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ANALYTICAL APPROXIMATION FOR FRACTIONAL ORDER LOGISTIC EQUATION

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Abstract: The aim of this paper is to obtain an approximate analytical solution of the fractional order logistic equation using the optimal homotopy asymptotic method (OHAM). OHAM uses optimally determined auxiliary constants to control and adjust the convergence of the series solution. We apply OHAM to the fractional order logistic equation for non-integer and integer derivatives. Additionally, we compare its performance with that of the Adams-Bashforth-Moulton method (ABFMM).

AMS Subject Classification: 65L99
Key Words: fractional order logistic equation, fractional differential equation, Caputo fractional derivative, optimal homotopy asymptotic method

1. Introduction

Fractional differential equations (FDEs) appear in a wide variety phenomena and applications. Thus, FDEs have become the focus of many different studies [1]. Fractional calculus (derivatives and integrals of arbitrary orders) is the generalization of ordinary differentiation and integration to an arbitrary non-integer order. Various processes seem to display the fractional order behaviour

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that varies with time or space or both. Most fractional differential equations do not have exact analytic solutions and consequently, approximate analytical and numerical techniques are required to solve the fractional differential equations. Approximate analytical solution techniques for FDEs such as variational iteration method [2], operational matrix method [3, 4, 5], Adomian decomposition method [6, 7], homotopy perturbation method [8, 9, 10, 11], tau method [12], collocation method [13, 14, 15], homotopy analysis method [16, 17, 18, 19, 20], and optimal homotopy analysis method [21] have been developed.

A fractional logistic model can be obtained through the application of fractional derivative operator on the logistic equation. The continous logistic model was proposed by Verhulst in 1838 [22, 23] and is a first-order ordinary differential equation. The discrete logistic model is a classic iteration equation that exhibits chaos in certain regions [24]. An example from population modelling that illustrates the periodic doubling and chaotic behaviour in a dynamic system is the Verhulst model [24]. The model describes population growth which is limited by certain factors such as population density [22, 23, 25]. The logistic equation is commonly applied in the population growth models and in modelling growth of tumors. The solution of the continuous logistic equation is $N(t) = N_0 e^{\rho t}$ where N_0 is the initial population [25].

In this paper, we consider the fractional order Logistic equation of the form;

$$D_{\ast}^{\alpha} u(t) = \rho u(t)(1 - u(t)), \qquad t > 0, \quad 0 < \alpha \leq 1, \quad \rho > 0, \tag{1}$$

where $D_{\ast}^{\alpha}$ is the Caputo fractional differential operator, introduced in Section 2, of order α. We also assume an initial condition;

$$u(0) = \mu, \quad \mu > 0. \tag{2}$$

For $\alpha = 1$, Eq. (1) is the standard Logistic equation

$$\frac{du(t)}{dt} = \rho u(t)(1 - u(t)).$$

The exact solution to this problem is

$$u(t) = \frac{\mu}{(1-\mu)e^{-\rho t} + \mu}.$$

The existence and uniqueness of the proposed problem (1) are discussed in [26, 27].

The OHAM has been successfully applied for obtaining an approximate analytical solution of fractional ordinary differential equation [28, 29]. The purpose of this paper is to obtain an approximate solution by a means of the optimal homotopy asymptotic method (OHAM) [30, 31, 32, 33] for the fractional order logistic equation (1) subject to the initial condition given in (2).

2. Basic Definitions and Preliminaries

This section states some basic definitions and properties of fractional integral and derivative. Many definitions have been proposed for the fractional calculus in the past two centuries. Some of these definitions include the Riemann–Liouville and Caputo definitions. The Caputo's definition is a modification of the Riemann–Liouville definition.

Definition 1. A real function $f(t)$, $t > 0$ is said to be in the space C_μ $(\mu > 0)$ if it can be written as $f(t) = t^p f_1(t)$ for some $p > \mu$ where $f_1(t)$ is continuous in $[0, \infty)$, and it is said to be in the space C_μ^m iff $f^m \in N \cup \{0\}$ [34].

Definition 2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\mu, \ \ \mu \geq -1$, is defined as [35]

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0, \quad t > 0,$$
$$J^0 f(t) = f(t),$$

where $\Gamma(z)$ is the well-known Gamma function.

Details and properties of the operator J^α can be found in [1]. Note that: For $f \in C_\mu, \ \mu \geq -1, \ \alpha, \beta \geq 0$ and $\lambda > -1$, we have

- $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,
- $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,
- $J^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)} t^{\lambda+\alpha}$.

Definition 3. Let $f \in C_{-1}^m$, $m \in N \cup \{0\}$. Then the Caputo fractional derivative of f of order $\alpha > 0$ is defined as [35, **?**, 37]

$$D_{\ast}^\alpha f(t) = J^{m-\alpha} f^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \quad t > 0,$$

where $m - 1 < \alpha \leq m$.

For $m-1<\alpha\leq m$, $m\in N$, and $f,g\in C_{\mu}^{m}$, $\mu\geq -1$, the following properties hold [1]

- $D_{*}^{\alpha}(af(t)+bg(t))=aD_{*}^{\alpha}f(t)+bD_{*}^{\alpha}g(t), \quad a,b\in R,$
- $D_{*}^{\alpha}J^{\alpha}f(t)=f(t),$
- $J^{\alpha}D_{*}^{\alpha}f(t)=f(t)-\sum_{j=0}^{k-1}f^{(j)}(0)\frac{t^j}{j!}, \quad t>0.$

3. OHAM Formulation for Fractional Order Differential Equation

The OHAM was introduced and developed by Marinca et al. [30, 31, 32, 33]. The formulation of this method for fractional differential equations is given in the following steps:

(a) Eq. (1) can be written in the form

$$D_{*}^{\alpha}u(t)+\mathcal{N}(u(t))+g(t)=0, \quad t\in\Omega, \tag{3}$$

subject to the initial conditions

$$\mathcal{B}(u,\frac{du}{dt})=0, \tag{4}$$

where D_{*}^{α} is the Caputo fractional differential operator considered as a linear operator, $\mathcal{N}$ is a nonlinear operator, Ω is the problem domain, t denotes the independent variable, $u(t)$ is an unknown function, $g(t)$ is a known function and $\mathcal{B}$ is a boundary operator.

(b) The optimal homotopy $\varphi(t;q):\Omega\times[0,1]\to R$ for the fractional differential equation is constructed which satisfies

$$(1-q)[D_{*}^{\alpha}(\varphi(t;q,C_j)+g(t)]-H(t,q,C_j)[D_{*}^{\alpha}u(t)+\mathcal{N}(u(t))+g(t)]=0, \tag{5}$$

where $q\in[0,1]$ is an embedding parameter, $H(t,q,C_i)$ is a nonzero auxiliary function for $q\neq 0$, and $H(t,0,C_j)=0$, $\varphi(t;q,C_j)$ is an unknown function, $C_j, j=1,2,\ldots,s$ are unknown parameters which will be determined later. When q increases from 0 to 1, the solution $\varphi(t;q,C_j)$ varies continuously from $u_0(t)$ to the exact solution $u(t,C_j)$.

The auxiliary function $H(t; q, C_i)$ can be written in the form

$$H(t; q, C_j) = qH_1(t, C_j) + q^2 H_2(t, C_j) + q^3 H_3(t, C_j) + \cdots , \tag{6}$$

where $H_i(t, C_j), i = 1, 2, \ldots$ are the auxiliary functions depending on t and on the parameters $C_j, j = 1, 2, \ldots, s$.

(c) Expanding $\varphi(t; q, C_j)$ in a Taylor series with respect to q, one has

$$\varphi(t; q, C_j) = u_0(t) + \sum_{i=1}^{\infty} u_i(t, q, C_j) q^i. \tag{7}$$

(d) Substituting Eqs. (7) and (6) into Eq. (5) and equating the coefficient of like powers of q, then the zero order deformation equation is obtained as given in Eq. (8). The zeroth-order, first-order and second-order deformation equations are given by

$$q^0 : D_{\square}^{\alpha}(u_0(t)) + f(t) = 0, \quad B(u_0, \frac{du_0}{dt}) = 0, \tag{8}$$

$$q^1 : D_{\square}^{\alpha}(u_1(t, C_j)) = H_1(t, C_j) N_0(u_0(t)), \quad B(u_1, \frac{du_1}{dt}) = 0, \tag{9}$$

$$\begin{aligned} q^2 : D_{\square}^{\alpha}(u_2(t, C_j)) =& D_{\square}^{\alpha}(u_1(t, C_j)) + H_2(t, C_j) N_0(u_0(t)) \\ &+ H_1(t, C_j)[D_{\square}^{\alpha}(u_1(t, C_J)) \\ &+ N_1(u_0(t), u_1(t, C_j))], \quad B(u_2, \frac{du_2}{dt}) = 0, \end{aligned} \tag{10}$$

and hence, the general governing equations for $u_k(t)$ are given by

$$\begin{aligned} D_{\square}^{\alpha}(u_k(t)) =& D_{\square}^{\alpha}(u_{k-1}(t, C_j)) + H_k(t, C_j) N_0(u_0(t)) \\ &+ \sum_{i=1}^{k-1} H_i(t, C_j)[D_{\square}^{\alpha}(u_{k-i}(t, C_j)) \\ &+ N_{k-i}(u_0(t), u_1(t, C_j), \cdots, u_{k-1}(t, C_j))], \\ & B(u_k, \frac{du_k}{dt}) = 0, \quad k = 3, 4, \ldots, \end{aligned} \tag{11}$$

where $N_m(u_0(t), u_1(t, C_j) \cdots, u_m(t, C_j))$ is the coefficient of q^m, obtained by expanding $N(\varphi(t; q, , C_j))$ in series with respect to the embedding parameter q,

$$\begin{aligned} N(\varphi(t; q, , C_j)) \quad = \quad & N_0(u_0(t)) + N_1(u_0(t), u_1(t, C_j))q \\ & + N_2(u_0(t), u_1(t, C_j), u_2(t, C_j))q^2 + \cdots . \end{aligned} \tag{12}$$

(e) By using the operator J^α on the above linear fractional order differential equations of different order problems with linear boundary conditions from the original problem, we get series of solution $u_k(t, C_j)$ for $k \geq 0$ easily.

(f) If the series (7) is convergent at $q = 1$, we have

$$u(t; C_j) = u_0(t) + \sum_{k=1}^{\infty} u_k(t; C_j) \quad j = 1, 2, \ldots, s. \tag{13}$$

Truncating the series solution (13) at level $k = m$, then the approximate solution of order m is given by

$$u^m(t, C_j) = u_0(t) + \sum_{i=1}^{m} u_i(t, C_j), \quad j = 1, 2, \ldots, s. \tag{14}$$

(g) Substituting Eq. (14) into Eq. (3), the following expression for the residual error

$$R(t; C_j) = D_{\Box}^{\alpha}(u^m(t; C_j)) + N(u^m(t; C_j)) + f(t), \quad j = 1, 2, \ldots, s \tag{15}$$

is obtained.

For the determination of the required auxiliary convergence-control parameters $C_j, j = 1, 2, \ldots, s$, there are many methods such as Galerkins method, the Ritz method, the collocation method and the least squares method. Here, we apply the method of least squares to compute the auxiliary convergence-control parameters.

In theory, at the mth order of approximation, the exact square residual error, $J_m(C_1, C_2, C_3, \ldots, C_m)$, is defined as

$$J_m(C_1, C_2, C_3, \ldots, C_s) = \int_a^b R^2(t; C_1, C_2, C_3, \ldots, C_s)dt, \tag{16}$$

where the values a and b depend on the given problem. Thus, at the given level of approximation m, the corresponding optimal values of convergence control parameters $C_1, C_2, C_3, \ldots, C_s$ are obtained by minimizing the exact square residual error, J_m, which corresponds to the following set of s algebraic equations

$$\frac{\partial J_m}{\partial C_i} = 0 \quad i = 1, 2, \ldots, s. \tag{17}$$

(f) When these parameters are determined, then the mth order approximate solution given by Eq. (14) will be constructed. The explanation and description of OHAM in this section draws upon [30, 31, 32, 33].

4. First-Order OHAM for Solving Fractional Order Logistic Equation

In this section, we apply the OHAM to construct the first-order approximate solutions for the fractional order logistic equation (FOLE) of the form (1) subject to the initial condition given in (2) after only one iteration.

The fractional differential operator $D_{\square}^{\alpha}(\varphi(t;q))$ is considered as a linear operator and the nonlinear operator corresponding to nonlinear fractional differential Eq. (1) becomes

$$\mathcal{N}(\varphi(t;q)) = -\rho\varphi(t;q)(1-\varphi(t;q)),$$

and the initial condition 2 becomes

$$\varphi(0;q) = \mu, \quad \mu > 0.$$

The initial approximation $u_0(t)$ can be obtained from Eq.(8):

$$D_{\square}^{\alpha}(u_0(t)) = 0, u_0(0) = \mu \tag{18}$$

which has the solution:

$$u_0(t) = \mu. \tag{19}$$

The first approximation $u_1(t, C_j)$ is obtained from Eq. (9) and this is written in the form

$$D_{\square}^{\alpha}(u_1(t, C_j)) = H(t, C_j)\mathcal{N}(u_0(t)), u_1(0) = 0 \tag{20}$$

where

$$\mathcal{N}(u_0(t)) = -\rho u_0(t)(1-u_0(t)) \tag{21}$$

Substituting Eq. (19) into Eq. (21), the nonlinear operator $\mathcal{N}$ becomes

$$\mathcal{N}(u_0(t)) = \rho\mu^2 - \rho\mu \tag{22}$$

The expression (22) is non-zero constant which is a polynomial of degree 0 and the optimal auxiliary function H from Eq. (20) is chosen such that the product HN and N be of the same form. Therefore we choose the optimal auxiliary function H of a polynomial form

$$H(t, C_J) = \frac{1}{\rho\mu^2 - \rho\mu}(C_1 + C_2 t + C_3 t^2 + \cdots + C_r t^{r-1}) \tag{23}$$

where r is an arbitrary positive integer number and $C_j, j = 1, 2, ..., r$ are unknown parameters. Inserting Eqs. (22) and (23) into Eq. (20) we obtain the equation:

$$D_{\square}^{\alpha}(u_1(t, C_j)) = -(C_1 + C_2 t + C_3 t^2 + \cdots + C_r t^{r-1}), u_1(0) = 0. \tag{24}$$

By solving the Eq. (24) we get

$$u_1(t) = \frac{\Gamma(1)C_1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{\Gamma(2)C_2}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{\Gamma(3)C_3}{\Gamma(\alpha+3)}t^{\alpha+2} + \cdots + \frac{\Gamma(r)C_r}{\Gamma(\alpha+r)}t^{\alpha+r-1}. \tag{25}$$

Now, substituting Eqs. (19) and (25) in (14) to get the first-order approximate solution of the form

$$\bar{u}(t, C_j) = \mu + \frac{\Gamma(1)C_1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{\Gamma(2)C_2}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{\Gamma(3)C_3}{\Gamma(\alpha+3)}t^{\alpha+2} + \cdots + \frac{\Gamma(r)C_r}{\Gamma(\alpha+r)}t^{\alpha+r-1}. \tag{26}$$

The residual given by Eq. 15 becomes

$$R(t, C_j) = D_{\square}^{\alpha}(\bar{u}(t, C_j)) - \rho\bar{u}(t, C_j)(1 - \bar{u}(t, C_j)), \quad j = 1, 2, \ldots, r. \tag{27}$$

For the parameters $C_j, j = 1, 2, \ldots, r$ can be determined from the system (17), In this way the first-order approximate solution (26) is determined.

5. Numerical Experiments

In order to illustrate the accuracy and validity of the of OHAM, we have chosen two examples of fractional order logistic equation of the form (1) subject to the initial condition given in (2). We compare our results with the exact solution in the case $\alpha = 1$ and with numerical results obtained by Adams-Bashforth-Moulton numerical method (ABFMM) in the case $0 < \alpha < 1$.

Example 1. Consider the fractional order logistic equation [38]

$$D_{\square}^{\alpha}u(t) = 0.5u(t)(1 - u(t)), \qquad t > 0, \quad 0 < \alpha \leq 1, \tag{28}$$

with initial condition

$$u(0) = 0.85. \tag{29}$$

By applying the procedure of OHAM as mentioned in the previous section for different values of α on the interval $[0, 3]$, we consider five optimal convergence-control parameters in Eq. (23) *i.e* ($r = 5$). Thus, the first-order approximate solution (26) becomes;

$$\bar{u}(t, C_j) \;=\; 0.85 + \frac{\Gamma(1)C_1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{\Gamma(2)C_2}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{\Gamma(3)C_3}{\Gamma(\alpha+3)}t^{\alpha+2}$$

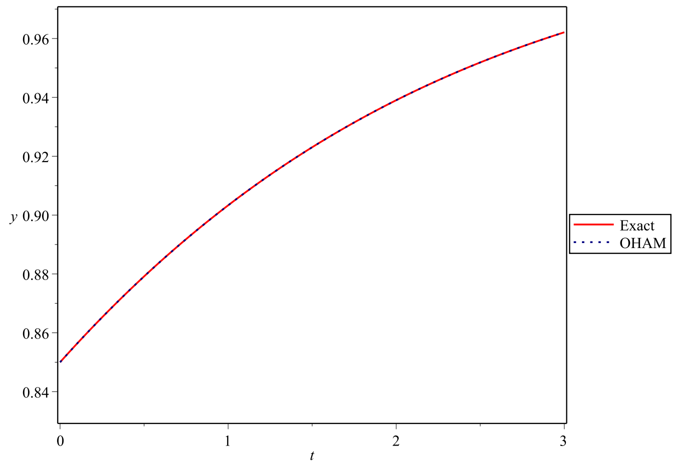

Figure 1: Comparison between the first-order OHAM solutions $\bar{u}(t)$ of Example 1 given by Eq. (30) with the Exact solution when $\alpha = 1$.

$$+\frac{\Gamma(4)C_4}{\Gamma(\alpha+4)}t^{\alpha+3} + \frac{\Gamma(5)C_5}{\Gamma(\alpha+5)}t^{\alpha+4}. \tag{30}$$

From the system (17), we can identify the convergence-control parameters $C_1, C_2, \ldots, C_5$, after that Substituting these values in expression (30) to get the first-order approximate solution using the OHAM.

Table 1 shows the optimal values of the convergence control parameters $C_1, C_2, \ldots, C_5$ in $\bar{u}(t)$ given in Eq. 30 for different values of α. Table 2 presents a comparison between first-order OHAM solutions and the exact solution when $\alpha = 1$. In Fig. 1 we plot the first-order OHAM solution, when $\alpha = 1$, and the exact solution. From the numerical results shown in Fig. 1 and Table 2, we can conclude that the approximate solutions obtained using first-order OHAM solutions are in a good agreement with the exact solution for all values of t. In Tables 3 and 4, we present a comparison between first-order OHAM solutions for Eq. (28) and the numerical results obtained by Adams-Bashforth-Moulton Method (ABFMM), with $h = 0.003$, for different values of α. In Fig. 2 we plot the approximate solution for different values of α. From the numerical results, we observe that the solutions obtained by OHAM are in an excellent agreement with the numerical results.

It is clear that the overall error of the first-order approximate solution $\bar{u}(t)$ decreases if the number of optimal convergence-control parameters involved in the optimal auxiliary function H increases.

Table 1: Values of convergence-control parameters for the first-order OHAM solution of Eq. (28) for different orders.

auxiliary parameter	$\alpha = 0.155$	$\alpha = 0.499$	$\alpha = 0.805$	$\alpha = 0.95$	$\alpha = 1$
C_1	0.0513981894	0.0592522961	0.0628217479	0.0635959707	0.0637613000
C_2	-0.0139123661	-0.0295982697	-0.0281966666	-0.0241095410	-0.0224273688
C_3	0.0115201046	0.0199025521	0.0113255667	0.0045485524	0.0021463669
C_4	-0.0044612794	-0.0071214854	-0.0032147792	-0.0006072773	0.0002436942
C_5	0.0006212748	0.0009544208	0.0003926817	0.0000526808	-0.0000513793

Table 2: Comparison between the first-order OHAM solutions $\bar{u}(t)$ of Example 1 given by Eq. (30) with the Exact solution when $\alpha = 1$.

t	Exact	OHAM	Absolute Error
0	0.85	0.85	0.
0.3	0.8681387195	0.8681389442	2.25×10^{-7}
0.6	0.8843823691	0.8843814886	8.81×10^{-7}
0.9	0.8988581326	0.8988575569	5.76×10^{-7}
1.2	0.9117024290	0.9117029232	4.94×10^{-7}
1.5	0.9230552212	0.9230562160	9.95×10^{-7}
1.8	0.9330555179	0.9330559212	4.03×10^{-7}
2.1	0.9418379992	0.9418373868	6.12×10^{-7}
2.4	0.9495306242	0.9495298247	8.00×10^{-7}
2.7	0.9562530660	0.9562533161	2.50×10^{-7}
3	0.9621158134	0.9621158134	3.18×10^{-10}

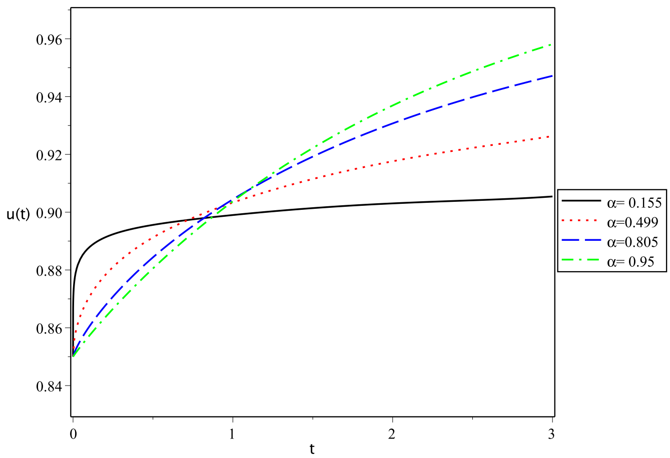

Figure 2: A first-order OHAM solution of Example 1 for different values of .

Example 2. Let us consider the fractional logistic equation

$$D_{\square}^{\alpha}u(t) = 0.5u(t)(1 - u(t)), \qquad t > 0, \quad 0 < \alpha \leq 1, \tag{31}$$

with initial condition;

$$u(0) = 0.5. \tag{32}$$

We give the optimal values of the convergence control parameters In Table 5.

Table 3: Comparison between the first-order OHAM solutions $\bar{u}(t)$ of Example 1 given by Eq. (30) with the numerical result (ABFMM) when $\alpha = 0.95$ and 0.805, (Relative Error=$|\bar{u}_{OHAM}(t) - u_{ABFMM}(t)|$)

	$\alpha = 0.95$			$\alpha = 0.805$		
t	OHAM	ABFMM	Relative error	OHAM	ABFMM	Relative Error
0	0.85	0.85	0	0.85	0.85	0
0.3	0.8695174678	0.8695161166	1.35×10^{-6}	0.8737998512	0.8737847058	1.51×10^{-5}
0.6	0.8856253470	0.8856209225	4.42×10^{-6}	0.8889841046	0.8889549969	2.91×10^{-5}
0.9	0.8995273701	0.8995272413	1.29×10^{-7}	0.9009446791	0.9009484848	3.81×10^{-6}
1.2	0.9116262182	0.9116296848	3.47×10^{-6}	0.9108386531	0.9108645266	2.59×10^{-5}
1.5	0.9221897985	0.9221927935	2.99×10^{-6}	0.9192329912	0.9192515752	1.86×10^{-5}
1.8	0.9314242487	0.9314240590	1.89×10^{-7}	0.9264538176	0.9264482960	5.52×10^{-6}
2.1	0.9394991012	0.9394964122	2.69×10^{-6}	0.9327090160	0.9326879776	2.10×10^{-5}
2.4	0.9465598493	0.9465579635	1.89×10^{-6}	0.9381521748	0.9381412742	1.09×10^{-5}
2.7	0.9527358458	0.9527366613	8.16×10^{-7}	0.9429246791	0.9429383376	1.37×10^{-5}
3	0.9581461009	0.9581455389	5.62×10^{-7}	0.9471877257	0.9471816122	6.11×10^{-6}

Table 4: Comparison between the first-order OHAM solutions $\bar{u}(t)$ of Example 1 given by Eq. (30) with the numerical result (ABFMM) when $\alpha = 0.499$ and 0.155, ((Relative Error=$|\bar{u}_{OHAM}(t) - u_{ABFMM}(t)|$)

	$\alpha = 0.499$			$\alpha = 0.155$		
t	OHAM	ABFMM	Relative error	OHAM	ABFMM	Relative Error
0	0.85	0.85	0	0.85	0.85	0
0.3	0.8835393627	0.8833820738	1.57×10^{-4}	0.8932334396	0.8929513149	2.82×10^{-4}
0.6	0.8942301353	0.8941491430	8.09×10^{-5}	0.8964478150	0.8964705525	2.27×10^{-5}
0.9	0.9013546287	0.9014371625	8.25×10^{-5}	0.8984507049	0.8986045735	1.54×10^{-4}
1.2	0.9068903585	0.9070034159	1.13×10^{-4}	0.9000643910	0.9001506459	8.62×10^{-5}
1.5	0.9114849224	0.9115135230	2.86×10^{-5}	0.9014103027	0.9013672829	4.30×10^{-5}
1.8	0.9153706261	0.9153018807	6.87×10^{-5}	0.9024831047	0.9023722390	1.11×10^{-4}
2.1	0.9186484494	0.9185627312	8.57×10^{-5}	0.9032929198	0.9032293602	6.36×10^{-5}
2.4	0.9214213883	0.9214201253	1.26×10^{-6}	0.9039197368	0.9039771212	5.74×10^{-5}
2.7	0.9238740100	0.9239585741	8.46×10^{-5}	0.9045401707	0.9046406579	1.00×10^{-4}
3	0.9263917529	0.9262382885	8.95×10^{-5}	0.9054430750	0.9052373046	2.06×10^{-4}

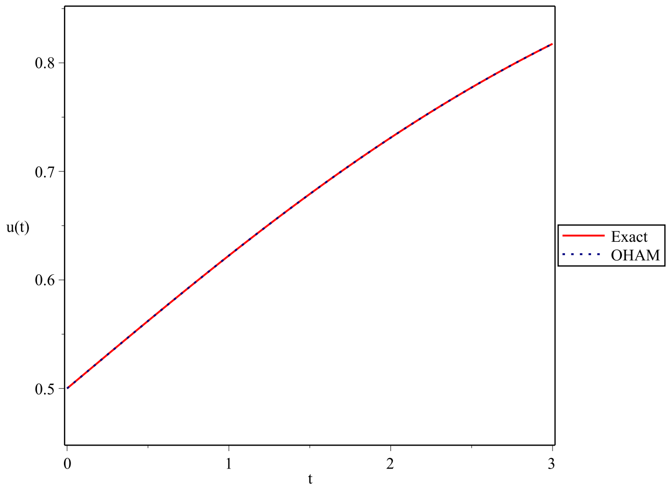

Figure 3: Comparison of OHAM approximate solution and exact solution in Example 2.

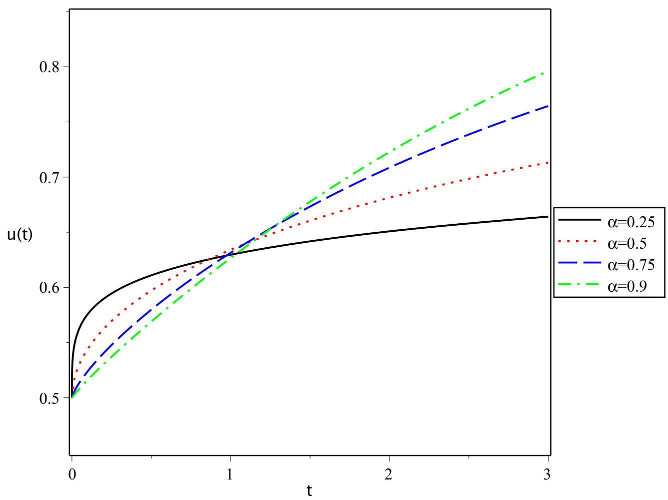

Figure 4: Comparison of OHAM approximate solution and exact solution in Example 2.

In Tables 6,7 and 8 we give the the numerical results for different values of α.

In Figs. 3 and 4 we plot the numerical results for different values of α.

Table 5: Values of convergence-control parameters for the first-order OHAM solution of Eq. (31) for different orders.

auxiliary parameter	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 1$
C_1	0.1233184514	0.1249993496	0.1251640476	0.1250355972	0.1249548446
C_2	-0.0120222051	-0.0099405243	-0.0038718632	-0.0008473779	0.0004506071
C_3	0.0075191523	0.0010391273	-0.0066768149	-0.0088588625	-0.0088487879
C_4	-0.0026735693	-0.0001017752	0.0019876238	0.0017628135	0.0008771890
C_5	0.0003574582	0.0000062302	-0.0002114601	-0.0001135881	0.0000526382

Table 6: Comparison between the first-order OHAM solutions $\bar{u}(t)$ of Example 2 given by Eq. (31) with the Exact solution when $\alpha = 1$.

t	Exact	OHAM	Absolute Error
0	0.5	0.5	0
0.3	0.5374298455	0.5374288935	9.52×10^{-7}
0.6	0.5744425170	0.5744461429	3.63×10^{-6}
0.9	0.6106392340	0.6106416979	2.46×10^{-6}
1.2	0.6456563065	0.6456542797	2.03×10^{-6}
1.5	0.6791786990	0.6791744502	4.25×10^{-6}
1.8	0.7109495025	0.7109476821	1.82×10^{-6}
2.1	0.7407748990	0.7407774288	2.53×10^{-6}
2.4	0.7685247835	0.7685281942	3.41×10^{-6}
2.7	0.7941296280	0.7941286033	1.02×10^{-6}
3	0.8175744765	0.8175744702	6.30×10^{-9}

Conclusion

The fractional order logistic equation was solved by means of optimal homotopy asymptotic method using only one iteration and the obtained result was compared with numerical results obtained by the Adams-Bashforth-Moulton numerical method. The obtained results suggest that the OHAM could be a useful, reliable and effective tool in solving fractional differential equations. The OHAM also provides us with a very simple way to control and adjust the convergence of the series solution using the auxiliary constants C_is which are optimally determined. Furthermore, by increasing the optimal convergence-control parameters forms of the auxiliary function $H(t, C_j)$, more accuracy can be obtained.

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Table 7: The results of the first-order approximate solution of Example 2 and the numerical solution (ABFMM), (Relative Error=$|\bar{u}_{OHAM}(t) - u_{ABFMM}(t)|$)

t	$\alpha = 0.25$			$\alpha = 0.5$		
	OHAM	ABFMM	Relative error	OHAM	ABFMM	Relative Error
0	0. 5	0. 5	0	0. 5	0. 5	0
0.3	0.5986900276	0.5985758401	1.14×10^{-4}	0.5760557755	0.5760556914	8.41×10^{-8}
0.6	0.6156670341	0.6156532937	1.37×10^{-5}	0.6059447979	0.6059446406	1.57×10^{-7}
0.9	0.6266491782	0.6267109386	6.18×10^{-5}	0.6278699757	0.6278698936	8.21×10^{-8}
1.2	0.6349884350	0.6350375110	4.91×10^{-5}	0.6455726344	0.6455727466	1.12×10^{-7}
1.5	0.6417748599	0.6417680395	6.82×10^{-6}	0.6605325711	0.6605327060	1.35×10^{-7}
1.8	0.6474875372	0.6474403974	4.71×10^{-5}	0.6735232466	0.6735232717	2.51×10^{-8}
2.1	0.6523917468	0.6523552305	3.65×10^{-5}	0.6850120798	0.6850120069	7.29×10^{-8}
2.4	0.6566821440	0.6566986270	1.65×10^{-5}	0.6953083813	0.6953083743	7.00×10^{-9}
2.7	0.6605477168	0.6605944364	4.67×10^{-5}	0.7046301004	0.7046301871	8.67×10^{-8}
3	0.6642062834	0.6641291970	7.71×10^{-5}	0.7131381451	0.7131380010	1.44×10^{-7}

Table 8: The results of the first-order approximate solution of Example 2 and the numerical solution (ABFMM), (Relative Error=$|\bar{u}_{OHAM}(t) - u_{ABFMM}(t)|$)

	$\alpha = 0.75$			$\alpha = 0.9$		
t	OHAM	ABFMM	Relative error	OHAM	ABFMM	Relative Error
0	0. 5	0. 5	0	0. 5	0. 5	0
0.3	0.5548093881	0.5548135874	4.20×10^{-6}	0.5438476810	0.5438483984	7.17×10^{-7}
0.6	0.5912172701	0.5912270246	9.75×10^{-6}	0.5812238170	0.5812249410	1.12×10^{-6}
0.9	0.6220229364	0.6220222870	6.49×10^{-7}	0.6157254716	0.6157250687	4.03×10^{-7}
1.2	0.6491169121	0.6491083950	8.52×10^{-6}	0.6478365192	0.6478355659	9.53×10^{-7}
1.5	0.6733331977	0.6733267617	6.44×10^{-6}	0.6777147237	0.6777144105	3.13×10^{-7}
1.8	0.6951660443	0.6951675193	1.47×10^{-6}	0.7054366386	0.7054371070	4.68×10^{-7}
2.1	0.7149590387	0.7149662536	7.21×10^{-6}	0.7310631114	0.7310637998	6.88×10^{-7}
2.4	0.7329709670	0.7329749694	4.00×10^{-6}	0.7546609461	0.7546612050	2.59×10^{-7}
2.7	0.7494007006	0.7493956820	5.02×10^{-6}	0.7763095474	0.7763092923	2.55×10^{-7}
3	0.7643955240	0.7643983742	2.85×10^{-6}	0.7961013553	0.7961013493	6.00×10^{-9}

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