NUMERICAL SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATION SYSTEM USING THE MÜNTZ-LEGENDRE POLYNOMIALS

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Abstract: In the paper, we apply a numerical approach to approximate solution of a system of the fractional differential equations based on the Müntz-Legendre polynomials. Features of these polynomials make them so appropriate for solving the fractional differential equation system. In the proposed method, a system of the fractional differential equations is transformed into a system of the algebraic equations which is move easier to solve. The numerical examples show accuracy and convergence of the present method.

1. Introduction

Fractional calculus plays main role in physics, chemistry and engineering which has attracted the interest of many researchers in recent years. In comparison to

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the differential equations of accurate order, the fractional differential equations
(FDEs) have an advantage that many phenomena in physics, chemistry and
engineering can be described very successfully by them. Moreover, this allows
one to apply the fractional differential equations to model these phenomena
more accurately. Some of these models and applications are presented in [1, 2,
3, 4, 5]. Also many researchers tried to investigate and describe these kinds of
the equations (see Ref. [6, 7]). Odibat suggested an analytical examination for
FDEs [8]. Jafari and Daftardar also proved existence and uniqueness solutions
of the FDEs [9]. Obtaining an exact solution of these problems is extremely
difficult thus there are a lot of numerical methods to approximate solutions of
the FDEs such as, Homotopy method [10], finite differences method [11, 12],
Adomian decomposition method [13], Collocation method [14], and also Chen
and et al. proposed a numerical method [15].

In this paper, we consider a system of the fractional differential equations,
as follows:

$$D_*^{\alpha_i} f_i(t) = g_i(t, f_1(t), f_2(t), \ldots, f_n(t)), \quad t \in [0, T], \qquad i = 1, 2, \ldots, n, \quad (1)$$

where $D_*^{\alpha_i} f_i(t)$ is derivative of order α_i for $f_i(t)$ and $0 < \alpha_i \leq 1$. Thus for the
system of Eq. (1), we consider the following initial condition:

$$f_i(0) = y_i, \qquad i = 1, 2, \ldots, n.$$

Note that $y_1, y_2, \ldots, y_n$ are constant numbers. There are several kinds of the
fractional derivatives of a function [16, 17], but Rieman-Liouville and Caputo
definitions are two of mostly used and famous for evaluating the fractional
derivatives. In present paper, we use Caputo derivative.
The Caputo fractional derivative of order $0 < \alpha < 1$ is defined in the following
form:

$$D_*^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} y'(\tau) d\tau. \tag{2}$$

Recall that the above formula for $\alpha = 1$ coincides with the usual derivative
operator of the first order.

We use The Müntz-Legendre polynomials [18, 19, 20] to approximate the
solution of the Eq. (1). The main advantage of using the Muntz.Legendre
polynomials is that their fractional derivative is still a Müntz-Legendre polyno-
mial. Furthermore, approximate solutions of the fractional deferential equations
with a collocation method based on the Müntz-Legendre polynomials show very
good agreement with the exact solution. Another considerable advantage of the

known polynomials in question is that the fractional derivatives of these poly-
nomials can be expressed in terms of the same polynomials that are appropriate
for solving the fractional differential equations.

A brief overview of the paper is as follows: first, the Jacobi polynomi-
als are defined. Then, the definition of Müntz-Legendre polynomials and ob-
taining them via a stable recurrence relation are presented. After that, a
method for evaluating the fractional derivatives of the Müntz-polynomials is
introduced. Finally, the method of solving the fractional derivatives based on
the Muntz.Legendre polynomials is explained and some numerical examples are
given to demonstrate the effectiveness and accuracy of the proposed method.

2. Preliminaries

2.1. Jacobi Polynomials

The Jacobi polynomials are extensively used for solving fractional differential
equations. They are orthogonal on the interval $[-1, 1]$ with respect to the weight
function $w^{(\alpha,\beta)}(t) = (1 - t)^\alpha (1 + t)^\beta$ where $\alpha, \beta > -1$. These polynomials can
be obtained through the following recurrent relation:

$$
\begin{aligned}
J_0^{(\alpha,\beta)}(t) =& 1, \\
J_1^{(\alpha,\beta)}(t) =& \frac{1}{2}\left((\alpha - \beta) + (\alpha + \beta + 2)t\right), \\
a_k^{(\alpha,\beta)} J_{k+1}^{(\alpha,\beta)}(t) =& b_k^{(\alpha,\beta)}(t) J_k^{(\alpha,\beta)}(t) - c_k^{(\alpha,\beta)} J_{k-1}^{(\alpha,\beta)}(t), \\
a_k^{(\alpha,\beta)} =& 2(k + 1)(k + \alpha + \beta + 1)(2k + \alpha + \beta), \\
b_k^{(\alpha,\beta)}(t) =& (2k + \alpha + \beta + 1)((2k + \alpha + \beta) \\
& \times (2k + \alpha + \beta + 2)t + \alpha^2 - \beta^2), \\
c_k^{(\alpha,\beta)} =& 2(k + \alpha)(k + \beta)(2k + \alpha + \beta + 2).
\end{aligned}
\tag{3}
$$

The initial derivative of the Jacobi polynomials can be obtained as follows:

$$
\frac{d}{dt} J_k^{(\alpha,\beta)}(t) = \frac{1}{2}(k + \alpha + \beta + 1) J_{k-1}^{(\alpha+1,\beta+1)}(t)
\tag{4}
$$

2.2. Müntz-Legendre Polynomials

Let $\Lambda_n = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be under condition $\mathrm{Re}(\lambda_k) > -\frac{1}{2}$ such that Müntz-Legendre polynomials on the interval $(0, 1]$ are defined as follows [18, 19]:

$$L_n(t) = L(\Lambda_n, t) = \sum_{k=0}^{n} C_{n,k} t^{\lambda_n}, \qquad C_{n,k} = \frac{\prod_{v=0}^{n-1}(\lambda_k + \overline{\lambda}_v + 1)}{\prod_{v=0, v \neq k}^{n}(\lambda_k - \lambda_v)} \tag{5}$$

Basic properties of the Müntz-Legendre polynomials:

$$
\begin{aligned}
(L_n, L_m) &= \int_0^1 L_n(t) L_m(t) dt = \frac{\delta_{mn}}{\lambda_n + \overline{\lambda}_n + 1}, \\
L_n(1) &= 1, \\
L_n'(1) &= \lambda_n + \sum_{k=0}^{n-1}(\lambda_k + \overline{\lambda}_k + 1),
\end{aligned}
\tag{6}
$$

Here, all λ_k are chosen so that $\lambda_k = \alpha k$, the shifted Müntz-Legendre polynomials on the interval $I = [0, R)$ are defined as follows (α is a positive real number):

$$L_{I,n}(t : \alpha) = \sum_{k=0}^{n} C_{n,k} \left(\frac{t}{R}\right)^{\alpha k}, \qquad C_{n,k} = \frac{(-1)^{n-k}}{\alpha^n k!(n-k)!} \prod_{v=0}^{n-1}((k+v)\alpha + 1) \tag{7}$$

Some of the shifted Müntz-Legendre polynomials' properties according to whatever has been said in [18] are:

$$L_{I,n}(R : \alpha) = 1, \qquad L_{I,n}'(R : \alpha) = \frac{\alpha n + \sum_{k=0}^{n-1}(2\alpha k + 1)}{R},$$

Regarding Eq. (6), we have:

$$\int_0^R L_{I,n}(t) L_{I,m}(t) dt = \frac{R \delta_{mn}}{1 + 2\alpha n},$$

Furthermore, following [14] a stable recurrence relation can be obtained for shifted Müntz-Legendre polynomials via the Jacobi polynomials, as follows:

$$L_{I,0}(t : \alpha) = 1, \qquad L_{I,1}(t : \alpha) = \left(\frac{1}{\alpha} + 1\right)\left(\frac{t}{R}\right)^{\alpha} - \frac{1}{\alpha},$$

$$a_{I,n}L_{I,n+1}(t : \alpha) = b_{I,n}(t)L_{I,n}(t : \alpha) - c_{I,n}L_{I,n-1}(t : \alpha), \tag{8}$$

$$a_{I,n} = a_n^{(0,\frac{1}{\alpha}-1)}, \qquad b_{I,n}(t) = b_n^{(0,\frac{1}{\alpha}-1)}\left(2\left(\frac{t}{R}\right)^{\alpha} - 1\right), \qquad c_{I,n} = c_n^{(0,\frac{1}{\alpha}-1)}$$

In addition, according to [19] fractional derivative of these polynomials can also be obtained with the following equation:

$$D_*^{\alpha}L_{I,n}(t : \alpha) = \frac{1 + \alpha n}{\alpha\Gamma(1-\alpha)R^{\alpha}} \int_0^R (1-\tau^{\frac{1}{\alpha}})^{-\alpha} J_{n-1}^{(1,\frac{1}{\alpha})}\left(2\left(\frac{t}{R}\right)^{\alpha}\tau - 1\right) d\tau. \tag{9}$$

3. Solution of the Fractional Differential Equations

In this section, we assume that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ then we present an approximate solution for the unknown fractional differential equations in Eq. (1) as follows:

$$f_i(t) \simeq \hat{f}_i(t) = \sum_{j=0}^{N} c_{ij}L_{I,j}(t : \alpha), \qquad i = 1, 2, \ldots, n, \tag{10}$$

In addition, the fractional derivative of the unknown functions can be found by:

$$D_*^{\alpha}f_i(t) \simeq D_*^{\alpha}\hat{f}_i(t) = \sum_{j=0}^{N} c_{ij}D_*^{\alpha}L_{I,j}(t : \alpha), \qquad i = 1, 2, \ldots, n, \tag{11}$$

where $D_*^{\alpha}L_{I,j}(t : \alpha)$ can be computed by the proposed Eq. (9) for evaluating the fractional derivative of the Müntz-Legendre polynomials and for the initial condition, we have:

$$f_i(0) \simeq \hat{f}_i(0) = \sum_{j=0}^{N} c_{ij}L_{I,j}(0 : \alpha) = y_i, \qquad i = 1, 2, \ldots, n, \tag{12}$$

in this case, a system of Eq. (1) with boundary conditions is transformed into the following relations,

$$
\begin{cases}
\displaystyle\sum_{j=0}^{N} c_{ij} D_*^\alpha L_{I,j}(t:\alpha) = g_i \left(t, \sum_{j=0}^{N} c_{1j} L_{I,j}(t:\alpha), \ldots, \sum_{j=0}^{N} c_{nj} L_{I,j}(t:\alpha) \right) \\
\displaystyle\sum_{j=0}^{N} c_{ij} L_{I,j}(0:\alpha) = y_i, \qquad i = 1,2,\ldots,n, \qquad t \in [0,R]
\end{cases}
$$

(13)

to find $\hat{f}_i(t)$, we should obtain coefficients c_{ij} ,for this purpose, we should satisfy the points $\theta_k = \frac{R}{2} - \frac{R}{2}\cos\left(\frac{\pi k}{N}\right)$, $k = 1,2,\ldots,N$ into above system of Eq. (13), and we have,

$$
\begin{cases}
\displaystyle\sum_{j=0}^{N} c_{ij} D_*^\alpha L_{I,j}(\theta_k:\alpha) = g_i \left(\theta_k, \sum_{j=0}^{N} c_{1j} L_{I,j}(\theta_k:\alpha), \ldots, \sum_{j=0}^{N} c_{nj} L_{I,j}(\theta_k:\alpha) \right) \\
\displaystyle\sum_{j=0}^{N} c_{ij} L_{I,j}(0:\alpha) = y_i, \qquad i = 1,2,\ldots,n, \qquad k = 1,2,\ldots,N.
\end{cases}
$$

(14)

in turn, Eq. (14) produce an algebraic system of $n(N+1)$ equations for the $n(N+1)$ unknown coefficients c_{ij} . These coefficients are obtained by solving this system of algebraic equations with one of the known methods. Substituting them into Eq. (10) leads to an approximated solution of $f_1(t)$, $f_2(t)$, ..., $f_n(t)$.

4. Numerical Examples

Example 1. Consider the following system of the fractional deferential equations:

$$
\begin{cases}
D_*^\alpha f_1(t) - f_1(t) - f_2(t) = 0 \\
D_*^\alpha f_2(t) + f_1(t) - f_2(t) = 0, \qquad t \in [0,2] \\
f_1(0) = 0, \quad f_2(0) = 1,
\end{cases}
$$

The exact solution of these equations for $\alpha = 1$ is given by $f_1(t) = e^t \sin(t)$ and $f_2(t) = e^t \cos(t)$. In Table 1, we list the maximum absolute errors obtained by $f_1(t)$, $f_2(t)$ with different values of N and $\alpha = 1$.

The approximate solutions obtained by $\hat{f}_1(t)$ and $\hat{f}_2(t)$ for different values of α and $N = 30$ are plotted in Figure 1.

Table 1: Maximum absolute error functions for Example 1

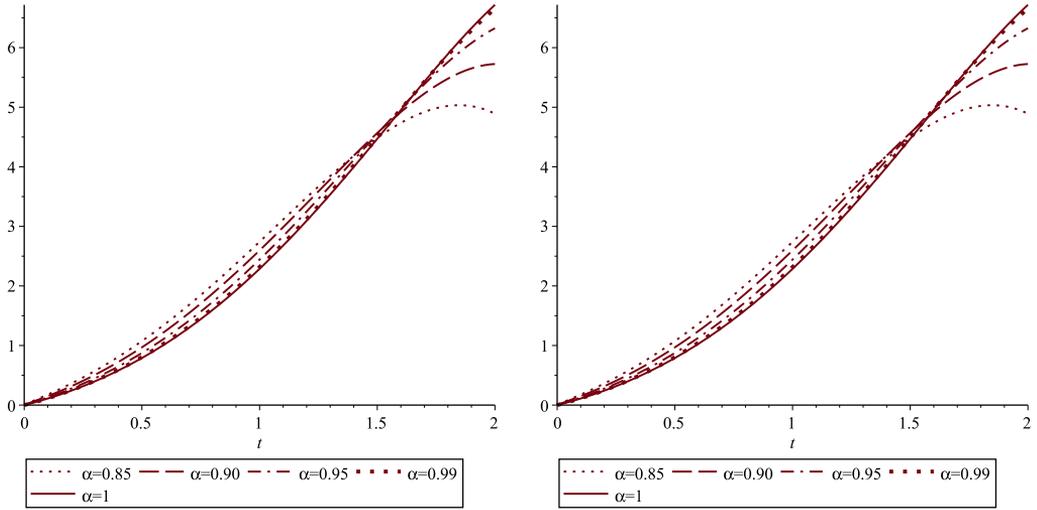| N	10	15	20	25	30
$\max|f_1(t) - \hat{f}_1(t)|$	7.5×10^{-8}	1.2×10^{-14}	1.5×10^{-21}	4.3×10^{-24}	2.5×10^{-27}
$\max|f_2(t) - \hat{f}_2(t)|$	7.6×10^{-8}	3.5×10^{-14}	1.8×10^{-21}	4.0×10^{-24}	2.3×10^{-27}

Figure 1: (a) The approximate solutions $f_1(t)$; (b) The approximate solutions $f_2(t)$.

Example 2. Consider the following system of the fractional deferential equations:

$$\begin{cases} D_*^\alpha f_1(t) = \frac{1}{2}f_1(t) \\ D_*^\alpha f_2(t) = (f_1(t))^2 + f_2(t), \qquad t \in [0, 2] \\ f_1(0) = 1, \quad f_2(0) = 0, \end{cases}$$

The exact solution of these equations for $\alpha = 1$ is given by $f_1(t) = e^{t/2}$ and $f_2(t) = te^t$. In Table 2, we list the maximum absolute errors obtained by $f_1(t)$, $f_2(t)$ with different values of N and $\alpha = 1$.

The approximate solutions obtained by $\hat{f}_1(t)$ and $\hat{f}_2(t)$ for different values of α and $N = 30$ are plotted in Figure 2.

Table 2: Maximum absolute error functions for Example 2

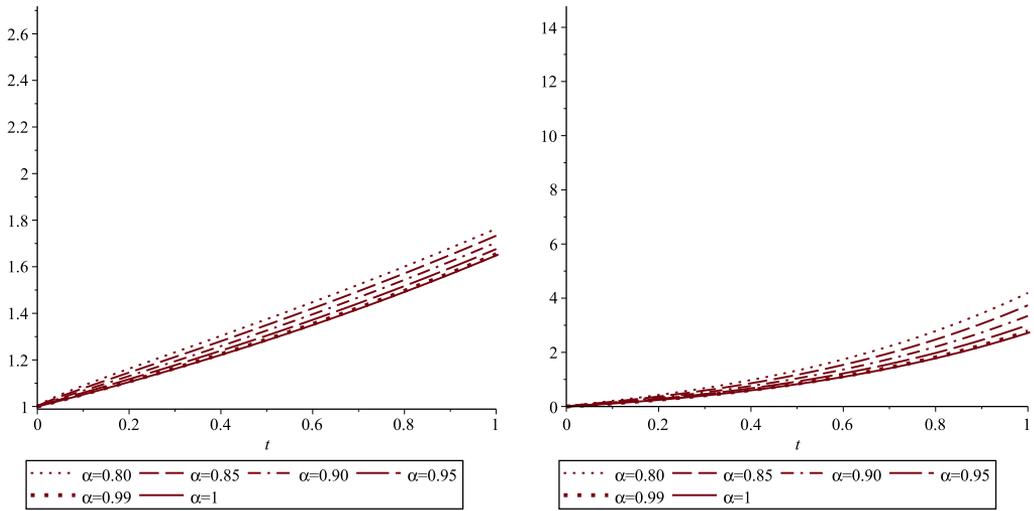| N	10	15	20	25	30
$\max|f_1(t) - \hat{f}_1(t)|$	2.5×10^{-13}	4.1×10^{-22}	1.7×10^{-31}	3.0×10^{-34}	3.6×10^{-37}
$\max|f_2(t) - \hat{f}_2(t)|$	3.1×10^{-8}	2.1×10^{-15}	3.2×10^{-23}	2.2×10^{-25}	2.3×10^{-27}

Figure 2: (a) The approximate solutions $f_1(t)$; (b) The approximate solutions $f_2(t)$.

5. Conclusion

In this paper, we presented a numerical method to obtain approximate solutions of the fractional deferential equations with a collocation method based on the Müntz-Legendre polynomials. The Müntz-Legendre polynomials are characterized for obtaining exact solutions of the fractional deferential equations. This is truly visible in results obtained through the numerical examples which their exact solutions are available.

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