International Journal of Pure and Applied Mathematics
Volume 115 No. 2 2017, 445-453
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v115i2.19

ijpam.eu

ON (m, n)-JORDAN $*$-DERIVATIONS AND RELATED MAPPINGS IN RINGS WITH INVOLUTION

Husain Alhazmi[1], Shakir Ali[2 §], Mohammad Salahuddin Khan[3]

[1,2]Department of Mathematics
Faculty of Science
King Abdulaziz University
Jeddah, 21589, SAUDI ARABIA

[3]Department of Applied Mathematics
Aligarh Muslim University
Aligarh, 202002, INDIA

Abstract: The aim of the present paper is to introduce the notions of (m, n) $*$-derivations and (m, n)-Jordan $*$-derivations, and to prove some related results in rings with involution. In particular, we prove that if $(m+n+1)!$-torsion free $*$-ring R admits an additive mapping $d : R \to R$ such that $d(x^{m+n+1}) = (m+n+1)x^n d(x)(x^*)^m$ for all $x \in R$, then d is an (m, n)-Jordan $*$-derivation on R, where $m \geq 0,\ n \geq 0$ with $m+n \neq 0$ are some fixed integers. Further, we prove that if R is an $(n+1)!$-torsion free, then an additive mapping T satisfies $T(x^{n+1}) = T(x)(x^*)^n$ for all $x \in R$ must be a Jordan left $*$-centralizer. As an application, Jordan left $*$-centralizers are characterized.

AMS Subject Classification: 16W10, 16W25, 16N60

Key Words: involution, torsion free ring, $*$-ring, semiprime $*$-ring, (m, n)-Jordan $*$-derivation, Jordan left $*$-centralizer

1. Introduction

Throughout this paper, R will denote an associative ring with the identity e. Recall that a ring R is said to be prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$,

Received: November 16, 2017
Revised: May 1, 2017
Published: July 14, 2017

[§]Correspondence author

and R is semiprime if $aRa = \{0\}$ implies $a = 0$. A ring R is said to be n-torsion free, where $n > 1$ is an integer and if $nx = 0$ implies $x = 0$ for all $x \in R$. An additive mapping $x \mapsto x^*$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$ is called an involution. A ring equipped with an involution $*$ is called a $*$-ring or ring with involution (see [11] for details). An additive mapping $d : R \to R$ is said to be a derivation (resp. Jordan derivation) on R if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. Following [7], an additive mapping $d : R \to R$ is said to be a Jordan left derivation of R if $d(x^2) = 2xd(x)$ holds for all $x \in R$. According [18], an additive mapping $T : R \to R$ is called a left centralizer in case $T(xy) = T(x)y$ for all $x, y \in R$. If R has the identity element, then $T : R \to R$ is a left centralizer if and only if T is of the form $T(x) = ax$ for all $x \in R$ and some fixed element $a \in R$. An additive mapping $T : R \to R$ is called a Jordan left centralizer in case $T(x^2) = T(x)x$ holds for all $x \in R$. The definition of a right centralizer and Jordan right centralizer should be self-explanatory. For T a left centralizer of R and S a right centralizer of R, the pair (S, T) is called double centralizer of R if $xT(y) = S(x)y$ holds for all $x, y \in R$ (see [10] for details). Following ideas from [5], Zalar [18] has proved that any left (resp. right) Jordan centralizers on a 2-torsion free semiprime ring is a left (resp. right) centralizer. The second named author together with Dar and Vukman [3], studied this problem in the setting of rings with involution. An additive mapping T is called a two sided centralizer in case $T : R \to R$ is a left and a right centralizer. In case $T : R \to R$ is a two sided centralizer, where R is a semiprime ring with extended centroid C, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ (see [4, Proposition 2.2.3]).

Let R be a $*$-ring. Following [1], an additive mapping $d : R \to R$ is said to be a $*$-derivation (resp. Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) for all $x, y \in R$. Note that the mapping $x \mapsto ax^* - xa$, where a is fixed element in R, is a Jordan $*$-derivation. Such Jordan $*$-derivation is said to be inner. The study of Jordan $*$-derivations has been motivated by the problem of the representativity of quadratic forms by bilinear forms (for the results concerning this problem we refer the reader to [12], [13] and [19]). It turns out that the question, whether each quadratic form can be represented by some bilinear form, is connected with the question, whether every Jordan $*$-derivation is inner, as shown by Šemrl [12]. In [6], Brešar and Vukman studied some algebraic properties of Jordan $*$-derivations. Further in [8], it was shown that in case of complex $*$-algebra every double centralizer (S, T) of A induces a Jordan $*$-derivation d, defined by $d(x) = T(x^*) - S(x)$ for all $x \in A$. Some recent results concerning these mappings can be found in [1], [8] and [14] where further references can be found.

The main purpose of this manuscript is to introduce the notions of (m,n) $*$-derivations and (m,n)-Jordan $*$-derivations(Definitions 1.3 & 1.4), and to prove some results related to (m,n)-Jordan $*$-derivations and Jordan left $*$-centralizers in rings with involution. In Section 2, we prove that if $(m+n+1)!$-torsion free $*$-ring R admits an additive mapping $d : R \to R$ such that $d(x^{m+n+1}) = (m+n+1)x^n d(x)(x^*)^m$ for all $x \in R$, then d is an (m,n)-Jordan $*$-derivation on R, where $m \geq 0,\ n \geq 0$ with $m+n \neq 0$ are some fixed integers(Theorem 2.1). Moreover, we establish the commutativity of $(m+n+1)!$-torsion free prime $*$-ring that satisfy certain $*$-differential identity(Theorem 2.2). Further in Section 3, we study Jordan left $*$-centralizers in $*$-rings, and prove that if R is an $(n+1)!$-torsion free, then an additive mapping T satisfies $T(x^{n+1}) = T(x)(x^*)^n$ for all $x \in R$ must be a Jordan left $*$-centralizer(Theorem 3.1). As an application, Jordan left $*$-centralizers are characterized(Corollary 3.3).

In [2] and [15], the following notions were introduced:

Definition 1. Let $m \geq 0,\ n \geq 0$ with $m+n \neq 0$ be some fixed integers. An additive mapping $d : R \to R$ is called an (m,n)-derivation if it satisfies

$$(m+n)d(xy) = 2md(x)y + 2nxd(y) \text{ for all } x, y \in R.$$

Definition 2. Let $m \geq 0,\ n \geq 0$ with $m+n \neq 0$ be some fixed integers. An additive mapping $d : R \to R$ is called an (m,n)-Jordan derivation if

$$(m+n)d(x^2) = 2md(x)x + 2nxd(x) \text{ holds for all } x \in R.$$

Motivated by the definitions of (m,n)-derivation and (m,n)-Jordan derivation, we introduce the following notions:

Definition 3. Let R be a $*$-ring, and let $m \geq 0,\ n \geq 0$ with $m+n \neq 0$ be some fixed integers. An additive mapping $d : R \to R$ is called an (m,n) $*$-derivation if it satisfies

$$(m+n)d(xy) = 2md(x)y^* + 2nxd(y) \text{ for all } x, y \in R.$$

Definition 4. Let R be a $*$-ring and let $m \geq 0,\ n \geq 0$ with $m+n \neq 0$ be some fixed integers. An additive mapping $d : R \to R$ is called an (m,n)-Jordan $*$-derivation if it satisfies

$$(m+n)d(x^2) = 2md(x)x^* + 2nxd(x) \text{ for all } x \in R.$$

Obviously, a (0,1)-Jordan $*$-derivation is a Jordan left derivation on R. Further, if R is 2-torsion free, then every (1,1)-Jordan $*$-derivation is a Jordan $*$-derivation. Thus, the notion of (m,n)-Jordan $*$-derivation includes the notion of Jordan left derivation as well as the notion of Jordan $*$-derivation.

Example 5. *Let* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{Z} \right\}$. *Then R is a ring under usual matrix operations. Taking* $\ast : R \to R$ *such that*

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{\ast} = \begin{pmatrix} 0 & b & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{for all } a, b \in \mathbb{Z}.$$

Next, consider the mapping $d : R \to R$ *such that*

$$d \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{for all } a, b \in \mathbb{Z}.$$

It can be easily check that for any nonnegative integers m and n, d is a nonzero (m,n)-Jordan $\ast$-derivation on R.

2. (m, n)-Jordan $\ast$-Derivations

The present section deals with the study of (m,n)-Jordan $\ast$-derivation and motivated by the recent work of Vukman [15]. In [16], Vukman and Kosi-Ulbl proved the following result: Let R be an $(m+n+2)!$-torsion free semiprime ring with identity element. Suppose there exists an additive mapping $d : R \to R$ such that $d(x^{m+n+1}) = (m+n+1)x^m d(x) x^n$ is satisfied for all $x \in R$. In this case, d is a derivation, which maps R into its center. In case R is a noncommutative prime ring, we have $d = 0$. This result motivated us to consider the following theorem.

Theorem 6. *Let $m \geq 0$, $n \geq 0$ with $m+n \neq 0$ be some fixed integers and R be an $(m+n+1)!$-torsion free $\ast$-ring. If $d : R \to R$ is an additive mapping such that $d(x^{m+n+1}) = (m+n+1)x^n d(x)(x^{\ast})^m$ for all $x \in R$, then d is an (m,n)-Jordan $\ast$-derivation on R.*

Proof. In view of our hypothesis, we have

$$d(x^{m+n+1}) = (m+n+1)x^n d(x)(x^{\ast})^m \text{ for all } x \in R. \tag{1}$$

Putting $x = e$ in (1), we get

$$d(e) = (m+n+1)d(e).$$

Since R is $(m+n+1)!$-torsion free, the above expression forces that $d(e)=0$. For any positive integer k, replacing x by $x+ke$ in (1) and using the fact that $d(e)=0$, we obtain

$$\begin{aligned}&\sum_{j=0}^{m+n}\binom{m+n+1}{j}d(x^{m+n+1-j})k^j\\ =\ \ &(m+n+1)\left(\sum_{j=0}^{n}\binom{n}{j}x^{n-j}k^j\right)d(x)\left(\sum_{j=0}^{m}\binom{m}{j}(x^*)^{m-j}k^j\right)\end{aligned}$$

for all $x\in R$. Using (1) and collecting the coefficient's of k^j for all $j=1,2,\ldots,m+n$, we find that

$$\sum_{j=1}^{m+n}p_j(x,e)k^j=0 \text{ for all } x\in R. \tag{2}$$

Putting $k=1,2,\ldots,m+n$ in (2), we obtain the system of $m+n$ homogeneous equations, whose matrix of coefficients is Vandermonde matrix

$$\begin{pmatrix}1 & 1 & \ldots & 1\\ 2 & 2^2 & \ldots & 2^{m+n}\\ \vdots & \vdots & \vdots & \vdots\\ (m+n) & (m+n)^2 & \ldots & (m+n)^{m+n}\end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than or equal to $m+n$, and R is $(m+n+2)!$-torsion free, it follows that $p_j(x,e)=0$ for all $j=1,2,\ldots,m+n$. In particular, for $p_{m+n-1}(x,e)=0$, we have

$$\begin{aligned}&\binom{m+n+1}{m+n-1}d(x^2)\\ =\ \ &(m+n+1)\left\{\binom{n}{n-1}\binom{m}{m}xd(x)+\binom{n}{n}\binom{m}{m-1}d(x)x^*\right\}\end{aligned}$$

for all $x\in R$. This implies that

$$(m+n+1)(m+n)d(x^2)=2(m+n+1)\{nxd(x)+md(x)x^*\} \text{ for all } x\in R. \tag{3}$$

Since R is $(m+n+1)!$-torsion free, so (3) reduces to

$$(m+n)d(x^2)=2md(x)x^*+2nxd(x) \text{ for all } x\in R.$$

This shows that d is an (m,n)-Jordan $*$-derivation of R. The theorem has now been proved. □

Corollary 7. *Let R be a 3!-torsion free $*$-ring. If $d : R \to R$ is an additive mapping such that $d(x^3) = 3xd(x)x^*$ for all $x \in R$, then d is a Jordan $*$-derivation on R.*

Next, we establish the commutativity of prime $*$-rings involving certain additive map.

Theorem 8. *Let $m \geq 0$, $n \geq 0$ with $m+n \neq 0$ be some fixed integers and R be an $(m+n+2)!$-torsion free semiprime $*$-ring. If $d : R \to R$ is an additive mapping such that $d(x^{m+n+1}) = (m+n+1)(x^*)^n d(x)(x^*)^m$ for all $x \in R$, then d maps R into its center. Moreover, if R is prime and d is nonzero, then R is commutative.*

Proof. We are given that $d : R \to R$ is an additive mapping such that

$$d(x^{m+n+1}) = (m+n+1)(x^*)^n d(x)(x^*)^m \text{ for all } x \in R. \tag{4}$$

Applying $*$ on both sides of the relation (4), we get

$$d(x^{m+n+1})^* = (m+n+1)x^m d(x)^* x^n \text{ for all } x \in R. \tag{5}$$

Now, we define a mapping $f : R \to R$ such that $f(x) = d(x)^*$ for all $x \in R$. Then (5) can be written as

$$f(x^{m+n+1}) = (m+n+1)x^m f(x)x^n \text{ for all } x \in R.$$

Since d and $*$ both are additive, so f is too. Hence, by [16, Theorem 2], we conclude that f is a derivation which maps R into its center. That is, $[f(x), y] = 0$ for all $x, y \in R$. This implies that $[d(x)^*, y] = 0$ for all $x, y \in R$. Again, we applying $*$ on both sides of the last relation, we get $[d(x), y^*] = 0$ for all $x, y \in R$ and henceforth, we conclude that d maps R into its center. Further, if R is prime and d is nonzero, then in view of [16, Theorem 2], we conclude that R is commutative. Thereby the proof is completed. □

3. Jordan Left $*$-Centralizers

Let R be a $*$-ring. Following [3], an additive mapping $T : R \longrightarrow R$ is said to be a left $*$-centralizer (resp. reverse left $*$-centralizer) if $T(xy) = T(x)y^*$(resp. $T(xy) = T(y)x^*$) holds for all $x, y \in R$. The definition of a right $*$-centralizer (resp. reverse right $*$-centralizer) should be self explanatory. Note that for some fixed element $a \in R$, the mapping $x \mapsto ax^*$ is a reverse left $*$-centralizer and

$x \mapsto x^{\star}a$ is a reverse right $*$-centralizer on R. An additive mapping $T : R \longrightarrow R$ is called a $*$-centralizer if T is both a left and right $*$-centralizer. An additive mapping $T : R \longrightarrow R$ is said to be a Jordan left $*$-centralizer (resp. right) if $T(x^2) = T(x)x^{\star}$ (resp. $T(x^2) = x^{\star}T(x)$) holds for all $x \in R$. In [3], Ali et al. proved that if a semiprime $*$ -ring such that $char(R) \neq 2$ admits an additive mapping $T; R \to R$ such that $T(x^2) = T(x)x^{\star}$ holds for all $x \in R$, then T is a reverse left $*$-centralizer that is, $T(xy) = T(y)x^{\star}$ for all $x, y \in R$. Inspired by the above mentioned result, here we consider the following situation.

Theorem 9. *Let $n \geq 1$ be a fixed integer, and let R be an $(n+1)!$-torsion free $*$-ring. If $T : R \to R$ is an additive mapping such that $T(x^{n+1}) = T(x)(x^{\star})^n$ for all $x \in R$, then T is a Jordan left $*$-centralizer of R.*

Proof. By assumption, we have

$$T(x^{n+1}) = T(x)(x^{\star})^n \text{ for all } x \in R. \tag{6}$$

Replacing x by $x + ke$ in (6) for any positive integer k, we get

$$\begin{aligned} T((x+ke)^{n+1}) &= T(x+ke)((x+ke)^{\star})^n \\ &= T(x)(x^{\star}+ke)^n + kT(e)(x^{\star}+ke)^n \end{aligned} \tag{7}$$

for all $x \in R$. Expanding both sides (7) and using (6), we obtain

$$\begin{aligned} & T\left(\binom{n+1}{1}x^n k + \ldots + \binom{n+1}{n-1}x^2k^{n-1} + \binom{n+1}{n}xk^n + k^{n+1}e\right) \\ = \;& T(x)\left\{\binom{n}{1}(x^{\star})^{n-1}k + \ldots + \binom{n}{n-1}x^{\star}k^{n-1} + k^n e\right\} \\ + \;& kT(e)\left\{(x^{\star})^n + \binom{n}{1}(x^{\star})^{n-1}k + \ldots + \binom{n}{n-1}x^{\star}k^{n-1} + k^n e\right\} \end{aligned}$$

for all $x \in R$. The above expression can be written as

$$kf_1(x,e) + k^2 f_2(x,e) + \ldots + k^n f_n(x,e) = 0 \text{ for all } x \in R,$$

where $f_i(x,e)$ are the coefficients of $k^{i'}$s for all $i = 1, 2, \ldots, n$. Henceforth using the same arguments as we have used in the proof of Theorem 6, we conclude that $f_i(x,e) = 0$ for all $i = 1, 2, \ldots, n$. In particular, for $f_n(x,e) = 0$, we have

$$\binom{n+1}{n}T(x) = \binom{n}{n}T(x) + \binom{n}{n-1}T(e)x^{\star} \text{ for all } x \in R.$$

Since R is $(n+1)!$-torsion free, the above relation reduces to

$$T(x) = T(e)x^{*} \text{ for all } x \in R.$$

This implies that

$$T(x^2) = T(e)x^{*2} = (T(e)x^{*})x^{*} = T(x)x^{*} \text{for all } x \in R.$$

Hence, T is a Jordan left $*$-centralizer of R. This finishes the proof of the theorem. □

As immediate consequences of Theorem 9, we obtain the following results:

Corollary 10. *Let $n \geq 1$ be a fixed integer, and let R be an $(n+1)!$-torsion free semiprime $*$-ring. If $T : R \to R$ is an additive mapping such that $T(x^{n+1}) = T(x)(x^{*})^n$ for all $x \in R$, then T is a reverse left $*$-centralizer on R.*

Proof. By Theorem 9, T is a Jordan left $*$-centralizer on R. In view of Proposition 2.3 in [3], we conclude that T is a reverse left $*$-centralizer on R. This proves the corollary. □

Corollary 11. *Let $n \geq 1$ be a fixed integer, and let R be an $(n+1)!$-torsion free semiprime $*$-ring. If $T : R \to R$ is an additive mapping such that $T(x^{n+1}) = T(x)(x^{*})^n$ for all $x \in R$, then $T(x) = qx^{*}$ for all $x \in R$, where $q \in Q_r(R)$, the right Martindale ring of quotients (see [4] for details).*

Proof. In view of above corollary, we conclude that T is a reverse left $*$-centralizer on R. That is, $T(xy) = T(y)x^{*}$ for all $x, y \in R$. Note that a map $\eta : R \to R$ defined by $\eta(x) = T(x)^{*}$ is a right R-module homomorphism (i.e., a left centralizer). Therefore, there exists $q \in Q_r(R)$ such that $\eta(x) = T(x)$ for all $x \in R$. Hence, $T(x) = qx^{*}$ for all $x \in R$. □

Acknowledgment: The authors are greatly indebted to the referee/s for a careful checking of the details and for helpful comments.

References

[1] S. Ali, On generalized $*$-derivation in $*$-rings, *Palest. J. Math.*, **1** (2012), no. 1, 32-37.

[2] S. Ali, A. Fošner, On generalized (m,n)-derivations and generalized (m,n)-Jordan derivations in rings, *Algebra Colloq.*, **21** (2014), no. 3, 411-420.

[3] S. Ali, N. A. Dar, J. Vukman, Jordan left *-centralizers of prime and semiprime rings with involution, *Beitr. Algebra Geom.*, **54** (2013), no. 2, 609-624.

[4] K. I. Beidar, W. S. Martindale III, A. V. Mikhalev, *Rings with generalized identities,* Marcel Dekker, Inc., New York, 1996.

[5] M. Brešar, Jordan derivations on semiprime rings, *Proc. Amer. Math. Soc.*, **104** (1988), no. 4, 1003-1006.

[6] M. Brešar, J. Vukman, On some additive mappings in rings with involution, *Aequationes Math.*, **38** (1989), no. 2-3, 178-185.

[7] M. Brešar, J. Vukman, On left derivations and related mappings, *Proc. Amer. Math. Soc.*, **110** (1990), no. 1, 7-16.

[8] M. Brešar, B. Zalar, On the structure of Jordan $*$-derivations, *Colloq. Math.*, **63** (1992), no. 2, 163-171.

[9] D. Bridges, J. Bergen, On the derivation of x^n in a ring, *Proc. Amer. Math. Soc.*, **90** (1984), no. 1, 25-29.

[10] R. C. Busby, Double centralizers and extensions of C^*-algebras, *Trans. Amer. Math. Soc.* 132 (1968), 79-99.

[11] I. N. Herstein, *Rings with involution*, The Univ. of Chicago Press, Chicago 1976.

[12] P. Šemrl, On Jordan $*$-derivations and an application, *Colloq. Math.*, **59** (1990), no. 2, 241-251.

[13] P. Šemrl, Quadratic functionals and Jordan $*$-derivations, *Studia Math.*, **97** (1991), no. 3, 157-165.

[14] J. Vukman, A note on Jordan $*$-derivations in semiprime rings with involution, *Int. Math. Forum*, **1** (2006), no. 13-16, 617-622.

[15] J. Vukman, On (m, n)-Jordan derivations and commutativity of prime rings, *Demonstratio Math.*, **41** (2008), no. 4, 773-778.

[16] J. Vukman, I. Kosi-Ulbl, On some equations related to derivations in rings, *Int. J. Math. Math. Sci.* (2005), no. 17, 2703-2710.

[17] J. Vukman, I. Kosi-Ulbl, A note on derivations in semiprime rings, *Int. J. Math. Math. Sci.* (2005), no. 20, 3347-3350.

[18] B. Zalar, On centralizers of semiprime rings, *Comment. Math. Univ. Carolin.*, **32** (1991), no. 4, 609-614.

[19] B. Zalar, Jordan $*$-derivation pairs and quadratic functionals on modules over $*$-rings, *Aequationes Math.*, **54** (1997), no. 1-2, 31-43.