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SOME COMMON FIXED POINT RESULTS FOR VARIOUS TYPES OF COMPATIBLE MAPPINGS IN MULTIPLICATIVE METRIC SPACES

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Abstract: In this paper, we prove some common fixed point theorems for the mappings satisfying rational contractions along with various types of compatible mappings in multiplicative metric spaces.

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Key Words: multiplicative metric space, compatible, A-compatible of type (E), S-compatible of type (E), weakly compatible, A-reciprocally continuous, S-reciprocally continuous

1. Introduction and Preliminaries

The set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual

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metric. To overcome this problem, in 2008, Bashirov et al. [2] introduced the concept of multiplicative metric spaces as follows:

Definition 1.1. Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(i) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) \cdot d(z, y)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

Then the mapping d together with X, that is, (X, d) is a multiplicative metric space.

Example 1.2. ([5]) Let $\mathbb{R}^n_+$ be the collection of all n-tuples of positive real numbers. Let $d^* : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be defined as follows:

$$d^*(x, y) = \left|\frac{x_1}{y_1}\right|^* \cdot \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*,$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ and $|\cdot|^* : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$|a|^* = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$$

Then it is obvious that all conditions of a multiplicative metric are satisfied. Therefore $(\mathbb{R}^n_+, d^*)$ is a multiplicative metric space.

Example 1.3. ([1]) Let $X = C^*[a, b]$ be the collection of all real-valued multiplicative continuous functions on $[a, b] \subset \mathbb{R}_+$. Then (X, d) is a multiplicative metric space with d defined by $d(f, g) = \sup_{x\in[a,b]} \left|\frac{f(x)}{g(x)}\right|$ for arbitrary $f, g \in X$.

Definition 1.4. ([5]) Let (X, d) be a multiplicative metric space. Then a sequence $\{x_n\}$ in X is said to be

(1) a *multiplicative convergent* to x if for every multiplicative open ball $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$, $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n \geq N$, that is, $d(x_n, x) \to 1$ as $n \to \infty$.

(2) a *multiplicative Cauchy sequence* if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$, that is, $d(x_n, x_m) \to 1$ as $n, m \to \infty$.

(3) We call a multiplicative metric space *complete* if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.

In 2012, Özavsar and Çevikel [5] gave the concept of multiplicative contraction mappings and proved some fixed point theorems of such mappings in a multiplicative metric space.

Definition 1.5. Let f be a mapping of a multiplicative metric space (X, d) into itself. Then f is said to be a *multiplicative contraction* if there exists a real number $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq d^{\lambda}(x, y) \quad \text{for all } x, y \in X.$$

Recently, in 2015, Kang et al. [3, 4] introduced the notion of compatible and its variants, weakly compatible and reciprocally continuous mappings in multiplicative metric spaces as follows:

Definition 1.6. Let A and S be two mappings of a multiplicative metric space (X, d) into itself. Then A and S are called

(1) *compatible* if

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

(2) *compatible of type* (E) if

$$\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ASx_n = St$$

and

$$\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} SAx_n = At,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

(3) *reciprocally continuous* if

$$\lim_{n\to\infty} d(ASx_n, At) = 1 \quad \text{and} \quad \lim_{n\to\infty} d(SAx_n, St) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

(4) *weakly compatible* if they commute at coincidence points, that is, $Ax = Sx$ for $x \in X$ implies $ASx = SAx$.

2. Various Types of Compatible Mappings and their Relationships

In 2011, Singh and Singh [7] introduced the notions of compatible of type (E), A-compatible of type (E) and S-compatible of type (E).

In 1999, Pant [6] introduced the concept of reciprocally continuous mappings. Also, Singh and Singh [7] introduced the notions of reciprocally continuous into A-reciprocally continuous and S-reciprocally continuous.

Now, we introduce these notions in setting of multiplicative metric spaces.

Definition 2.1. Let A and S be two mappings of a multiplicative metric space (X, d) into itself. Then A and S are called

(1) *S-compatible of type (E)* if

$$\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} SAx_n = At,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

(2) *A-compatible of type (E)* if

$$\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ASx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

(3) *S-reciprocally continuous* if

$$\lim_{n\to\infty} SAx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

(4) *A-reciprocally continuous* if

$$\lim_{n\to\infty} ASx_n = At,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Remark 2.2. 1. It is easy to see that compatible of type (E) implies both A-compatible of type (E) and S-compatible of type (E), however A-compatible of type (E) or S-compatible of type (E) do not imply compatible of type (E) (see Example 2.3)

2. Clearly, the reciprocal continuity of A, S implies both A-reciprocal continuity and S-reciprocal continuity, however A-reciprocal contionuity or S-reciprocal continuity do not imply reciprocal continuity (see Example 2.4).

Example 2.3. Let $X = [0,1]$ and (X,d) be a complete multiplicative metric space, where d is defined by $d(x,y) = e^{|x-y|}$ for all $x,y \in X$. Let A and $S : X \to X$ be two mappings defined by

$$Ax = \begin{cases} 1 & \text{if } x \in [0,\frac{1}{2}] - \{\frac{1}{4}\}, \\ 0 & \text{if } x = \frac{1}{4}, \\ \frac{1-x}{2} & \text{if } x \in (\frac{1}{2},1], \end{cases} \qquad Sx = \begin{cases} \frac{1}{5} & \text{if } x \in [0,\frac{1}{2}] - \{\frac{1}{4}\}, \\ 1 & \text{if } x = \frac{1}{4}, \\ \frac{x}{2} & \text{if } x \in (\frac{1}{2},1]. \end{cases}$$

Since A and S are not continuous at $x = \frac{1}{2}, \frac{1}{4}$. Suppose that $x_n \to \frac{1}{2}$, $x_n > \frac{1}{2}$ for all n. Then, we have $Ax_n = \frac{1-x_n}{2} \to \frac{1}{4} = t$ and $Sx_n = \frac{x_n}{2} \to \frac{1}{4} = t$. Consequently, we have $AAx_n = A\big(\frac{1-x_n}{2}\big) \to 1$, $ASx_n = A\big(\frac{x_n}{2}\big) \to 1$, $S(t) = 1$ and $SSx_n = S\big(\frac{x_n}{2}\big) \to \frac{1}{5}$, $SAx_n = S\big(\frac{1-x_n}{2}\big) \to \frac{1}{5}$, $A(t) = 0$. Hence the pair A and S are A-compatible of type (E) but not compatible of type (E) because it is not S-compatible of type (E).

Example 2.4. Let $X = [0,1]$ and (X,d) be a complete multiplicative metric space, where d is defined by $d(x,y) = e^{|x-y|}$ for all $x,y \in X$. Let $A, S : X \to X$ be two mappings defined by

$$Ax = \begin{cases} 1 & \text{if } x \in [0,\frac{1}{2}), \\ 1-x & \text{if } [\frac{1}{2},1], \\ \frac{1-x}{2} & \text{if } x \in (\frac{1}{2},1], \end{cases} \qquad Sx = \begin{cases} 0 & \text{if } x \in [0,\frac{1}{2}), \\ x & \text{if } [\frac{1}{2},1], \\ x/2 & \text{if } x \in (\frac{1}{2},1]. \end{cases}$$

Since A and S are not continuous at $x = \frac{1}{2}, \frac{1}{4}$. Suppose that $x_n \to \frac{1}{2}$, $x_n > \frac{1}{2}$ for all n. Then, we have $Ax_n = 1 - x_n \to \frac{1}{2} = t$ and $Sx_n = x_n \to \frac{1}{2} = t$. Consequently, we have $ASx_n = A(x_n) = 1 - x_n \to \frac{1}{2}$, $A(t) = \frac{1}{2}$, $SAx_n = S(1-x_n) \to 0$, $S(t) = \frac{1}{2}$. It follows that $\lim_{n\to\infty} ASx_n = \frac{1}{2} = A(t)$ and $\lim_{n\to\infty} SAx_n = 0 \neq \frac{1}{2} = S(t)$. Therefore, the pair A and S are A-reciprocally continuous. However the pair is neither S-reciprocally continuous nor reciprocally continuous.

Proposition 2.5. *Let A and S be two mappings of a multiplicative metric space (X,d) into itself. Suppose that $\{x_n\}$ is a sequence in X such that $Ax_n \to$ and $Sx_n \to t$ for some $t \in X$. If one of the following conditions is satisfied:*

(a) the pair A and S are S-compatible of type (E) and S-reciprocally continuous;

(b) the pair A and S are A-compatible of type (E) and A-reciprocally continuous.

Then $At = St$ and if there exists $u \in X$ such that $Au = Su = t$, then $ASu = SAu$.

Proof. Assume that (a) hold and $\{x_n\}$ is a sequence in X such that $Ax_n \to$ and $Sx_n \to t$. Then $\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} SAx_n = At$ and $\lim_{n\to\infty} SAx_n = St$ implies that $At = St$. Now if there exists $u \in X$ such that $Au = Su = t$, then $ASu = At = St = SAu$.

Similarly, assume that (b) hols, we oatain the same result. □

3. Main Results

Now we prove the following theorem for mappings satisfying rational contraction along with compatible and weakly compatible mappings as follows:

Theorem 3.1. *Let A, B, S and T be mappings of a complete multiplicative metric space (X, d) into itself satisfying the following conditions:*

$$(C_1) \qquad SX \subset BX, \quad TX \subset AX;$$

$$(C_2) \qquad d(Sx, Ty) \leq [\varphi(M(x, y))]^{\lambda}$$

for each $x, y \in X$ and $\lambda \in (0, 1)$, where $\varphi : [1, \infty) \to [1, \infty)$ is a monotone increasing continuous function such that $\varphi(1) = 1$, $\varphi(t) < t$ for all $t > 1$, $\varphi(t) = \sqrt{2t-1}$ for all $t \geq 1$ and

$$\begin{aligned} M(x, y) = \max \Big\{ & d(Ax, By), d(Ax, Sx), d(By, Ty), \\ & (d(Ax, Ty) \cdot d(Sx, By))^{1/2}, \\ & \frac{d(Ax, Ty) \cdot d(Sx, By)}{1 + d(Ax, By)}, \frac{d(Ax, Sx) \cdot d(By, Ty)}{1 + d(Ax, By)}, \\ & \frac{1 + d(Ax, Ty) \cdot d(By, Sx)}{1 + d(Ax, Sx) \cdot d(By, Ty)} \cdot d(Ax, Sx) \Big\}. \end{aligned}$$

Suppose that one of the following conditions is satisfied:

(a) either A or S is continuous, A and S are compatible, and B and T are weakly compatible.

(b) either B or T is continuous, B and T are compatible, and A and S are weakly compatible.

Then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $SX \subset BX$, for point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Sx_0 = Bx_1$. Now for this point x_1 there exists a point $x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$. Similarly, we can inductively define a sequence $\{y_n\}$ such that

$$Sx_{2n} = Bx_{2n+1} = y_{2n}; \quad Tx_{2n+1} = Ax_{2n+2} = y_{2n+1} \quad \text{for } n \geq 0.$$

Now from (C_2), we obtain

$$d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1}) \leq \varphi(M(x_{2n}, x_{2n+1}))^{\lambda},$$

where by the multiplicative triangle inequality

$$\begin{aligned}
&M(x_{2n}, x_{2n+1})\\
&= \max\Big\{ d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}),\\
&\quad (d(Ax_{2n}, Tx_{2n+1}) \cdot d(Sx_{2n}, Bx_{2n+1}))^{1/2},\\
&\quad \frac{d(Ax_{2n}, Tx_{2n+1}) \cdot d(Sx_{2n}, Bx_{2n+1})}{1 + d(Ax_{2n}, Bx_{2n+1})}, \frac{d(Ax_{2n}, Sx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Ax_{2n}, Bx_{2n+1})},\\
&\quad \frac{1 + d(Ax_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sx_{2n})}{1 + d(Ax_{2n}, Sx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})} \cdot d(Ax_{2n}, Sx_{2n}) \Big\}\\
&= \max\Big\{ d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}),\\
&\quad (d(y_{2n-1}, y_{2n+1}) \cdot d(y_{2n}, y_{2n}))^{1/2},\\
&\quad \frac{d(y_{2n-1}, y_{2n+1}) \cdot d(y_{2n}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})}, \frac{d(y_{2n-1}, y_{2n}) \cdot d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n})},\\
&\quad \frac{1 + d(y_{2n-1}, y_{2n+1}) \cdot d(y_{2n}, y_{2n})}{1 + d(y_{2n-1}, y_{2n}) \cdot d(y_{2n}, y_{2n+1})} \cdot d(y_{2n-1}, y_{2n}) \Big\}\\
&= \max\Big\{ d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), (d(y_{2n-1}, y_{2n}) \cdot d(y_{2n}, y_{2n+1}))^{1/2},\\
&\quad \frac{d(y_{2n-1}, y_{2n}) \cdot d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n})}, \frac{d(y_{2n-1}, y_{2n}) \cdot d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n})},\\
&\quad \frac{1 + d(y_{2n-1}, y_{2n}) \cdot d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n}) \cdot d(y_{2n}, y_{2n+1})} \cdot d(y_{2n-1}, y_{2n}) \Big\}\\
&= \max\left\{ d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), (d(y_{2n-1}, y_{2n}) \cdot d(y_{2n}, y_{2n+1}))^{1/2} \right\}.
\end{aligned}$$

If $d(y_{2n-1}, y_{2n}) < d(y_{2n}, y_{2n+1})$, then we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq [\varphi(M(x_{2n}, x_{2n+1}))]^{\lambda} \\ &= [\varphi(d(y_{2n}, y_{2n+1}))]^{\lambda} \\ &< d^{\lambda}(y_{2n}, y_{2n+1}), \end{aligned}$$

which is a contraction. Hence $d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$ and $M(x_{2n}, x_{2n+1}) = d(y_{2n-1}, y_{2n})$. Thus we have

$$d(y_{2n}, y_{2n+1}) < d^{\lambda}(y_{2n-1}, y_{2n}), \tag{3.1}$$

where $\lambda \in (0, 1)$.

Similarly, we have

$$d(y_{2n+2}, y_{2n+1}) = d(Sx_{2n+2}, Tx_{2n+1}) \leq [\varphi(M(x_{2n+2}, x_{2n+1}))]^{\lambda},$$

where

$$\begin{aligned} &M(x_{2n+2}, x_{2n+1}) \\ &= \max\Big\{d(Ax_{2n+2}, Bx_{2n+1}), d(Ax_{2n+2}, Sx_{2n+2}), d(Bx_{2n+1}, Tx_{2n+1}), \\ &\quad (d(Ax_{2n+2}, Tx_{2n+1}) \cdot d(Sx_{2n+2}, Bx_{2n+1}))^{1/2}, \\ &\quad \frac{d(Ax_{2n+2}, Tx_{2n+1}) \cdot d(Sx_{2n+2}, Bx_{2n+1})}{1 + d(Ax_{2n+2}, Bx_{2n+1})}, \\ &\quad \frac{d(Ax_{2n+2}, Sx_{2n+2}) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Ax_{2n+2}, Bx_{2n+1})}, \\ &\quad \frac{1 + d(Ax_{2n+2}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sx_{2n+2})}{1 + d(Ax_{2n+2}, Sx_{2n+2}) \cdot d(Bx_{2n+1}, Tx_{2n+1})} \cdot d(Ax_{2n+2}, Sx_{2n+2})\Big\} \\ &= \max\Big\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), \\ &\quad (d(y_{2n+1}, y_{2n+1}) \cdot d(y_{2n+2}, y_{2n}))^{1/2}, \\ &\quad \frac{d(y_{2n+1}, y_{2n+1}) \cdot d(y_{2n+2}, y_{2n})}{1 + d(y_{2n+1}, y_{2n})}, \frac{d(y_{2n+1}, y_{2n+2}) \cdot d(y_{2n}, y_{2n+1})}{1 + d(y_{2n+1}, y_{2n})}, \\ &\quad \frac{1 + d(y_{2n+1}, y_{2n+1}) \cdot d(y_{2n}, y_{2n+2})}{1 + d(y_{2n+1}, y_{2n+2}) \cdot d(y_{2n}, y_{2n+1})} \cdot d(y_{2n+1}, y_{2n+2})\Big\} \\ &= \max\left\{d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), (d(y_{2n}, y_{2n+1}) \cdot d(y_{2n+2}, y_{2n+1}))^{1/2}\right\}. \end{aligned}$$

If $d(y_{2n+1}, y_{2n}) \leq d(y_{2n+1}, y_{2n+2})$, then we have

$$\begin{aligned} d(y_{2n+2}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq [\varphi(d(y_{2n+1}, y_{2n+2}))]^{\lambda} \\ &< d^{\lambda}(y_{2n+1}, y_{2n+2}), \end{aligned}$$

which is a contraction. Hence we have

$$d(y_{2n+2}, y_{2n+1}) < d^{\lambda}(y_{2n+1}, y_{2n}), \tag{3.2}$$

where $\lambda \in (0,1)$.

It follows from (3.1) and (3.2) that, for all $n \in \mathbb{N}$,

$$\begin{aligned} d(y_n, y_{n+1}) &\leq d^{\lambda}(y_{n-1}, y_n) \\ &\leq d^{\lambda^2}(y_{n-2}, y_{n-1}) \\ &\leq \cdots \leq d^{\lambda^n}(y_0, y_1). \end{aligned}$$

Let $m, n \in \mathbb{N}$ with $m > n$, we get

$$\begin{aligned} d(y_m, y_n) &\leq d(y_m, y_{m-1}) \cdots d(y_{n+1}, y_n) \\ &\leq d^{\lambda^{m-1} + \cdots + \lambda^n}(y_1, y_0) \\ &\leq d^{\frac{\lambda^n}{1-\lambda}}(y_1, y_0) \\ &\to 1 \quad \text{as } m, n \to \infty. \end{aligned}$$

Hence $\{y_n\}$ is a multiplicative Cauchy sequence. Since X is complete so $\{y_n\} \to z \in X$. Therefore, subsequences $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Tx_{2n+1}\}$ also converges to $z \in X$. Thus we get

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z. \tag{3.3}$$

Next, we show that z is a common fixed point of A, B, S and T under the condition (a).

Case 1. Suppose that A is continuous. Then it follows that $\lim_{n\to\infty} ASx_{2n} = \lim_{n\to\infty} AAx_{2n} = Az$. Since A and S are compatible, it follows from (3.3) that

$$\lim_{n\to\infty} d(SAx_{2n}, Az) = \lim_{n\to\infty} d(SAx_{2n}, ASx_{2n}) = 1,$$

this is, $\lim_{n\to\infty} SAx_{2n} = Az$.

First we claim that $Az = z$. Let $Az \neq z$. Then putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in inequality (C_2), we get

$$d(SAx_{2n}, Tx_{2n+1}) \leq [\varphi(M(Ax_{2n}, x_{2n+1}))]^{\lambda},$$

where

$$\begin{aligned} &M(Ax_{2n}, x_{2n+1}) \\ &= \max\Big\{ d(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\ &\quad (d(AAx_{2n}, Tx_{2n+1}) \cdot d(SAx_{2n}, Bx_{2n+1}))^{1/2}, \\ &\quad \frac{d(AAx_{2n}, Tx_{2n+1}) \cdot d(SAx_{2n}, Bx_{2n+1})}{1 + d(AAx_{2n}, Bx_{2n+1})}, \\ &\quad \frac{d(AAx_{2n}, SAx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(AAx_{2n}, Bx_{2n+1})}, \\ &\quad \frac{1 + d(AAx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SAx_{2n})}{1 + d(AAx_{2n}, SAx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})} \cdot d(AAx_{2n}, SAx_{2n}) \Big\}. \end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned} &\lim_{n\to\infty} M(Ax_{2n}, x_{2n+1}) \\ &= \max\Big\{ d(Az, z), 1, 1, d(Az, z), \frac{d(Az, z) \cdot d(Az, z)}{1 + d(Az, z)}, \\ &\quad \frac{1}{1 + d(Az, z)}, \frac{1 + d(Az, z).d(z, Az)}{2} \Big\} \\ &= \frac{1 + d^2(z, Az)}{2}. \end{aligned}$$

Hence we have

$$d(Az, z) \leq \left[\varphi\left(\frac{1 + d^2(z, Az)}{2}\right)\right]^{\lambda} \leq d^{\lambda}(z, Az),$$

by property φ, which implies that $Az = z$.

Next we claim that $Sz = z$. Let $Sz \neq z$. Then putting $x = z$ and $y = x_{2n+1}$ in inequality (C_2), we have

$$d(Sz, Tx_{2n+1}) \leq [\varphi(M(z, x_{2n+1}))]^{\lambda},$$

where

$$\begin{aligned}&M(z,x_{2n+1})\\&=\max\Big\{d(Az,Bx_{2n+1}),d(Az,Sz),d(Bx_{2n+1},Tx_{2n+1}),\\&\qquad(d(Az,Tx_{2n+1})\cdot d(Sz,Bx_{2n+1}))^{1/2},\\&\qquad\frac{d(Az,Tx_{2n+1})\cdot d(Sz,Bx_{2n+1})}{1+d(Az,Bx_{2n+1})},\frac{d(Az,Sz)\cdot d(Bx_{2n+1},Tx_{2n+1})}{1+d(Az,Bx_{2n+1})},\\&\qquad\frac{1+d(Az,Tx_{2n+1})\cdot d(Bx_{2n+1},Sz)}{1+d(Az,Sz)\cdot d(Bx_{2n+1},Tx_{2n+1})}\cdot d(Az,Sz)\Big\}.\end{aligned}$$

Taking $n\to\infty$ and using $Az=z$, we have

$$\begin{aligned}&\lim_{n\to\infty}M(z,x_{2n+1})\\&=\max\left\{1,d(z,Sz),1,d^{1/2}(Sz,z),\frac{d(z,Sz)}{2},\frac{d(z,Sz)}{2},d(z,Sz)\right\}\\&=d(z,Sz).\end{aligned}$$

Hence we have

$$d(Sz,z)=\lim_{n\to\infty}d(Sz,Tx_{2n+1})\le[\varphi(d(z,Sz))]^{\lambda}<d^{\lambda}(z,Sz),$$

which implies that $Sz=z$.

On the other hand, since $z=Sz\in S(X)\subset B(X)$, there exist $u\in X$ such that $z=Sz=Bu$.

Next we claim that $Tu=z$. Let $Tu\neq z$. Then using $z=Sz=Az=Bu$ and putting $x=z$ and $y=u$ in inequality (C_2), we have

$$d(z,Tu)=d(Sz,Tu)\le[\varphi(M(z,u))]^{\lambda},$$

where

$$\begin{aligned}M(z,u)&=\max\Big\{d(Az,Bu),d(Az,Sz),d(Bu,Tu),\\&\qquad(d(Az,Tu)\cdot d(Sz,Bu))^{1/2},\\&\qquad\frac{d(Az,Tu)\cdot d(Sz,Bu)}{1+d(Az,Bu)},\frac{d(Az,Sz)\cdot d(Bu,Tu)}{1+d(Az,Bu)},\\&\qquad\frac{1+d(Az,Tu)\cdot d(Bu,Sz)}{1+d(Az,Sz)\cdot d(Bu,Tu)}\cdot d(Az,Sz)\Big\}\\&=\max\left\{1,1,d(z,Tu),d^{1/2}(z,Tu),\frac{d(z,Tu)}{2},\frac{d(z,Tu)}{2},1\right\}\\&=d(z,Tu),\end{aligned}$$

so $Tu = z = Bu$. Since B an T are weakly compatible, we have $Tz = TBu = BTu = Bz$.

Next, we prove that $Tz = z$. Putting $x = z$ and $y = z$ in inequality (C_2), we have

$$d(z, Tz) = d(Sz, Tz) \le [\varphi(M(z, z))]^{\lambda},$$

where

$$\begin{aligned} M(z,z) = \max\Big\{ & d(Az,Bz), d(Az,Sz), d(Bz,Tz), \\ & (d(Az,Tz)\cdot d(Sz,Bz))^{1/2}, \\ & \frac{d(Az,Tz)\cdot d(Sz,Bz)}{1+d(Az,Bz)}, \frac{d(Az,Sz)\cdot d(Bz,Tz)}{1+d(Az,Bz))}, \\ & \frac{1+d(Az,Tz)\cdot d(Bz,Sz)}{1+d(Az,Sz)\cdot d(Bz,Tz)}\cdot d(Az,Sz)\Big\} \\ = \max & \left\{1, 1, d(z,Tz), d^{1/2}(z,Tz), \frac{d(z,Tz)}{2}, \frac{d(z,Tz)}{2}, 1\right\} \\ = d(z, & Tz). \end{aligned}$$

This implies that $d(z, Tz) = 1$ and so $z = Tz$. Hence we obtain $z = Sz = Az = Tz = Bz$ and hence z is a common fixed point of A, B, S and T.

Case 2. Suppose that S is continuous. Then it follows that $\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} SSx_{2n} = Sz$. Since A and S are compatible, it follows from (3.3) that

$$\lim_{n\to\infty} d(Sz, ASx_{2n}) = \lim_{n\to\infty} d(SAx_{2n}, ASx_{2n}) = 1,$$

this is, $\lim_{n\to\infty} ASx_{2n} = Sz$.

First we claim that $Sz = z$. Let $Sz \neq z$. Then putting $x = Sx_{2n}$ and $y = x_{2n+1}$ in inequality (C_2), we get

$$d(SSx_{2n}, Tx_{2n+1}) \le [\varphi(M(Sx_{2n}, x_{2n+1}))]^{\lambda},$$

where

$$
\begin{aligned}
&M(Sx_{2n}, x_{2n+1})\\
&= \max\Big\{d(ASx_{2n}, Bx_{2n+1}), d(ASx_{2n}, SSx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}),\\
&\quad (d(ASx_{2n}, Tx_{2n+1}) \cdot d(SSx_{2n}, Bx_{2n+1}))^{1/2},\\
&\quad \frac{d(ASx_{2n}, Tx_{2n+1}) \cdot d(SSx_{2n}, Bx_{2n+1})}{1 + d(ASx_{2n}, Bx_{2n+1})},\\
&\quad \frac{d(ASx_{2n}, SSx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(ASx_{2n}, Bx_{2n+1})},\\
&\quad \frac{1 + d(ASx_{2n}, Tx_{2n+1}) \cdot d(Bx_{2n+1}, SSx_{2n})}{1 + d(ASx_{2n}, SSx_{2n}) \cdot d(Bx_{2n+1}, Tx_{2n+1})} \cdot \cdot d(ASx_{2n}, SSx_{2n})\Big\}.
\end{aligned}
$$

Taking $n \to \infty$, we have

$$
\begin{aligned}
&\lim_{n\to\infty} M(Sx_{2n}, x_{2n+1})\\
&= \max\Big\{d(Sz, z), 1, 1, d(Sz, z), \frac{d(Sz, z) \cdot d(Sz, z)}{1 + d(Sz, z)},\\
&\quad \frac{1}{1 + d(Sz, z)}, \frac{1 + d(Sz, z) \cdot d(z, Sz)}{2}\Big\}\\
&= \frac{1 + d^2(z, Sz)}{2}.
\end{aligned}
$$

Hence we have

$$
d(Sz, z) \le \left[\varphi\left(\frac{1 + d^2(z, Sz)}{2}\right)\right]^{\lambda} \le d^{\lambda}(z, Sz),
$$

which implies that $Sz = z$. Since $z = Sz \in S(X) \subset B(X)$, there exist $v \in X$ such that $z = Sz = Bv$.

Next we claim that $Tv = z$. Let $Tv \neq z$. Then putting $x = Sx_{2n}$ and $y = v$ in inequality (C_2), we have

$$
d(SSx_{2n}, Tv) \le [\varphi(M(Sx_{2n}, v))]^{\lambda},
$$

where

$$\begin{aligned}
&M(Sx_{2n}, v)\\
&= \max\Big\{d(ASx_{2n}, Bv), d(ASx_{2n}, SSx_{2n}), d(Bv, Tv),\\
&\qquad (d(ASx_{2n}, Tv)\cdot d(SSx_{2n}, Bv))^{1/2},\\
&\qquad \frac{d(ASx_{2n}, Tv)\cdot d(SSx_{2n}, Bv)}{1+d(ASx_{2n}, Bv)}, \frac{d(ASx_{2n}, SSx_{2n})\cdot d(Bv, Tv)}{1+d(ASx_{2n}, Bv)},\\
&\qquad \frac{1+d(ASx_{2n}, Tv)\cdot d(Bv, SSx_{2n})}{1+d(ASx_{2n}, SSx_{2n})\cdot d(Bv, Tv)}\cdot d(ASx_{2n}, SSx_{2n})\Big\}.
\end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned}
&\lim_{n\to\infty} M(Sx_{2n}, v)\\
&= \max\left\{1, 1, d(z, Tv), d^{1/2}(z, Tv), \frac{d(z, Tv)}{2}, \frac{d(z, Tv)}{2}, 1\right\}\\
&= d(z, Tv).
\end{aligned}$$

Hence we have

$$d(z, Tv) = \lim_{n\to\infty} d(SSx_{2n}, Tv) \le [\varphi(d(z, Tv))]^{\lambda},$$

so $Tv = z = Bv$. Since B and T are weakly compatible, we have $Tz = TBv = BTv = Bz$.

Next, we prove that $Tz = z$. Putting $x = x_{2n}$ and $y = z$ in inequality (C_2), we have

$$d(Sx_{2n}, Tz) \le [\varphi(M(x_{2n}, z))]^{\lambda},$$

where

$$\begin{aligned}
M(x_{2n}, z) = \max\Big\{&d(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n}), d(Bz, Tz),\\
&(d(Ax_{2n}, Tz)\cdot d(Sx_{2n}, Bz))^{1/2},\\
&\frac{d(Ax_{2n}, Tz)\cdot d(Sx_{2n}, Bz)}{1+d(Ax_{2n}, Bz)}, \frac{d(Ax_{2n}, Sx_{2n})\cdot d(Bz, Tz)}{1+d(Ax_{2n}, Bz)},\\
&\frac{1+d(Ax_{2n}, Tz)\cdot d(Bz, Sx_{2n})}{1+d(Ax_{2n}, Sx_{2n})\cdot d(Bz, Tz)}\cdot d(Ax_{2n}, Sx_{2n})\Big\}.
\end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} M(x_{2n}, z) &= \max\left\{d(z,Tz), 1, 1, d(z,Tz), \frac{d(z,Tz)\cdot d(z,Tz)}{1+d(z,Tz)},\right.\\ &\qquad \left.\frac{1}{1+d(z,Tz)}, \frac{1+d(z,Tz)\cdot d(Tz,z)}{2}\right\}\\ &= \frac{1+d^2(z,Tz)}{2}.\end{aligned}$$

This implies that $d(z,Tz) = 1$ and so $z = Tz$. Hence we obtain $z = Tz = Bz$.

On the other hand, since $z = Tz \in T(X) \subset A(X)$, there exist $w \in X$ such that $z = Tz = Aw$.

Next we prove that $z = Sw$. Let $z \neq Sw$. Then using $Tz = Bz = z$ and putting $x = w$ and $y = z$ in (C_2), we have

$$d(Sw, z) = d(Sw, Tz) \leq [\varphi(M(w,z))]^{\lambda},$$

where

$$\begin{aligned}M(w,z) = \max\Big\{&d(Aw,Bz), d(Aw,Sw), d(Bz,Tz),\\ &(d(Aw,Tz)\cdot d(Sw,Bz))^{1/2},\\ &\frac{d(Aw,Tz)\cdot d(Sw,Bz)}{1+d(Aw,Bz)}, \frac{d(Aw,Sw)\cdot d(Bz,Tz)}{1+d(Aw,Bz)},\\ &\frac{1+d(Aw,Tz)\cdot d(Bz,Sw)}{1+d(Aw,Sw)\cdot d(Bz,Tz)}\cdot d(Aw,Sw)\Big\}\\ = \max&\left\{1, d(z,Sw), 1, d^{1/2}(Sw,z), \frac{d(z,Sw)}{2}, \frac{d(z,Sw)}{2}, d(z,Sw)\right\}\\ = d&(z,Sw).\end{aligned}$$

This implies that $d(Sw,z) = 1$ and so $Sw = z = Aw$. Since A and S are compatible, we have $Az = SAw = ASw = Sz$ Hence $z = Sz = Az = Tz = Bz$ and hence z is a common fixed point of A, B, S and T.

Finally, we prove that A, B, S and T have a unique common fixed point.

Suppose that $w \in X$ is another common fixed point of A, B, S and T. Then putting $x = z$ and $y = w$ in inequality (C_2), we have

$$d(z,w) = d(Sz,Tw) \leq [\varphi(M(z,w))]^{\lambda},$$

where

$$\begin{aligned}
M(z,w) = \max\Big\{ & d(Az,Bw), d(Az,Sz), d(Bw,Tw), \\
& (d(Az,Tw)\cdot d(Sz,Bw))^{1/2}, \\
& \frac{d(Az,Tw)\cdot d(Sz,Bw)}{1+d(Az,Bw)}, \frac{d(Az,Sz)\cdot d(Bw,Tw)}{1+d(Az,Bw)}, \\
& \frac{1+d(Az,Tw)\cdot d(Bw,Sz)}{1+d(Az,Sz)\cdot d(Bw,Tw)}\cdot d(Az,Sz)\Big\} \\
= \max\Big\{ & d(z,w), 1, 1, d(z,w), \frac{d(z,w)\cdot d(z,w)}{1+d(z,w)}, \\
& \frac{1}{1+d(z,w)}, \frac{1+d(z,w)\cdot d(w,z)}{2}\Big\} \\
= & \frac{1+d^2(z,w)}{2}.
\end{aligned}$$

This implies that $d(z,w)=1$ and so $z=w$. Therefore, z is a unique common fixed point of A, B, S and T.

Similarly if the condition (b) holds, then we obtain the same result.

This completes the proof. □

Next we prove the following theorem for compatible mappings of type (E) along with the notion of reciprocally continuous mappings.

Theorem 3.2. *Let A, B, S and T be mappings of complete multiplicative metric space (X,d) into itself satisfying (C_1) and (C_2).*

Assume that one of the following conditions is satisfied:

(a) A and S are A-compatible of type (E) and A-reciprocally continuous, B and T are B-compatible of type (E) and B-reciprocally continuous.

(b) A and S are is S-compatible of type (E) and S-reciprocally continuous, B and T are T-compatible of type (E) and T-reciprocally continuous.

Then A, B, S and T a unique common fixed point in X.

Proof. By the proof of Theorem 3.1, the sequence $\{y_n\}$ is a multiplicative Cauchy sequence. By the completeness of X, the sequence $\{y_n\}$ converges to a point $z \in X$. Consequently the subsequences $\{Sx_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Tx_{2n+1}\}$ also converges to $z \in X$, that is,

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z.$$

Suppose that A and S are A-compatible of type (E) and A-reciprocally continuous. Then

$$AAx_{2n} \to Sz, \quad ASx_{2n} \to Sz \quad \text{and} \quad ASx_{2n} \to Az.$$

Hence $Az = Sz$.

Now, suppose that $z \neq Sz$. Then putting $x = z$ and $y = x_{2n+1}$ in (C_2), we have

$$d(Sz, Tx_{2n+1}) \leq [\varphi(M(z, x_{2n+1}))]^{\lambda},$$

where

$$\begin{aligned} &M(z, x_{2n+1}) \\ &= \max\Big\{d(Az, Bx_{2n+1}), d(Az, Sz), d(Bx_{2n+1}, Tx_{2n+1}), \\ &\quad (d(Az, Tx_{2n+1}) \cdot d(Sz, Bx_{2n+1}))^{1/2}, \\ &\quad \frac{d(Az, Tx_{2n+1}) \cdot d(Sz, Bx_{2n+1})}{1 + d(Az, Bx_{2n+1})}, \frac{d(Az, Sz) \cdot d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Az, Bx_{2n+1})}, \\ &\quad \frac{1 + d(Az, Tx_{2n+1}) \cdot d(Bx_{2n+1}, Sz)}{1 + d(Az, Sz) \cdot d(Bx_{2n+1}, Tx_{2n+1})} \cdot d(Az, Sz)\Big\}. \end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned} &\lim_{n\to\infty} M(z, x_{2n+1}) \\ &= \max\Big\{d(Sz, z), 1, 1, d(Sz, z), \frac{d(Sz, z) \cdot d(Sz, z)}{1 + d(Sz, z)}, \\ &\quad \frac{1}{1 + d(Sz, z)}, \frac{1 + d(Sz, z) \cdot d(z, Sz)}{2}\Big\} \\ &= \frac{1 + d^2(z, Sz)}{2}. \end{aligned}$$

Hence we have

$$d(Sz, z) \leq \left[\varphi\left(\frac{1 + d^2(z, Sz)}{2}\right)\right]^{\lambda} \leq d^{\lambda}(z, Sz),$$

which implies that $Sz = z$. Hence $z = Az = Sz$. Since $S(X) \subset B(X)$, there exists $u \in X$ such that $z = Sz = Bu$.

Let $Tu \neq Bu$. Then by putting $x = z$ and $y = u$ in (C_2), we have

$$d(Bu, Tu) = d(Sz, Tu) \leq [\varphi(M(z, u))]^{\lambda},$$

where

$$\begin{aligned} M(z,u) = \max\Big\{ & d(Az,Bu), d(Az,Sz), d(Bu,Tu), \\ & d(Az,Tu)\cdot d(Sz,Bu))^{1/2}, \\ & \frac{d(Az,Tu)\cdot d(Sz,Bu)}{1+d(Az,Bu)}, \frac{d(Az,Sz)\cdot d(Bu,Tu)}{1+d(Az,Bu)}, \\ & \frac{1+d(Az,Tu)\cdot d(Bu,Sz)}{1+d(Az,Sz)\cdot d(Bu,Tu)}\cdot d(Az,Sz)\Big\} \\ = \max & \Big\{1,1,d(Bu,Tu), d^{1/2}(Bu,Tu), \frac{d(Bu,Tu)}{2}, \frac{d(Bu,Tu)}{2}, 1\Big\} \\ = d(B & u,Tu). \end{aligned}$$

Hence $Bu = Tu = z$. Since B and T are B-compatible of type (E) and B-reciprocally continuous, by Proposition 2.5, we obtain $Bz = BTu = TBu = Tz$.

Moreover putting $x = z$ and $y = z$ in (C_2), we have

$$d(z,Tz) = d(Sz,Tz) \le [\varphi(M(z,z))]^{\lambda},$$

where

$$\begin{aligned} M(z,z) = \max\Big\{ & d(Az,Bz), d(Az,Sz), d(Bz,Tz), \\ & (d(Az,Tz)\cdot d(Sz,Bz))^{1/2}, \\ & \frac{d(Az,Tz)\cdot d(Sz,Bz)}{1+d(Az,Bz)}, \frac{d(Az,Sz)\cdot d(Bz,Tz)}{1+d(Az,Bz)}, \\ & \frac{1+d(Az,Tz)\cdot d(Bz,Sz)}{1+d(Az,Sz)\cdot d(Bz,Tz)}\cdot d(Az,Sz)\Big\} \\ = \max\Big\{ & d(z,Tz), 1, 1, d(z,Tz), \frac{d(z,Tz)\cdot d(z,Tz)}{1+d(z,Tz)}, \\ & \frac{1}{1+d(z,Tz)}, \frac{1+d(z,Tz)\cdot d(Tz,z)}{2}\Big\} \\ = \frac{1+d^2(z,Tz)}{2}. & \end{aligned}$$

Hence we have

$$d(z,Tz) \le \left[\varphi\left(\frac{1+d^2(z,Tz)}{2}\right)\right]^{\lambda} \le d^{\lambda}(z,Tz),$$

which implies that $Tz = z$. Hence $z = Az = Sz = Bz = Tz$, that is, z is a common fixed point of A, B, S and T.

Similarly, we can prove z is a common fixed point of A, B, S and T if the condition (b) holds.

The uniqueness easily follows from (C_2). This complete the proof. □

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