdots y_{m+k}^{\gamma_{m+k}} = {}_s y^{\gamma_{s,k_s}} \cdot \left({}_sy'\right)^{\gamma_{n-s,k'_s}}$$

and

$$d\mu_{n,k}(y) = d\mu_{s,k_s}({}_sy)\, d\mu_{n-s,k'_s}\left({}_sy'\right).$$

If $m_s > 0\,(k_s > 0)$, then we assume

$$\{n_1, ..., n_s\} \cap \{1, ..., m\} = \{j_1, ..., j_{m_s}\}, \quad j_1 < ... < j_{m_s},$$

$$(\{n_1, ..., n_s\} \cap \{m+1, ..., m+k\} = \{m+i_1, ..., m+i_{k_s}\}, i_1 < ... < i_{k_s}).$$

Then, obviously

$${}_sy = \left(y_{j_1}, ..., y_{j_{m_s}}, y_{m+i_1}, ..., y_{m+i_{k_s}}\right)$$

and

$$d_sy = dy_{j_1}...dy_{j_{m_s}} dy_{m+i_1}...dy_{m+i_{k_s}}.$$

Let $s \in \{1, ..., m+k-1\}$, m_s, k_s, and the coordinates of the point ${}_sy$ be fixed. Then ${}_sy' = \left(y_{j'_1}, ..., y_{j'_{m'_s}}, y_{m+i'_1}, ..., y_{m+i'_{k'_s}}\right)$ is also determined uniquely. Assume $y_{i'_1} = y_{m+i'_1}$.

When $G \subseteq R^+_{n,k}$ is a measurable set, and $p \geq 1$

$$L_{p,\gamma_{n,k}}(G) =$$

$$= \left[f - meas.: \left\| K(f) : L_{q,\gamma_{n,k}}(G) \right\| = \left(\int_G |f(y)|^p \, d\mu_{\gamma_{n,k}}(y) \right)^{1/p} < +\infty \right]$$

is a space of functions summable in the p-th degree on the set G.

Further, we will repeatedly use the following easily provable properties of the generalized shift operator $T^y = T^y_{\mu_{n,k}}$ [4], [5]:

T1) the operator $T^y_{\mu_{n,k}}$ is sef-adjoint, i.e.

$$\int\limits_{R^+_{m+k;k}} \upsilon(s)\, T^s_{\gamma_{n,k}} u(x)\, d\mu_{\gamma_{n,k}}(s) = \int\limits_{R^+_{m+k;k}} u(s)\, T^s_{\gamma_{n,k}} \upsilon(x)\, d\mu_{\gamma_{n,k}}(s),$$

$$T2)T^y 1 = 1; \quad T3)T^y (Cf) = CT^y (f), C \in R;$$

$$T4) \text{ if } |f| \le |g|, \text{ then } T^y(|f|) \le T^y(|g|);$$

$$T5) \text{ if } p > 1, \text{then } \left(|T^y(f)|\right)^p \le T^y(|f|)^p;$$

$$T6) \quad \left(\int_{R^+_{m+k,k}} \left(T^s_{\gamma_{n,k}}(|f(x)|)\right)^p d\mu(y)\right)^{1/p} \le \left\| f : L_{q,\gamma_{n,k}}\left(R^+_{n,k}\right)\right\|;$$

$$T7) \quad T^{({}_sy,{}_sy')}_{\gamma_{n,k}} f\left({}_sx,{}_sx'\right) = T^{{}_sy}_{\gamma_{s,k_s}}\left(T^{{}_sy'}_{\gamma_{n-s,k'_s}} f\left({}_sx,{}_sx'\right)\right) =$$

$$T^{{}_sy'}_{\gamma_{n-s,k'_s}}\left(T^{{}_sy}_{\gamma_{s,k_s}} f\left({}_sx,{}_sx'\right)\right).$$

We need the following known inequalities.

Holder's inequality. Let $1 < p < \infty$, $q = \frac{p}{p-1}$ then

$$\left| \int_{R^+_{n,k}} f_1(y) f_2(y) d\mu_{\gamma_{n,k}}(y) \right| \le$$

$$\le \left(\int_{R^+_{n,k}} |f_1(y)|^p d\mu_{\gamma_{n,k}}(y) \right)^{1/p} \left(\int_{R^+_{n,k}} |f_2(y)|^p d\mu_{\gamma_{n,k}}(y) \right)^{1/q}$$

and the existence of the left hand side follows from the existence of the right one.

Minkovskii's inequality. Let $1 \le p < \infty$, $\mu(x)$ and $\nu(y)$ be Radon-Stietjes non-negative measures on the sets G and Ω, respectively. Then

$$\left[\int_G \left(\int_\Omega |\varphi(x,y)| d\nu(y)\right)^p d\mu(x)\right]^{1/p} \le \int_G \left[\int_\Omega |\varphi(x,y)|^p d\mu(x)\right]^{1/p} dv(y).$$

Young's inequality. Let $1 \le p, q, r \le \infty$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$ and

$$(Kf)(x) = \int_{R^+_{n,k}} K(x-y) f(y) d\mu_{\gamma_{n,k}}(y),$$

then

$$\left\| K(f) : L_{q,\gamma_{n,k}}\left(R^{+}_{n,k}\right)\right\| \le \left\| K : L_{r,\gamma_{n,k}}\left(R^{+}_{n,k}\right)\right\| \left\| f : L_{p,\gamma_{n,k}}\left(R^{+}_{n,k}\right)\right\| .$$

Hardy-Littlerwood's theorems on maximal functions. Let $1 \le p \le q \le \infty$, $\overline{u}(t)$ and $\overline{v}(t)$ be positive functions on the interval $(0, +\infty)$. Then for validity of the inequality

$$\left(\int_0^\infty \left| \overline{u}(\tau) \int_t^\infty f(t)\, d\tau \right|^q dt\right)^{1/q} \le k \left(\int_0^\infty |f(t)\, \overline{v}(t)|^p \, dt\right)^{1/p},$$

with the constant k independent of f it is necessary and sufficient to fulfill the conditions

$$\sup_{\xi>0} \left(\int_0^\xi |\overline{u}(t)|^q \, dt\right)^{1/q} \left(\int_\xi^\infty |\overline{v}(t)|^{-p'} \, dt\right)^{1/p'} < \infty$$

3. Main Results

The operator A is said to be subadditive if for any λ, $\mu > 0$ and any functions f and g from the domain of definition of the operator A

$$|A(\lambda f + \mu g)(x)| \le \lambda |A(f)(x)| + \mu |A(g)(x)| .$$

Definition 3.1. Let $1 \le p \le q < +\infty$. Say that the subadditive operator A belongs to the class $\overline{K}_{\gamma_{n,k}}(p,q)$ if $A : L_{p,\gamma_{n,k}}\left(R^{+}_{n,k}\right) \to L_{q,\gamma_{n,k}}\left(R^{+}_{n,k}\right)$ is bounded and for any function $u \in L_{p,\gamma_{n,k}}\left(R^{+}_{n,k}\right)$ with a compact support

$$|Au(x)| \le \int_{R^{+}_{m+k,k}} |y|^{-\beta}\, T^{y}_{\gamma_{n,k}} |u(x)|\, d\mu_{\gamma_{n,k}}(y), \quad x \notin \sup pu,$$

where $\beta = (m + k + |\gamma_{n,k}|)\left((1/p') + q^{-1}\right)$, and C is independent of u.

In the case $p = q$ the operators $A \in \overline{K}_{\gamma_{n,k}}(p,q)$ may be singular [9]. Let us consider

1. Poisson's $B_{\gamma_{n,k}}$-integral:

$$\left(U_{\gamma_{n,k}} f\right)(x) \overset{df}{=} \sup_{t>0} \left(U_{\gamma_{n,k}} f\right)(x,t),$$

$$\left(U_{\gamma_{n,k}}f\right)(x,t)=a_{\nu}\int\limits_{R^{+}_{m+k,k}} t\left(t^2+|y|^2\right)^{-\frac{m+k+1+\left|\gamma_{n,k}\right|}{2}} f(y)\,d\mu_{\gamma_{n,k}}(s);$$

2. $B_{\gamma_{n,k}}$-maximal function:

$$M_{\gamma_{n,k}}f(x)=\sup_{r>0}|B(0,r)|^{-1}_{\gamma_{n,k}}\int\limits_{B(0,r)} T^{y}_{\gamma_{n,k}}|f(x)|\,d\mu_{\gamma_{n,k}}(y),$$

$$B(0,r)=\left\{y\in R^{+}_{m+k,k}:|y|<\varepsilon\right\},\quad |B(0,r)|_{\gamma_{n,k}}\int\limits_{B(0,r)} d\mu_{\gamma_{n,k}}(y);$$

3. $B_{\gamma_{n,k}}$-fractional maximal function

$$M^{\alpha}_{\gamma_{n,k}}f(x)=\sup_{r>0}|B(0,r)|^{\frac{\alpha}{m+k+\gamma_{n,k}}-1}_{\gamma_{n,k}}\int\limits_{B(0,r)} T^{y}_{\gamma_{n,k}}|f(x)|\,d\mu_{\gamma_{n,k}}(y);$$

4. The Riesz $B_{m+k,2v}$ potential

$$I^{\alpha}_{\gamma_{n,k}}f(x)=\int\limits_{R^{+}_{m+k,k}} T^{y}|x|^{\alpha-(m+k)-\left|\gamma_{n,k}\right|} f(y)\,d\mu_{\gamma_{n,k}}(y).$$

It is known [13] that $U_{\gamma_{n,k}}$, $M_{\gamma_{n,k}}\in\overline{K}_{\gamma_{n,k}}(p,p)$ if $p>1$ and

$$M^{\alpha}_{\gamma_{n,k}},I^{\alpha}_{\gamma_{n,k}}\in\overline{K}_{\gamma_{n,k}}(p,q)\ \text{ as } 1<p<q<\infty$$

and

$$\alpha=(m+k+|\gamma_{n,k}|)\left(\frac{1}{p}-\frac{1}{q}\right).$$

In what follows, C is a positive number various in different inequalities.

When $s\in\{1,...,m+k\}$, $A^{*}_{p,\gamma_{n,k}}({}_{s}x)$ denotes the totality of all functions measurable on the set $R^{+}_{m+k,k}$ and belonging to $L_{p,\gamma_{n,k}}\left(\left\{x\in R^{+}_{m+k,k}:|{}_{s}x|\le\xi\right\}\right)$ for every number $\xi>0$.

Further $\alpha_{q,s}=(s+|\gamma_{k_s}|)/q$.

For the function $u\in A^{*}_{p,\gamma_{n,k}}({}_{s}x)$ we introduce the characteristics

$$\Omega^{*({}_{s}x)}_{p,\mu_{n,k}}(u,\xi)=\left\{\int_{\left\{x\in R^{+}_{m+k,k}:|{}_{s}x|\le\xi\right\}}|u(x)|^{p}\,d\mu_{\gamma_{n,k}}(x)\right\}^{1/p},\xi>0,$$

and the set

$$J^*_{p,\gamma_{n,k}}({}_sx) = \left\{ u \in A^*_{p,\gamma_{n,k}}({}_sx) : \int\limits_{\xi}^{+\infty} t^{\alpha_{q,s}-1} \Omega^{*({}_sx)}_{p,\mu_{n,k}}(u,t)\,dt < +\infty, \quad \forall \xi > 0 \right\}.$$

By definition, the non-negating function $\alpha(t)$, $0 < t < \infty$ belongs to the set N^* if for small $\varepsilon > 0$ for almost all $t \in (\varepsilon, +\infty)$ $\alpha(t) > 0$ and $\forall \varepsilon > 0$ the integral $\int\limits_{\varepsilon}^{+\infty} \alpha(t)\,dt$ converges.

Let $1 \le p < \infty$ and $\varphi \in N^*$. Introduce the space

$$I^{*({}_sx)}_{p,\mu_{n,k}}(\varphi) = \left\{ u - meas. \left\| u : I^{*({}_sx)}_{p,\mu_{n,k}}(\varphi) \right\|^p \overset{df}{=} \int\limits_0^{\infty} \left(\Omega^{*({}_sx)}_{p,\mu_{n,k}}(u,\xi) \right)^p \varphi(\xi)\,d\xi < \infty \right\}.$$

Let

$$L_{p,\gamma_{n,k}}(\omega : G) =$$
$$\left[f - meas. : \left\| f : L_{p,\gamma_{n,k}}(G) \right\| = \left(\int\limits_G |f(y)\,\omega(y)|^p \, d\mu_{\gamma_{n,k}}(y) \right)^{1/p} < +\infty \right].$$

$L_{p,\gamma_{n,k}}(G)$ is a space with the weight $\omega(y)$.

The main results of the paper are given in the following three theorems.

Theorem 3.2. *Let $A \in \overline{K}_{\gamma_{n,k}}(p,q)$, $s \in \{1,...,m+k\}$ and $u \in J^*_{p,\mu_{n,k}}({}_sx)$. Then for almost all $x \in R^+_{m+k,k}$ there exists $\upsilon(x) = A(u)(x)$ and it holds the estimation*

$$\Omega^{*({}_sx)}_{p,\mu_{n,k}}(\upsilon,\xi) \le c\xi^{\alpha_{q,s}} \int\limits_{\xi}^{+\infty} t^{-(\alpha_{q,s}+1)} \Omega^{*({}_sx)}_{p,\mu_{n,k}}(u,t)\,dt, \quad \xi > 0, (\Omega)^*,$$

where the constant c is independent of u and ξ.

Note that the estimation $(\Omega)^*$ is new in the case of ordinary shift in all coordinates, as well.

Theorem 3.3. *Let $1 < p \le q < \infty$, $A \in \overline{K}_{\gamma_{n,k}}(p,q)$, $\varphi, \psi \in N^*$, $s \in \{1,...,m+k\}$ and the following condition be fulfilled:*

$$\sup_{t>0}\left(\int_0^t \left[\psi^{\frac{1}{q}}(\xi)\,\xi^{\alpha_{q,s}}\right]^q d\xi\right)^{1/q} \times \left(\int_t^\infty \left[\varphi^{1/p}(\xi)\,(\xi)\,\xi^{(\alpha_{q,s}+1)}\right]^{-p} d\xi\right)^{1/p'} < \infty. \,(\varphi,\psi)^*$$

Then the operator A acts from the space $I^{*({}_sx)}_{p,\mu_{n,k}}(\varphi)$ to $I^{*({}_sx)}_{q,\mu_{n,k}}(\psi)$ and it holds the inequality

$$\left\|Au : I^{*({}_sx)}_{p,\mu_{n,k}}(\psi)(\psi)\right\| \le c\left\|Au : I^{*({}_sx)}_{p,\mu_{n,k}}(\psi)(\varphi)\right\|,$$

where c is independent of the function u.

Theorem 3.4. *Let $1 < p \le q < \infty$, $A \in \overline{K}_{\gamma_{n,k}}(p,q)$, $s \in \{1,...,m+k\}$, $\omega(t)$ and $\omega_1(t)$ be functions non-negative and non-decreasing on the interval $(0,+\infty)$ such that*

$$c^{-1}\omega^p(\xi) \le \int_\xi^\infty \omega^p(t)\,t^{-1}dt \le c\omega^p(\xi)$$

and

$$c_1^{-1}\omega_1^p(\xi) \le \int_\xi^\infty \omega_1^p(t)\,t^{-1}dt \le c_1\omega_1^p(\xi)\,. \qquad (\omega)^*$$

Then if the condition

$$\sup_{t>0}\left(\int_0^t (\omega_1(\xi)\,\xi^{\alpha_{q,s}})^q \frac{d\xi}{\xi}\right)^{1/q} \times \left(\int_t^\infty (\omega(\xi)\,\xi^{\alpha_{q,s}})^{-p'} \frac{d\xi}{\xi}\right)^{1/p'} < \infty, \quad (\omega,\omega_1)^*$$

is fulfilled, then the operator A acts from the space $L_{p,\gamma_{n,k}}\left(\omega(|{}_sx|);R^+_{n,k}\right)$ to $L_{q,\gamma_{n,k}}\left(\omega_1(|{}_sx|);R^+_{n,k}\right)$ and it holds the inequality

$$\left\|Au : L_{q,\gamma_{n,k}}\left(\omega_1(|{}_sx|);R^+_{n,k}\right)\right\| \le c\left\|Au : L_{p,\gamma_{n,k}}\left(\omega(|{}_sx|);R^+_{n,k}\right)\right\|$$

where c is independent of the function u.

4. Point-to-point Integral Inequalities in Terms of $\Omega^{*({}_s x)}_{p,\mu_{n,k}}$ Characteristics

Let $\chi_E(t)$ be a characteristical function of the set $E \subset [0,+\infty)$ and $s \in \{1,...,m+k-1\}$. Assume

$$u^*_{\xi,s}(y) = \chi_{[\xi,+\infty)}(|{}_s y|)\,|u(y)|, \qquad y \in R^+_{n,k}, \quad \xi > 0;$$

$$u^{*0}_{\xi,s}({}_s y) = \left(\int\limits_{R^+_{s',k'_s}} u^{*p}_{\xi,s}\left({}_s y, {}_s y'\right) d\mu_{s',k'_s}\left({}_s y'\right) \right)^{1/p}, \quad {}_s y \in R^+_{m_s+k_s,k_s}.$$

$$I^*_{\xi,s}(u,\xi)(x) = \int\limits_{R^+_{m+k,k}} \frac{u^*_{\xi,s}(y)\,d\mu_{n,k}(y)}{(|{}_s x' -_s y'| + |{}_s y| + \xi)^\beta}, \quad x \in R^+_{n,k}.$$

Also, if $s = m+k$, then we assume

$$u^*_{\xi,s}(y) = u^*_{\xi,s}(y) \equiv \chi_{[\xi,+\infty)}(|y|)\,|u(y)|, \quad y \in R^+_{n,k}, \quad \xi > 0;$$

$$I^*_{\xi,s}(u,\xi)(x) = \int\limits_{R^+_{m+k,k}} \frac{u^*_{\xi,s}(y)}{(|y|+\xi)^\beta} d\mu_{n,k}(y), \quad x \in R^+_{n,k}.$$

Everywhere in what follows, $\sigma_s = m_s - \beta p' \;\; \alpha_{q,s} = (s + |\gamma_{s,k_s}|)/q$.

Lemma 4.1. *Let $s \in \{1,...,m+k\}$, $1 < p \le q < +\infty$, $u \in A^*_{p,\gamma_{n,k}}({}_s x)$ and $\xi > 0$. Then*

a) $\forall x \in \left\{ x \in R^+_{m+k,k} : |{}_s x| \le \xi \right\}$,

$$I^*_{\beta,s}(u,\xi)(x) \le$$

$$\le c \int\limits_{\left\{R^+_{s,k_s} : |{}_s y| \ge \xi\right\}} \left(\sum_{l=1}^{k'_s} x_{i'_1} + |{}_s y| \right)^{|y_{s'},k'_s|/p'} |{}_s y|^{(\sigma_s - k'_s)/p'} u^{*0}_{\xi,s}({}_s y)\, d\mu_{s,k_s}({}_s y),$$

if $1 \le s \le m+k-1$,

$$I^*_{\beta,s}(u,\xi)(x) \le c \int\limits_{\left\{R^+_{n,k} : |y| \ge \xi\right\}} |y|^{-\beta}\,|u(y)|\, d\mu_{n,k}(y) \quad \textit{if } \; s = m+k.$$

b)

$$\left\| I^{*}_{\beta,s}(u,\xi) : L_{p,\gamma_{n,k}}\left(R^{+}_{n,k} : |_{s}x| \le \xi\right)\right\| \le$$

$$\le c\left(\xi^{s+|\gamma_{s,k_s}|}\right)^{1/q} \int\limits_{R^{+}_{m_s+k_s,k_s}} (|_{s}y|+\xi)^{\frac{|\gamma'_s,k'_s|+s'}{r}-\beta} u^{*0}_{\xi,s}(_{s}y)\, d\mu_{s,k_s}(_{s}y)$$

where the constant c is independent of u and ξ.

Lemma 4.2. *Let $s \in \{1,...,m+k\}$ and $u \in J^{*}_{p,\mu_{n,k}}(_{s}x)$, then*

$$\int\limits_{R^{+}_{m_s+k_s,k_s}} (|_{s}y|+\xi)^{\frac{|\gamma'_s,k'_s|+s'}{r}-\beta} u^{*0}_{\xi,s}(_{s}y)\, d\mu_{s,k_s}(_{s}y) \le \int\limits_{\xi}^{+\infty} t^{-(\alpha^{*}_{p,s}+1)}\Omega^{*(_{s}x)}_{p,\mu_{n,k}}(u,t)\, dt,$$

where the constant c is independent of u and ξ.

Proof of lemma 4.1.

Prove item a) of lemma 1. Let $s \in \{1,...,m+k\}$ and

$$x \in \left\{x \in R^{+}_{m+k,k} : |_{s}x| \le \xi\right\}.$$

Introduce the denotations

$$d\mu\left(y_{i'_l}\right) = \left(y_{i'_l}\right)^{\gamma_{i'_l}} dy_{i'_l}, \quad l = 1,...,k'_s,$$

$$a_j = \left(\sum_{l=1}^{j}\left|x_{i'_1} - y_{i'_l}\right| + |_{s}x|\right), \quad j = 1,...,k'_s.$$

Sequential application of the estimation

$$\Lambda = \int\limits_{R^{+}} (a_j+d)^{\sigma} d\mu\left(y_{i'_l}\right)$$
$$\le c\left(\left(x_{i'_l}\right)^{\gamma_{i'_l}}(d+a_{i-1})^{\sigma+1} + \left(d+x_{i'_l}+a_{i-1}\right)^{\sigma+\gamma_{i'_l}+1}\right) \quad (4.1)$$

where $d > 0$, $j \ge 1$ leads to the proof of the inequality

$$B = \left\{\int\limits_{R^{+}_{m'_s+k'_s,k'_s}} \frac{d\mu_{s',k'_s}(_{s}y')}{(|_{s}x' -_{s} y'| + |_{s}y|)^{\beta p'}}\right\}^{1/p'}$$

$$\leq c\left(\sum_{l=1}^{k'_s} x_{i'_1}+|_sy|\right)^{\left(\sum_{l=1}^{k'_s}\gamma_{i'_l}\right)/p'} |_sy|^{(\sigma-k'_s)/p'}.$$

Using the Foubini theorem and then the Holder inequality in the inner integral, we get

$$I^*_{\xi,s}(u,\xi)(x)=\int_{R^+_{m+k,k}}\frac{u^*_{\xi,s}(y)\,d\mu_{n,k}(y)}{(|_sx'-_sy'|+|_sy|+\xi)^\beta}=$$

$$=\int_{R^+_{m_s+k_s,k_s}}\left\{\int_{R^+_{m'_s+k'_s,k'_s}}\frac{u^*_{\xi,s}(y)\,d\mu_{s',k'_s}(_sy')}{(|_sx'-_sy'|+|_sy|)^\beta}\right\}d\mu_{s,k_s}(_sy)\leq$$

$$\leq\int_{R^+_{m_s+k_s,k_s}}\left\{\int_{R^+_{m'_s+k'_s,k'_s}}\frac{d\mu_{s',k'_s}(_sy')}{(|_sx'-_sy'|+|_sy|)^{\beta p'}}\right\}^{1/p'}\times$$

$$\times\left(\int_{R^+_{m'_s+k'_s,k'_s}}u^p_{\xi,s}(_sy,_sy')\,d\mu_{s',k'_s}(_sy')\right)^{1/p}d\mu_{s,k_s}(_sy)=$$

$$=\int_{R^+_{m_s+k_s,k_s}}\left(\sum_{l=1}^{k'_s}x_{i'_l}+|_sy|\right)^{\left(\sum_{l=1}^{k'_s}\gamma_{i'_l}\right)/p'}|_sy|^{(\sigma-k'_s)/p'}\,u^{*0}_{\xi,s}(_sy)\,d\mu_{s,k_s}(_sy).$$

Allowing for estimation B, item a) of lemma 4.1 is proved in the case $s\in\{1,...,m+k-1\}$. The case $s=m+k$ directly follows from the definition of $I^*_{\beta,s}(u,\xi)$.

Now let us prove item b) of lemma 4.1 in the case $s\in\{1,...,m+k-1\}$. First of all note that having applied the Young inequality, where $r>1$, $r^{-1}=(p')^{-1}+q^{-1}$, we get

$$A=\left(\int_{R^+_{m_s+k_s,k_s}}\left(\int_{R^+_{m'_s+k'_s,k'_s}}\frac{u^*_{\xi,s}(y)\,d\mu_{s',k'_s}(_sy')}{(|_sx'-_sy'|+|_sy|+\xi)^\beta}\right)^q d\mu_{s',k'_s}(_sx')\right)^{1/q}$$

$$\leq c\left(\int\limits_{R^+_{m'_s+k'_s,k'_s}} \frac{d\mu_{s',k'_s}\left({}_sy'\right)}{\left(|{}_sy'|+|{}_sy|+\xi\right)^{\beta r}}\right)^{1/r} u^{*0}_{\xi,s}\left({}_sy\right) \leq c\left(|{}_sy|+\xi\right)^{\frac{|y'_s,k'_s|+s'}{r}-\beta} u^{*0}_{\xi,s}\left({}_sy\right)$$

$$= A'^{*}_{\xi,s}.$$

Taking into account the last one, having used the Foubini theorem on reduction of multiple integral to repeated integrals, and using the passage

$$\left(\int\limits_1\int\limits_2\left\{\int\limits_{1'}\int\limits_{2'}(...)\right\}^q\right)^{1/q} = \left(\int\limits_1\left[\int\limits_2\left\{\int\limits_{1'}\int\limits_{2'}(...)\right\}^q\right]^{(1/q)q}\right)^{1/q} \leq$$

$$\leq \left(\int\limits_1\left[\int\limits_{1'}\left\{\int\limits_2\left(\int\limits_{2'}(...)\right)^q\right\}^{1/q}\right]^q\right)^{1/q},$$

we get

$$\left\|I^*_{\xi,s}\left(u,\xi\right) : L_{q,\gamma_{n,k}}\left(R^+_{n,k} : |{}_sx| \leq \xi\right)\right\|$$

$$= \left(\int\limits_{\left\{R^+_{m_s+k_s,k_s}:|{}_sx|\leq\xi\right\}} d\mu_{s,k_s}\left({}_sx\right) \int\limits_{R^+_{m'_s+k'_s,k'_s}} d\mu_{s',k'_s}\left({}_sx'\right) \times\right.$$

$$\left.\times\left(\int\limits_{R^+_{m_s+k_s,k_s}} d\mu_{s,k_s}\left({}_sy\right) \int\limits_{R^+_{m'_s+k'_s,k'_s}} \frac{u^*_{\xi,s}\left(y\right) d\mu_{s',k'_s}\left({}_sy'\right)}{\left(|{}_sx'-{}_s y'|+|{}_sy|+\xi\right)^{\beta}}\right)^q\right)^{1/q} \leq$$

$$\leq c\left(\int\limits_{\left\{R^+_{m_s+k_s,k_s}:|{}_sx|\leq\xi\right\}} \left(\int\limits_{R^+_{m_s+k_s,k_s}} A d\mu_{s,k_s}\left({}_sy\right)\right)^q d\mu_{s,k_s}\left({}_sx\right)\right)^{1/q} \leq$$

$$\leq \left(\int\limits_{\left\{R^+_{m_s+k_s,k_s}:|{}_sx|\leq\xi\right\}} \left(\int\limits_{R^+_{m_s+k_s,k_s}} A'^{*}_{\xi,s} d\mu_{s,k_s}\left({}_sy\right)\right)^q d\mu_{s,k_s}\left({}_sx\right)\right)^{1/q} =$$

$$= \int\limits_{R^+_{m_s+k_s,k_s}} A'^{*}_{\xi,s} d\mu_{s,k_s}\left({}_s y\right) \left(\int\limits_{\left\{R^+_{m_s+k_s,k_s}:\left|{}_s x\right|\le\xi\right\}} d\mu_{s,k_s}\left({}_s x\right) \right)^{1/q} =$$

$$= \int\limits_{R^+_{m_s+k_s,k_s}} \left(\left|{}_s y\right|+\xi\right)^{\frac{\left|\gamma_{s'},k'_s\right|+s'}{r}-\beta} u^{*0}_{\xi,s}\left({}_s y\right) d\mu_{s,k_s}\left({}_s y\right)$$

$$\times \left(\int\limits_{\left\{R^+_{m_s+k_s,k_s}:\left|{}_s x\right|\le\xi\right\}} d\mu_{s,k_s}\left({}_s x\right) \right)^{1/q} =$$

$$= \left(\xi^{s+\left|\gamma_{s,k_s}\right|}\right)^{1/q} \int\limits_{R^+_{m_s+k_s,k_s}} \left(\left|{}_s y\right|+\xi\right)^{\frac{\left|\gamma_{s'},k'_s\right|+s'}{r}-\beta} u^{*0}_{\xi,s}\left({}_s y\right) d\mu_{s,k_s}\left({}_s y\right).$$

As

$$\left(\int\limits_{\left\{R^+_{m_s+k_s,k_s}:\left|{}_s y\right|\le\xi\right\}} d\mu_{s,k_s}\left({}_s x\right) \right)^{1/q} = \left(\xi^{s+\left|\gamma_{s,k_s}\right|}\right)^{1/q}.$$

This proves item b) of lemma 4.1 in the case $s \in \{1, ..., m+k-1\}$.

Now let $s = m+k$. For further reasonings we need to pass to spherical coordinates.

Let $S^+_{n,k} \in \left\{x \in R^+_{n,k} : |x| = 1\right\}$ be a unit sphere of $R^+_{n,k}$. Then passing to spherical coordinates centered in the origin of coordinates

$$\left(y \to (r,\theta), \theta \in S^+_{n,k}, r \ge 0\right),$$

we get

$$d\mu_{n,k}(y) = y^{\gamma_{n,k}} dy = \left(\frac{y}{|y|}\right)^{\gamma_{n,k}} dy = \theta^{\gamma_{n,k}} r^{n-1+\left|\gamma_{n,k}\right|} dr d\sigma(\theta),$$

where $d\sigma(\theta)$ is the element of the area of the surface of sphere $S^+_{n,k}$.

Then allowing for the estimation

$$B = \left(\int\limits_{R^+_{m+k,k}} d\mu_{n,k}(x) \right)^{1/q}$$

$$= \left(\int\limits_{S^+_{n,k}} \theta^{\gamma_{n,k}} d\sigma(\theta) \int\limits_0^\xi r^{m+k+|\gamma_{n,k}|-1} dr \right)^{1/q} \le c\xi^{\left(m+k+|\gamma_{n,k}|\right)/q},$$

we get

$$\left\| I^*_{\xi,s}(u,\xi) : L_{q,\gamma_{n,k}} \left(R^+_{n,k} : |_s x| \le \xi \right) \right\| =$$

$$= \left(\int\limits_{\{R^+_{m+k,k}:|x|\le\xi\}} \left(\int\limits_{R^+_{m+k,k}} \frac{u^*_{\xi,s}(y)}{(|y|+\xi)^\beta} d\mu_{n,k}(y) \right)^q d\mu_{n,k}(x) \right)^{1/q} \le$$

$$\le B \int\limits_{R^+_{m+k,k}} \frac{u^*_{\xi,s}(y)}{(|y|+\xi)^\beta} d\mu_{n,k}(y) = B \int\limits_{R^+_{m+k,k}} \frac{\chi_{[\xi,+\infty)}(|y|)\,|u(y)|}{(|y|+\xi)^\beta} d\mu_{n,k}(y) \le$$

$$\le c\xi^{\left(m+k+|\gamma_{n,k}|\right)/q} \int\limits_{\{R^+_{n,k}:|y|\ge\xi\}} \frac{|u(y)|}{(|y|+\xi)^\beta} d\mu_{n,k}(y).$$

Taking into account that if $s = m+k$, then $\left|\gamma_{s',k'_s}\right| + s' = 0$, this proves item b) and completely Lemma 4.1.

Prove lemma 4.2. Let $r > 1$ and $r^{-1} = (p')^{-1} + q^{-1}$. Assume

$$\tau = \left(\left|\gamma_{s',k'_s}\right| + s' \right)/r - \beta, \quad \beta' = \tau + \left(|\gamma_{s,k_s}| + s\right) - 1.$$

It is easy to show

$$\tau = -\left(|\gamma_{s,k_s}| + s\right)\left((p')^{-1} + q^{-1} \right) \text{ and } \beta' = -\left(|\gamma_{s,k_s}| + s\right)\left(q^{-1} - p^{-1}\right) - 1.$$

Let $s \in \{1, ..., m+k-1\}$. Let us consider

$$\int\limits_{R^+_{m_s+k_s,k_s}} (|_s y| + \xi)^{\frac{\left|\gamma_{s',k'_s}\right|+s}{r} - \beta} u^{*0}_{\xi,s}(_s y)\, d\mu_{s,k_s}(_s y) \le$$

$$\leq \int\limits_{\left\{R^+_{s,k_s}:\xi\leq|_sy|\right\}} |_sy|^\tau\, u^{*0}_{\xi,s}\left({}_sy\right) d\mu_{s,k_s}\left({}_sy\right).$$

Let $\eta > 1$, then $(\eta-1)\frac{1}{r^{-\eta+1}}\int\limits_r^{+\infty} t^{-\eta}dt = 1$.

Allowing for this, passing to spherical coordinates, then

$$d\mu_{s,k_s}\left({}_sy\right) = y^{\gamma_{s,k_s}}d_sy = \theta^{\gamma_{s,k_s}}r^{s-1+\left|\gamma_{s,k_s}\right|}drd\sigma\left(\theta\right) = \theta^{\gamma_{s,k_s}}r^{s-1+\left|\gamma_{s,k_s}\right|}drd\sigma\left(\theta\right)$$

and using the Dirichlet formula, we have

$$\int\limits_{\left\{R^+_{s,k_s}:\xi\leq|_sy|\right\}} |_sy|^\tau\, u^{*0}_{\xi,s}\left({}_sy\right) d\mu_{s,k_s}\left({}_sy\right)$$

$$= \int\limits_{S_{s,k_s}}\int\limits_{\xi}^{+\infty} r^\tau u^{*0}_{\xi,s}\left(r\theta\right)\theta^{\gamma_{s,k_s}}r^{s-1+\left|\gamma_{s,k_s}\right|}drd\sigma\left(\theta\right)$$

$$= \int\limits_{S_{s,k_s}}\left(\int\limits_{\xi}^{+\infty} r^{\beta'}u^{*0}_{\xi,s}\left(r\theta\right)dr\right)\theta^{\gamma_{s,k_s}}d\sigma\left(\theta\right) =$$

$$= (\eta-1)\int\limits_{S_{s,k_s}}\left(\int\limits_{\xi}^{+\infty}\left(\frac{1}{r^{-\eta+1}}\int\limits_r^{+\infty}t^{-\eta}dt\right)r^{\beta'}u^{*0}_{\xi,s}\left(r\theta\right)dr\right)\theta^{\gamma_{s,k_s}}d\sigma\left(\theta\right) =$$

$$= (\eta-1)\int\limits_{S_{s,k_s}}\left(\int\limits_{\xi}^{+\infty}t^{-\eta}\left(\int\limits_{\xi}^{t}r^{\beta'+\eta-1}u^{*0}_{\xi,s}\left(r\theta\right)dr\right)dt\right)\theta^{\gamma_{s,k_s}}d\sigma\left(\theta\right) =$$

$$= (\eta-1)\int\limits_{S_{s,k_s}}\times\left(\int\limits_{\xi}^{+\infty}t^{-\eta}\times\right.$$

$$\left.\times\left(\int\limits_{\xi}^{1}r^{\beta'+\eta-1\left(s+\left|\gamma_{s,k_s}\right|-1\right)\frac{1}{p}}u^{*0}_{\xi,s}\left(r\theta\right)dr^{\left(s+\left|\gamma_{s,k_s}\right|-1\right)\frac{1}{p}}dr\right)dt\right)\theta^{\gamma_{s,k_s}}d\sigma\left(\theta\right)\leq$$

$$\leq\left(\beta''=\beta'+\eta-1-\left(s+\left|\gamma_{s,k_s}\right|-1\right)/p=-\left(\left|\gamma_{s,k_s}\right|+s\right)q^{-1}-1+\eta-\left(p'\right)^{-1}\right)=$$

$$= (\eta-1)\int\limits_{S_{s,k_s}}\left(\int\limits_{\xi}^{+\infty} t^{-\eta}\left(\int\limits_{\xi}^{t}\left(u^{*0}_{\xi,s}(r\theta)\right)^p r^{\left(s+\left|\gamma_{s,k_s}\right|-1\right)}dr\right)^{1/p}\times\right.$$

$$\left.\times\left(\int\limits_{\xi}^{t} r^{\beta''p'}dr\right)^{1/p'}dt\right)\theta^{\gamma_{s,k_s}}d\sigma(\theta)\le\int\limits_{\xi}^{+\infty} t^{-\eta}\left(\int\limits_{\xi}^{t} r^{\beta''p'}dr\right)^{1/p}\times$$

$$\times\left\{\int\limits_{S_{s,k_s}}\left(\int\limits_{\xi}^{t}\left(u^{*0}_{\xi,s}(r\theta)\right)^p r^{\left(s+\left|\gamma_{s,k_s}\right|-1\right)}dr\right)^{1/p}\theta^{\gamma_{s,k_s}}d\sigma(\theta)\right\}dt\le$$

$$\le c\int\limits_{\xi}^{+\infty} t^{-\eta}t^{\beta''+(p')^{-1}}\times$$

$$\times\left\{\int\limits_{S_{s,k_s}}\left(\int\limits_{\xi}^{t}\left(u^{*0}_{\xi,s}(r\theta)\right)^p r^{\left(s+\left|\gamma_{s,k_s}\right|-1\right)}dr\right)\theta^{\gamma_{s,k_s}}d\sigma(\theta)\right\}^{1/p}dt=$$

$$=\Big(-\eta+\beta''+\left(p'\right)^{-1}=-\eta-\left(\left|\gamma_{s,k_s}\right|+s\right)q^{-1}-1+\eta-\left(p'\right)^{-1}+$$

$$+\left(p'\right)^{-1}=\left(\left|\gamma_{s,k_s}\right|+s\right)q^{-1}-1=-\left(\alpha_{q,s}+1\right)\Big)=$$

$$\le\int\limits_{\xi}^{+\infty} t^{-(\alpha_{q,s}+1)}\times\left\{\int\limits_{S_{s,k_s}}\left(\int\limits_{\xi}^{t}\left(u^{*0}_{\xi,s}(r\theta)\right)^p r^{\left(s+\left|\gamma_{s,k_s}\right|-1\right)}dr\right)\theta^{\gamma_{s,k_s}}d\sigma(\theta)\right\}^{1/p}dt=$$

(here we take into account that
$\left(u^{*0}_{\xi,s}(r\theta)\right)^p=\int\limits_{R^+_{s',k'_s}} u^{*0}_{\xi,s}\left((r\theta),_s y'\right)d\mu_{s',k'_s}\left(_s y'\right)$)

$$\int\limits_{\xi}^{+\infty} t^{-(\alpha_{q,s}+1)}\times\left\{\int\limits_{S_{s,k_s}}\left(\int\limits_{\xi}^{t}\int\limits_{R^+_{s',k'_s}} u^{*p}_{\xi,s}\left((r\theta)^p,_s y'\right)d\mu_{s',k'_s}\left(_s y'\right)r^{\left(s+\left|\gamma_{s,k_s}\right|-1\right)}dr\right)\times\right.$$

$$\times\theta^{\gamma_{s,k_s}}d\sigma(\theta)\}^{1/p}dt \le \int_{\xi}^{+\infty} t^{-(\alpha_{q,s}+1)} \times \left\{ \int_{\{R^+_{s,k_s}:\xi\le|_sy|\le t\}} |u(y)|_s^p\, d\mu_{n,k}(y) \right\}^{1/p} dt \le$$

$$\le c\int_{\xi}^{+\infty} t^{-(\alpha_{q,s}+1)}\Omega^{*(_sx)}_{p,\mu_{n,k}}(u,t)\,dt.$$

Lemma 4.2 was proved in the case $s\in\{1,...,m+k-1\}$.

Let $s=m+k$, then taking into account that $\left|\gamma_{s',k'_s}\right|+s'=0$ and

$$u^{*0}_{\xi,s}(y)=u^{*}_{\xi,s}(y)\equiv\chi_{[\xi,+\infty)}(|y|)\,|u(y)|\,,\quad y\in R^+_{n,k},\quad \xi>0,$$

we get

$$\int_{R^+_{m_s+k_s,k_s}} (|_sy|+\xi)^{\frac{\left|\gamma_{s',k'_s}\right|+s'}{r}-\beta}\, u^{*0}_{\xi,s}(_sy)\,d\mu_{s,k_s}(_sy)=$$

$$\int_{\{R^+_{m+k,k}:|y|\ge\xi\}} |y|^{-\beta}\,|u(y)|\,d\mu_{n,k}(y)\,.$$

Now passing to spherical coordinates, we get $(\alpha=m+k+|\gamma_{n,k}|-1)$

$$\int_{\{R^+_{m+k,k}:|y|\ge\xi\}} |y|^{-\beta}\,|u(y)|\,d\mu_{n,k}(y)=\int_{S^+_{n,k}}\theta^{\gamma_{n,k}}d\sigma(\theta)\int_{\xi}^{\infty}|u(r\theta)|\,r^{m+k+|\gamma_{n,k}|-1-\beta}dr=$$

$$=c\int_{S^+_{n,k}}\theta^{\gamma_{n,k}}d\sigma(\theta)\int_{\xi}^{\infty}|u(r\theta)|\,r^{\alpha-\beta}\left(r^{\eta-1}\int_{r}^{\infty}t^{-\eta}dt\right)dr=$$

$$=\int_{\xi}^{\infty}t^{-\eta}\left(\int_{S^+_{n,k}}\theta^{\gamma_{n,k}}d\sigma(\theta)\left[\int_{t}^{\infty}|u(r\theta)|\,r^{\alpha-\beta}r^{\eta-1}dr\right]\right)dt=$$

$$=\int_{\xi}^{\infty}t^{-\eta}\left(\int_{S^+_{n,k}}\theta^{\gamma_{n,k}}d\sigma(\theta)\left[\int_{t}^{\infty}|u(r\theta)|\,r^{\alpha/p}r^{\alpha/p'-\beta+\eta-1}dr\right]\right)dt=$$

$$= \int_{\xi}^{\infty} t^{-\eta} \left(\int_{S^+_{n,k}} \theta^{\gamma_{n,k}} d\sigma(\theta) \left[\int_t^{\infty} |u(r\theta)|^p r^{\alpha} dr \right]^{1/p} \times \left[\int_t^{\infty} r^{(\alpha/p'-\beta+\eta-1)p'} dr \right]^{1/p'} dt \right) =$$

$$= c \int_{\xi}^{\infty} t^{-\eta} t^{\alpha/p'-\beta+\eta-1+1/p'} \left(\int_{S^+_{n,k}} \theta^{\gamma_{n,k}} d\sigma(\theta) \left[\int_t^{\infty} |u(r\theta)|^p r^{\alpha} dr \right]^{1/p} \right) dt =$$

$$= c \int_{\xi}^{\infty} t^{-(\alpha_{q,s}+1)} \left(\int_{S^+_{n,k}} \theta^{\gamma_{n,k}} d\sigma(\theta) \left[\int_t^{\infty} |u(r\theta)|^p r^{\alpha} dr \right]^{1/p} \right) dt \le$$

$$\le \int_{\xi}^{\infty} t^{-(\alpha_{q,s}+1)} \Omega^{*({}_s x)}_{p,\mu_{n,k}}(u,t)\, dt.$$

This completely proves lemma 4.2.

5. Proof of Main Theorems

Prove theorem 3.2 Let $A \in \overline{K}_{\gamma_{n,k}}(p,q)$, $s \in \{1,...,m+k\}$ and $u \in J^*_{p,\mu_{n,k}}({}_s x)$. Then $u(y) \in L_{p,\gamma_{n,k}}\left(\left\{x \in R^+_{m+k,k} : |{}_s x| \le \xi\right\}\right)$ for every $\xi > 0$. Take $\xi \in (0,\infty)$ and represent the function $u(y)$ in the form of the sum $u = u^*_{\xi,s}(y) + \overline{u}^*_{\xi,s}(y)$, where $\overline{u}^*_{\xi,s}(y) = u(y) - u^*_{\xi,s}(y)$. Obviously $\overline{u}^*_{\xi,s}(y) = \chi_{[0,\xi]}(|{}_s y|)\, u(y)$. Then $\overline{u}^*_{\xi,s}(y) \in L_{p,y}\left(R^+_{m+k,k}\right)$ and $A\overline{u}^*_{\xi,s}(x)$ exists almost for all $x \in R^+_{m+k,k}$. Now let us prove that $Au^*_{\xi,s}(x)$ converges for all points $x \in \left\{R^+_{m+k,k} : |{}_s x| < \xi\right\}$.

Note that if $\beta > 0$, then $T^y\left(|x|^{-\beta}\right) \le c\,|x-y|^{-\beta}$ and furthermore, if

$$x \in \left\{R^+_{m+k,k} : |{}_s x| < \xi\right\} \text{ and } |{}_s y| \ge \frac{\xi}{2}, \text{ then}$$

$$c\left(\left|{}_s x' -{}_s y'\right| + |{}_s y| + \xi\right)^{-\beta} \le |x-y|^{-\beta} \le c_1\left(\left|{}_s x' -{}_s y'\right| + |{}_s y| + \xi\right)^{-\beta}.$$

Taking into account this condition $A \in \overline{K}_{\gamma_{n,k}}(p,q)$ and self-adjointness of the operator T^y; by item a) of lemma 4.1 we get

$$\left|A\left(u^*_{\xi,s}(x)\right)\right| \le \int_{R^+_{m+k,k}} |y|^{-\beta}\, T^y_{\mu_{\gamma_{n,k}}}\left(\left|u^*_{\xi,s}(x)\right|\right) d\mu_{\gamma_{n,k}}(y) =$$

$$= \int\limits_{R^{+}_{m+k,k}} \left|u^{*}_{\xi,s}(y)\right| T^{y}_{\mu_{\gamma_{n,k}}}\left(|x|^{-\beta}\right) d\mu_{\gamma_{n,k}}(y) \leq c \int\limits_{R^{+}_{m+k,k}} \left|u^{*}_{\xi,s}(y)\right| |x-y|^{-\beta}\, d\mu_{\gamma_{n,k}}(y) \leq$$

$$\leq c \int\limits_{R^{+}_{m+k,k}} \frac{\left|u^{*}_{\xi,s}(y)\right| d\mu_{\gamma_{n,k}}(y)}{\left(|_{s}x' -_{s} y'| + |_{s}y|\right)^{\beta}} \leq c I^{*}_{\beta,s}(u,\xi)(x) < +\infty$$

Now let us prove estimation (Ω^{*}) of theorem 3.2. Take $\xi > 0$.
We have

$$\Omega^{(_{s}x)}_{q,\mu_{n,k}}(u,\xi) \leq \left\{ \int\limits_{\left\{x \in R^{+}_{m+k,k}: |_{s}x| \leq \xi\right\}} \left|A\left(u^{*}_{2\xi,s}(x)\right)\right|^{q} d\mu_{\gamma_{n,k}}(x) \right\}^{1/q} +$$

$$+ \left\{ \int\limits_{\left\{x \in R^{+}_{m+k,k}: |_{s}x| \leq \xi\right\}} \left|A\left(\overline{u}^{*}_{2\xi,s}(y)\right)\right|^{q} d\mu_{\gamma_{n,k}}(x) \right\}^{1/q} = i_{1} + i_{2}$$

By the condition $A \in \overline{K}_{\nu}(p,q)$ we get

$$i_{2} = \left\{ \int\limits_{\left\{x \in R^{+}_{m+k,k}: |_{s}x| \leq \xi\right\}} \left|A\left(\overline{u}^{*}_{2\xi,s}(x)\right)\right|^{q} d\mu_{\gamma_{n,k}}(x) \right\}^{1/q} \leq$$

$$\leq c \left\{ \int\limits_{\left\{x \in R^{+}_{m+k,k}: |_{s}x| \leq \xi\right\}} \left|\overline{u}^{*}_{2\xi,s}(x)\right|^{p} d\mu_{\gamma_{n,k}}(x) \right\}^{1/q} \leq$$

$$\leq c\Omega^{(_{s}x)}_{p,\mu_{n,k}}(u,\xi) \leq c\xi^{\alpha_{q,s}} \int\limits_{\xi}^{+\infty} t^{-(\alpha_{q,s}+1)} \Omega^{*(,x)}_{p,\mu_{n,k}}(u,t)\, dt.$$

Using the estimation

$$\left|Au^{*}_{2\xi,s}(x)\right| \leq c I^{*}_{\beta,s}(u,2\xi)(x)$$

and applying lemma 4.1 and lemma 4.2, we get

$$i_1 \leq c\xi^{\alpha_{q,s}} \int\limits_{\xi}^{+\infty} t^{-(\alpha_{q,s}+1)} \Omega^{*(,x)}_{p,\mu_{n,k}}(u,t)\,dt.$$

Theorem 3.2 is proved.

For further reasoningss we need two lemmas.

Lemma 5.1. *Let $\phi \in N^*$ and $\omega^p(t) = \int\limits_t^{+\infty} \varphi(\xi)\,d\xi \quad t > 0$. Then*

$$I^{*({}_s x)}_{p,\mu_{n,k}}(\phi) = L_{p,\nu}\left(\omega(|{}_s x|), R^+_{m+k,k}\right)$$

and the corresponding norms are equivalent.

Proof.

$$\int\limits_{\xi}^{\infty} \left(\Omega^{*({}_s x)}_{p,\mu_{n,k}}(u,\xi)\right)^p \varphi(\xi)\,d\xi = \int\limits_0^{\infty} \int\limits_{R^+_{m+k,k}} \left|u\left({}_s y, {}_s y'\right)\right|^p \chi_{[\xi,\infty]}(|{}_s y|)\,d\mu_{\gamma_{n,k}}(y)\,\varphi(\xi)\,d\xi =$$

$$= \int\limits_{R^+_{m+k,k}} \left|u\left({}_s y, {}_s y'\right)\right|^p \left(\int\limits_0^{\infty} \chi_{[\xi,\infty]}(|{}_s y|)\,\varphi(\xi)\,d\xi\right) d\mu_{\gamma_{n,k}}(y) =$$

$$= \int\limits_{R^+_{m+k,k}} \left|u\left({}_s y, {}_s y'\right)\right|^p \left(\int\limits_{|{}_s y|}^{\infty} \varphi(\xi)\,d\xi\right) d\mu_{\gamma_{n,k}}(y) = \int\limits_{R^+_{m+k,k}} |u(y)|^p\, \omega^p(|{}_s y|)\,d\mu_{\gamma_{n,k}}(y).$$

□

Lemma 5.2. *Let $\phi \in N^*$ and the integral*

$$\int\limits_{\xi}^{x} \left[\varphi^{1/p}(t)\,t^{(\alpha_{q,s}+1)}\right]^{-p'} dt < \infty,$$

converge, then $I^{*(,x)}_{\rho,\mu_{n},k}(\varphi) \subset J^{*}_{\rho,\mu_{n},k}({}_s x)$.

Proof.

$$\int\limits_{\xi}^{+\infty} t^{-(\alpha_{q,s}+1)}\Omega^{*({}_sx)}_{\rho,\mu_n,k}(u,t)dt = \int\limits_{\xi}^{+\infty} \Omega^{*(x)}_{\rho,\mu_n,k}(u,t)\,[\varphi(t)]^{1/p}\,[\varphi(t)]^{-1/p}\,t^{-(\alpha_{q,s}+1)}dt \le$$

$$\le \left(\int\limits_{\xi}^{\infty} \Omega^{*(,x)}_{\rho,\mu_n,k}(u,t))^p\varphi(t)dt\right)^{1/p} \times \left(\int\limits_{\xi}^{\infty} \left([\varphi(t)]^{-1/p}\,t^{-(\alpha_{q,s}+1)}\right)^{p'} dt\right)^{1/p'} < +\infty$$

The lemma is proved. □

Proof of theorem 3.3.

Convergence of the integral in the conditions of lemma 4.4 whence $I^{*({}_sx)}_{\rho,\mu_n,k}(\varphi) \subset J^{*}_{\rho,\mu_n,k}({}_sx)$, follows from the condition $(\varphi,\psi)^*$ of theorem 3.3. Consequently, it holds estimation $(\Omega)*$ of theorem 3.2. Assume

$$\overline{u}(\xi) = \xi^{\alpha_{q,s}}\psi^{\frac{1}{q}}(\xi)\ ,\ \text{and}\ \overline{v}(\xi) = \varphi^{\frac{1}{q}}(\xi)\xi^{(1+\alpha_{q,s})}$$

then

$$\psi(\xi) = (\overline{u}(\xi)\xi^{\alpha_{p,s}})^q \ \text{ and } \varphi(\xi) = (\overline{v}(\xi)\xi^{-\alpha_{q,s}}+1)^p.$$

Then using estimation $(\Omega)^*$ and applying the Hardy theorem, we get

$$\left(\int\limits_{0}^{\infty} \left(\Omega^{*(,x)}_{\rho,\mu_n,k}(v,\xi)\right)^q \psi(\xi)d\xi\right)^{\frac{1}{q}}$$

$$\le c\left(\int\limits_{0}^{\infty}\left(\xi^{\alpha_{p,s}}\int\limits_{\xi}^{+\infty} t^{-(\alpha_{q,s}+1)}\Omega^{*(,x)}_{\rho,\mu_n,k}(u,t)dt\right)^q \psi(\xi)d\xi\right)^{\frac{1}{q}} =$$

$$= c\left(\int\limits_{0}^{\infty}\left(\xi^{\alpha_{p,s}}\psi^{\frac{1}{q}}(\xi)\int\limits_{\xi}^{+\infty} t^{-(\alpha_{q,s}+1)}\Omega^{*({}_sx)}_{\rho,\mu_n,k}(u,t)dt\right)^q d\xi\right)^{\frac{1}{q}} =$$

$$= c\left(\left(\int\limits_{0}^{\infty} (\overline{u}(\xi)\int\limits_{\xi}^{+\infty} t^{-(\alpha_{q,s}+1)}\Omega^{*({}_sx)}_{\rho,\mu_n,k}(u,t)dt\right)^q\right)^{\frac{1}{q}} \le$$

$$\leq c\left(\int_0^\infty \overline{v}^p(\xi)\left(t^{-(\alpha_{q,s}+1)}\Omega^{*(_s x)}_{\rho,\mu_n,k}(u,\xi)\right)^p d\xi\right)^{\frac{1}{p}} =$$

$$= c\left(\int_0^\infty \overline{v}^p(\xi)\xi^{-(1+\alpha_{q,s})p}\left(\Omega^{*(_s x)}_{\rho,\mu_n,k}(u,\xi)\right)^p d\xi\right)^{\frac{1}{p}} = \left(\int_0^\infty \varphi(\xi)\left(\Omega^{*(_s x)}_{\rho,\mu_n,k}(u,\xi)\right)^p d\xi\right)^{\frac{1}{p}}.$$

Theorem 3.3 is proved.

Let all conditions of theorem 3.4 be fulfilled.

Assume $\varphi(\xi) = \omega(\xi)^p \xi^{-1}$ and $\psi(\xi) = \omega_1^q(\xi)\xi^{-1}$. Then the condition $(\varphi,\psi)^*$ of theorem 3.3 takes the form

$$\sup_{t>0}\left(\int_0^1 (\omega_1(\xi)\xi^{\alpha_{q,s}})^q \frac{d\xi}{\xi}\right)^{1/q}\left(\int_t^\infty (\omega(\xi)\xi^{\alpha_{q,s}})^{-p'}\frac{d\xi}{\xi}\right)^{1/p'} < \infty.$$

Therefore theorem 3.4 is the corollary of theorem 3.3.

Note that theorem 3.4 may be proved without the condition $(\omega)^*$ as will (see [11]).

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