International Journal of Pure and Applied Mathematics
Volume 115 No. 2 2017, 301-310
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v115i2.8
ijpam.eu

STEADY-STATE HEAT CONDUCTION PROBLEM IN A THICK CIRCULAR PLATE AND ITS THERMAL STRESSES

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Abstract: The present paper deals with the determination of a quasi-static thermal stresses in a thick circular plate subjected to arbitrary temperature on the outer circular edge, with lower and upper face are at zero temperature. The governing heat conduction equation has been solved by using finite Fourier sine transform technique. The results are obtained in series form in terms of Bessels functions. The results for displacement and stresses have been computed numerically and illustrated graphically.

AMS Subject Classification: 35B07, 35G30, 35K05, 44A10
Key Words: quasi-static, thermal stresses, heat conduction problem, thick circular plate, steady-state

1. Introduction

Nowacki [1] has determined the steady-state thermal stresses in a circular plate subjected to an axisymmetric temperature distribution on the upper surface with zero temperature on the lower surface and with the circular edge thermally insulated. Roy Choudhary [2] has succeeded in determining the quasi-static thermal stresses in a thin circular plate subjected to a transient temperature along the circumference of a circle over the upper face with the lower face at zero temperature and a fixed circular edge thermally insulated. Okumura and Noda [3] presented an explicit solution composed of two thermoelastic potential functions in cylindrical coordinates for steady-state, axially

Received: January 10, 2017
Revised: April 27, 2017
Published: July 14, 2017

asymmetric thermal stresses in an isotropic long hollow cylinder. Kulkarni et al. [4] studied the quasi-static thermal stresses in a thick circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature and the fixed circular edge thermally insulated. Ghadle et al. [5, 6, 7] have been solved some direct and inverse thermoelastic problems on thick circular plate and annular disc. Ghadle et al. [8] have solved non-homogeneous heat conduction problem and obtained thermal deflection due to internal heat generation in a thin hollow circular disk. Gaikwad [9] analysed thermoelastic deformation of a thin hollow circular disk due to partially distributed heat supply. Recently, Gaikwad [10] discussed the two-dimensional steady-state temperature distribution of a thin circular plate due to uniform internal energy generation.

This paper deals with the realistic problem of the quasi-static thermal stresses in a thick circular plate subjected to arbitrary temperature on the outer circular edge, with lower and upper face are at zero temperature. The governing heat conduction equation has been solved by using finite Fourier sine transform technique. The results are obtained in series form in terms of Bessels functions. The results for displacement and stresses have been computed numerically and illustrated graphically.

2. Formulation of the Problem

Consider a thick circular plate of radius a and thickness h occupying space D defined by $0 \leq r \leq a$, $0 \leq z \leq h$. Initially the plate is at zero temperature. Let the plate be subjected to an arbitrary temperature $f(z)$ on the outer circular edge $(r = a)$. The lower surface $(z = 0)$ and the upper surface $(z = h)$ are at zero temperature Assume that the boundary of the circular plate is free from traction. Under these more realistic prescribed conditions, the quasi-static thermal stresses are required to be determined.

The differential equation governing the displacement potential function $\phi(r, z)$ is given by Noda [11] as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = K\tau \tag{1}$$

where K is the restraint coefficient and temperature change $\tau = T - T_i$, T_i is initial temperature. Displacement function ϕ is known as Goodier's thermoelastic displacement potential.

The steady-state temperature of the plate satisfies the heat condition equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad \text{in} \qquad 0 \le r \le a,\, 0 \le z \le h \tag{2}$$

subject to the boundary conditions,

$$T = f(z) \qquad \text{at } r = a \tag{3}$$

$$T = 0 \qquad \text{at } z = 0 \text{ and } z = h \tag{4}$$

The displacement function in the cylindrical coordinate system are represented by the Michell's function defined in Noda [11] as,

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \tag{7}$$

$$u_z = \frac{\partial \phi}{\partial z} + 2\,(1-\nu)\,\nabla^2 M - \frac{\partial^2 M}{\partial z^2} \tag{8}$$

The Michell's function M must satisfy

$$\nabla^2 \nabla^2 M = 0 \tag{9}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{10}$$

The components of the stresses are represented by the thermoelastic displacements potential ϕ and Michell's function M as

$$\sigma_{rr} = 2G\left[\frac{\partial^2 \phi}{\partial r^2} - K\tau + \frac{\partial}{\partial z}\left(\nu \nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right)\right] \tag{11}$$

$$\sigma_{\theta\theta} = 2G\left[\frac{1}{r}\frac{\partial \phi}{\partial r} - K\tau + \frac{\partial}{\partial z}\left(\nu \nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right)\right] \tag{12}$$

$$\sigma_{zz} = 2G\left[\frac{\partial^2 \phi}{\partial z^2} - K\tau + \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right] \tag{13}$$

and

$$\sigma_{rz} = 2G\left[\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial z}\left((1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right] \tag{14}$$

where G and ν are the Shear modulus and Poisson's ratio respectively.
The boundary conditions on the traction free surface of thick circular plate are

$$\sigma_{zz} = \sigma_{rz} = 0 \qquad \text{at } z = h \tag{15}$$

Equations (1) to (15) constitute the mathematical formulation of the problem under consideration.

3. Solution of the Problem

To obtain the expressions for temperature $T(r,z)$,

Assume

$$T(r,z)=\sum_{n=1}^{\infty}A_n I_0(\alpha_n r)\sin(\alpha_n z) \tag{16}$$

where $\alpha_1,\alpha_2,\alpha_3,\ldots$ are the positive roots of the transcendental equation

$$\sin(\alpha_n h)=0, \qquad n=1,2,\ldots$$

and $I_0(\alpha_n r)$ denotes the modified Bessel's function of the first kind of order zero and A_n is constant. The constant A_n can be found from the nature of temperature on the outer curved surface.

From the boundary condition (3), we have

$$f(z)=\sum_{n=1}^{\infty}A_n I_0(\alpha_n a)\sin(\alpha_n z), \qquad \text{in} \quad 0\le z\le h \tag{17}$$

Now we introduce the finite Fourier sine transform over the variable z in the range $0\le z\le h$ and its inverse transform defined in Sneddon [13] as

$$\overline{f}(z)=\frac{2}{\pi}\int_0^h f(z)\sin(\alpha_n z)dz \tag{18}$$

where $\overline{f}(z)$ is called the finite Fourier sine transform of an arbitrary function $f(z)$.

The Fourier sine inversion is defined as

$$f(z)=\int_0^h \overline{f}(z)\sin(\alpha_n z)dz \tag{19}$$

The constant A_n are determined as

$$A_n=\frac{2\,\overline{f}(z)}{h\,I_0(\alpha_n a)} \tag{20}$$

By Eqs. (20) and (16) the required expression for temperature function is obtained as

$$T(r,z)=\left(\frac{2}{h}\right)\sum_{n=1}^{\infty}\frac{\overline{f}(z)}{I_0(\alpha_n a)}I_0(\alpha_n r)\sin(\alpha_n z) \tag{21}$$

Since $T_i=0$, the temperature change $\tau=T$.

Now suitable form of M satisfying (9) is given by

$$M = \left(\frac{K}{h}\right) \sum_{n=1}^{\infty} \frac{\overline{f}(z)}{I_0(\alpha_n a)} \left[B_n I_0(\alpha_n r) + C_n r I_1(\alpha_n r)\right] \cos(\alpha_n z) \tag{22}$$

where B_n and C_n are arbitrary constants, which can be determined from the boundary condition (15).
Assuming displacement function $\phi(r, z)$ which satisfies (1) as

$$\phi(r, z) = \sum_{n=1}^{\infty} D_n \frac{r I_1(\alpha_n r)}{I_0(\alpha_n a)} \sin(\alpha_n z) \tag{23}$$

and using ϕ in (1), one obtains

$$D_n = \frac{K\overline{f}(z)}{2\alpha_n} \tag{24}$$

Eq. (23), therefore becomes

$$\phi(r, z) = \left(\frac{K}{h}\right) \sum_{n=1}^{\infty} \frac{r\overline{f}(z)}{\alpha_n I_0(\alpha_n a)} I_1(\alpha_n r) \sin(\alpha_n z) \tag{25}$$

Displacement Components

Now using equations (21), (22) and (25) in equations (7), (8) and (11) to (14), One obtains the expressions for displacement and thermal stresses as

$$\begin{aligned} u_r = \left(\frac{K}{h}\right) \sum_{n=1}^{\infty} \left[\frac{\overline{f}(z)}{I_0(\alpha_n a)}\right] &\left\{[r I_0(\alpha_n r)] - B_n \alpha_n^2 I_1(\alpha_n r)\right. \\ &\left. - C_n \alpha_n^2 r I_0(\alpha_n r)\right\} \sin(\alpha_n z) \end{aligned} \tag{26}$$

$$\begin{aligned} u_z = \left(\frac{K}{h}\right) \sum_{n=1}^{\infty} \left[\frac{\overline{f}(z)}{I_0(\alpha_n a)}\right] &\left\{-r I_1(\alpha_n r) + B_n \alpha_n^2 I_0(\alpha_n r)\right. \\ &\left. + C_n \alpha_n [4(1-\nu) I_0(\alpha_n r) + \alpha_n r I_1(\alpha_n r)]\right\} \cos(\alpha_n z) \end{aligned} \tag{27}$$

Quasi-Static Thermal Stresses

Now using Eqs. (23)-(26) and (29) in (11)-(14), one obtains the expressions for stresses as

$$\begin{aligned} \sigma_{rr} = \left(\frac{2GK}{h}\right) \sum_{n=1}^{\infty} \left[\frac{\overline{f}(z)}{I_0(\alpha_n a)}\right] &\left\{- [2 I_0(\alpha_n r) - \alpha_n r I_1(\alpha_n r)]\right. \\ &+ B_n \alpha_n^3 \left[\frac{I_1(\alpha_n r)}{\alpha_n r} - I_0(\alpha_n r)\right] \\ &\left. - C_n \alpha_n^2 \left[(1-2\nu) I_0(\alpha_n r) + \alpha_n r I_1(\alpha_n r)\right]\right\} \sin(\alpha_n z) \end{aligned} \tag{28}$$

$$\sigma_{\theta\theta} = \left(\frac{-2GK}{h}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(z)}{I_0(\alpha_n a)}\right]\Big\{-I_0(\alpha_n r) + \ B_n\alpha_n^3\left(\frac{I_1(\alpha_n r)}{\alpha_n r}\right) \\ +C_n\alpha_n^2(1-2\nu)I_0(\alpha_n r)\Big\}\sin(\alpha_n z) \tag{29}$$

$$\sigma_{zz} = \left(\frac{2GK}{h}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(z)}{I_0(\alpha_n a)}\right]\{-\left[2I_0(\alpha_n r)+\alpha_n r I_1(\alpha_n r)\right] \\ +\ B_n\alpha_n^3 I_0(\alpha_n r) + C_n\alpha_n^2\left[2(2-\nu)I_0(\alpha_n r)+\alpha_n r I_1(\alpha_n r)\right]\Big\}\sin(\alpha_n z) \tag{30}$$

$$\sigma_{rz} = \left(\frac{K}{h}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(z)}{I_0(\alpha_n a)}\right]\Big\{-\alpha_n r I_0(\alpha_n r) + B_n\alpha_n^3 I_1(\alpha_n r) \\ +C_n\alpha_n^2[2(1-\nu)I_1(\alpha_n r)+\alpha_n r I_0(\alpha_n r)]\Big\}\cos(\alpha_n z) \tag{31}$$

Now in order to satisfy the boundary conditions given in the equation (15), we use equations (33)-(35) for B_n and C_n one obtain

$$B_n = (1-\nu)K\overline{f}(z)\frac{a}{\alpha_n^2}\frac{\alpha_n a I_0^2(\alpha_n a) - \alpha_n a I_1^2(\alpha_n a) + I_0(\alpha_n a)I_1(\alpha_n a)}{(2-2\nu+\alpha_n^2 a^2)I_1^2(\alpha_n a) - \alpha_n^2 a^2 I_0^2(\alpha_n a)}$$

and

$$C_n = K\overline{f}(z)\frac{a^2}{2}\frac{I_1^2(\alpha_n a) - I_0^2(\alpha_n a)}{(2-2\nu+\alpha_n^2 a^2)I_1^2(\alpha_n a) - \alpha_n^2 a^2 I_0^2(\alpha_n a)}$$

4. Numerical Calculations

SPECIAL CASE Setting

$$f(z) = (z-h)z \tag{32}$$

Apply the finite Fourier transform to the equation (37), one obtains

$$\overline{f}(z) = \frac{4}{\pi\alpha_n^3}[\cos(\alpha_n h) - 1] \tag{33}$$

The numerical calculation has been carried out for a aluminium (pure) circular plate with the parameters $a = 1$ m, $h = 0.4$ m, k= 204 W/mk, $\nu = 0.35$. The first five positive root of the transcendental equation $\sin(\alpha_n h) = 0$ as defined in Ozisik [12] are $\alpha_1 = 3.14$, $\alpha_2 = 6.28$, $\alpha_3 = 9.42$, $\alpha_4 = 12.56$, $\alpha_5 = 15.70$.

In order to examine the influence of steady state temperature field on the thick plate, one performed the numerical calculations for $r = 0, 0.1, 0.2, \ldots 1m$ and $z = 0, 0.1, 0.2, 0.3, 0.4m$. The numerical calculation has been carried out with the help of computational mathematical software MATLAB-2007 and the graphs are plotted with the help of MATLAB.

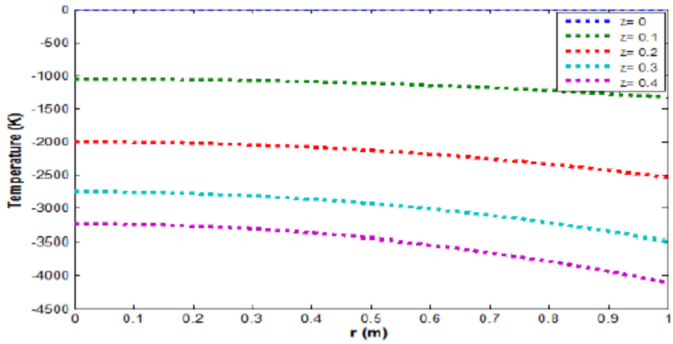

Figure 1: Temperature distribution function

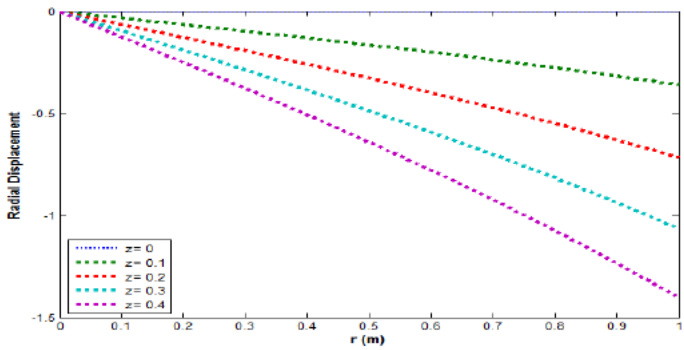

Figure 2: Radial displacement function

5. Discussion of the Results

In this paper a thick circular plate is considered which is free from traction and determined the expressions for temperature, displacement and stress function due to steady state temperature field. The thermoelastic behavior is examined such as temperature, displacement and stresses with the help of arbitrary temperature $f(z)$ is applied on the outer circular boundary $(r = a)$.

Figure 1 shows the variation of temperature distribution function change along the radial direction with different values of z. The temperature is maximum at the center $(r = 0)$ and decreases towards the outer circular edge $(r = 1)$ with increase the radius. Figure 2 shows the variation of radial displacement change along the radial direction with different values of z. The radial displacement is maximum at the center $(r = 0)$ and decreases non-uniformly towards

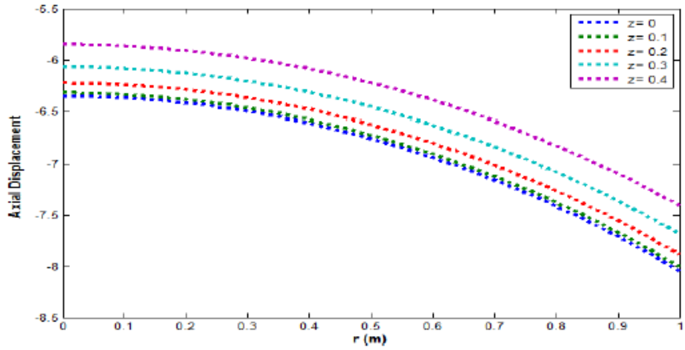

Figure 3: Axial displacement function

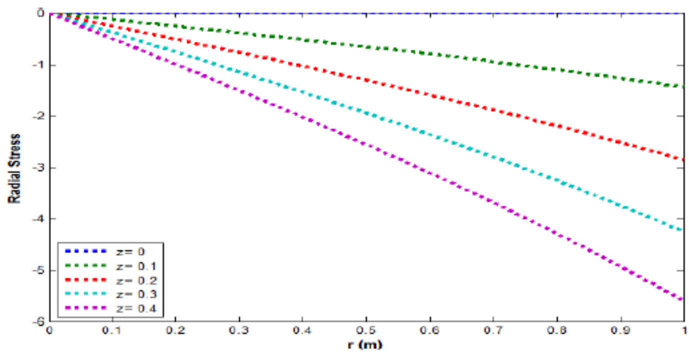

Figure 4: Radial stress function

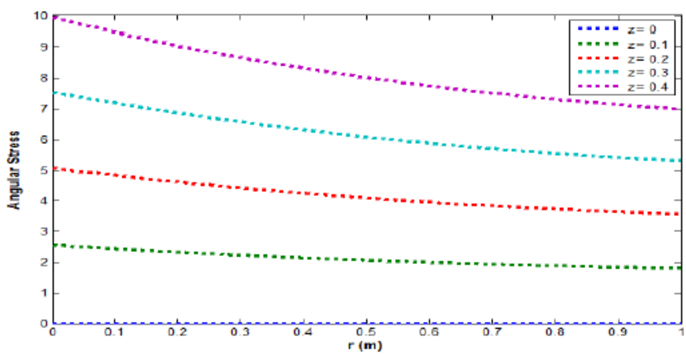

Figure 5: Angular stress function

the outer circular edge ($r = 1$) with increase the radius. It develops the compressive stresses in radial direction. Figure 3 shows the variation of axial displacement change along the radial direction with different values of z. The axial displacement decreases towards the outer circular edge ($r = 1$) with increase the radius and is maximum at the center ($r = 0$). Figure 4 shows the variation of radial stress change along the radial direction with different values of z. The radial stress is maximum at the center ($r = 0$) and decreases non-uniformly towards the outer circular edge ($r = 1$) with increase the radius. It develops the compressive stresses in radial direction. Figure 5 shows the variation of angular stress change along the radial direction with different values of z. The temperature is maximum at the center ($r = 0$) and decreases towards the outer circular edge ($r = 1$) with increase the radius. It develops tensile stresses in radial direction.

6. Concluding Remarks

In the present article, we analyzed the steady-state heat conduction problem in a thick circular plate which is free from traction subjected to an arbitrary temperature on the outer circular edge and determined the expressions for temperature, displacement and thermal stresses in radial direction with different axial thickness. The present method is based on the direct method, using the finite Fourier sine transform and using their inversion. As a special case, a mathematical model is constructed for Aluminum (pure), thick circular plate with the material properties specified as above and examined the behaviors in the steady-state for the temperature change, the displacement and the thermal stresses. The numerical calculations and graphs are plotted with the help of MATLAB-2007.

We observed that displacement and stress components occur near heated region. Radial displacement and radial stress develops the compressive stresses in radial direction. Angular stress develops the tensile stresses in radial direction. Also, it can be observed from the figures of radial displacement and radial stress that the direction of heat flow and direction of body displacement are same and they are propositional to each other. In the plane state of stress the stress components σ_{zz} and σ_{rz} are zero. The results obtained here are more useful in engineering problems particularly in the determination of state of strain in thick circular plate. Also any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions (26) to (31).

Acknowledgments

The authors are thankful to University Grant Commission, Western Regional Office, Pune, India, to provide the partial financial assistance under Minor Research Project Scheme File No: 47/892/14 (WRO).

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