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ON THE COMPOSITION OPERATORS ON BANACH WEIGHTED HARDY SPACES

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Abstract: In this paper we consider composition operators on weighted Hardy spaces and the aim of the paper is to investigate the numerical range of composition operators.

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1. Introduction

Let $\{\beta(n)\}_n$ be a sequence of positive numbers with $\beta(0) = 1$ and let $1 < p < \infty$. Let $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ be such that

$$\|f\|_p = \|f\|_{H^p(\beta)} = \left(\sum_{n=0}^{+\infty} |\hat{f}(n)|^p \beta(n)^p\right)^{1/p} < \infty.$$

The notation $f(z) = \sum_{n=0}^{+\infty} \hat{f}(n) z^n$ shall be used weather or not the series converges for any value of z. The space of such formal power series is called the weighted Hardy space, which is denoted by $H^p(\beta)$. In the case $p = 2$, the classical Hardy

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space, Bergman space and the Dirichlet space are weighted Hardy spaces with $\beta(n) = 1, \beta(n) = (n+1)^{-\frac{1}{2}}$ and $\beta(n) = (n+1)^{\frac{1}{2}}$, respectively. The spaces $H^p(\beta)$ are reflexive Banach spaces with the norm $\|.\|_p$ and the dual of $H^p(\beta)$ is $H^q(\beta^{\frac{p}{q}})$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta^{\frac{p}{q}} = \{\beta(n)^{\frac{p}{q}}\}_n$.

Recall that for $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n$ in $H^q(\beta^{p/q})$, note that

$$||g||_q^q = ||g||_{H^q(\beta^{p/q})}^q = \sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta(n)^p < \infty.$$

If $\lim \dfrac{\beta(n+1)}{\beta(n)} = 1$ or $\liminf \beta(n)^{\frac{1}{n}} = 1$, then $H^p(\beta)$ consists of functions analytic on the open unit disk U.

A complex number λ is said to be a bounded point evaluation on $H^p(\beta)$ if the functional of point evaluation at λ, e_λ, is bounded. A complex number λ is a bounded point evaluation on $H^p(\beta)$ if and only if $\left\{ \dfrac{\lambda^n}{\beta(n)} \right\}_n \in l^q$.

Let φ be an analytic self map of U. A composition operator C_φ maps an analytic function $f \in H^p(\beta)$ into $(C_\varphi f)(z) = f(\varphi(z))$. The function φ is called the composition map. Some sources on formal power series and composition operators include [1–7].

2. Main Result

This work represents the necessary and sufficient conditions for the closedness of the numerical range of a compact composition operator acting on weighted Hardy spaces $H^p(\beta)$.

In the following we define some definitions that will be used in the main theorem.

Definition 2.1. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^p(\beta)$ and define $f^*(z) = \sum_{n=0}^{\infty} |\hat{f}(n)|^{(p-q)/q} \hat{f}(n) z^n$.

Note that $||f^*||_q^q = ||f^*||_{H^q(\beta^{p/q})}^q = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p = ||f||_p^p$ and obviously one can see that $< f, f^* >= ||f||_p^p$. Also

Definition 2.2. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n$ belongs to $H^q(\beta^{p/q})$ and define ${}^*g(z) = \sum_{n=0}^{\infty} |\hat{g}(n)|^{(q-p)/p}\, \hat{g}(n)z^n$.

Notice that $||{}^*g||_p^p = \sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta(n)^p = ||g||_q^q < \infty$ and so ${}^*g \in H^p(\beta)$. Obviously, one can see that ${}^*(f^*) = f$ for all f in $H^p(\beta)$ and $({}^*g)^* = g$ for all g in $(H^p(\beta))^*$. Also, clearly $< g, {}^* g >= ||g||_q^q$.

Definition 2.3. If T is a bounded linear operator on $H^p(\beta)$, the numerical range of T is denoted by $W(T)$ that is defined by

$$W(T) = co\{< Tf, f^* >: f \in H^p(\beta) \text{ and } ||f||_p = 1\}.$$

Note that clearly $W(T) = \{< T({}^*g), g >: g \in (H^p(\beta))^* \text{ and } ||g||_q = 1\}$.

Theorem 2.4. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\liminf \beta(n)^{\frac{1}{n}} = 1$, $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q} = \infty$, and C_φ be compact on $H^p(\beta)$. Suppose that there exists ξ_0 in ∂U such that the limit of ${}^*e_\lambda(\varphi(\lambda))$ exists and is finite as $\lambda \to \xi_0$. Then $0 \in W(C_\varphi)$ if and only if $W(C_\varphi)$ is closed.

Proof. First note that since $\liminf \beta(n)^{\frac{1}{n}} = 1$, thus for each λ in the open unit disk, the functional of evaluation at λ, e_λ, is a bounded linear functional and we have

$$e_\lambda(z) = \sum_{n=0}^{\infty} \frac{\bar{\lambda}^n}{\beta(n)^p} z^n,$$

and

$$\|e_\lambda\|^q = \sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q}.$$

Now suppose that $\{h_n\}_n$ is a sequence of unit vectors in $H^p(\beta)$. By weakly compactness of ball$H^p(\beta)$, a subsequence of $\{h_n\}_n$ is weakly convergent. For simplicity we suppose that $\{h_n\}_n$ converges weakly to a vector h in $H^p(\beta)$. Then we have

$$\begin{aligned}
| < C_\varphi h_n, h_n^* > - < C_\varphi h, h^* > | &\leq < C_\varphi h_n, h_n^* > - < C_\varphi h, h_n^* > | \\
&\quad + | < C_\varphi h, h_n^* > - < C_\varphi h, h^* > | \\
&= | < C_\varphi (h_n - h), h_n^* > | \\
&\quad + | < C_\varphi h, (h_n^* - h^*) > | \\
&\leq ||C_\varphi(h_n - h)|| \;\; ||h_n^*||
\end{aligned}$$

$$+ |< C_\varphi h, (h_n^* - h^*) >|.$$

which converges to 0 since C_φ is completely continuous and $h_n \to h$ weakly. Hence

$$< C_\varphi h_n, h_n^* > \to ||h||_p^p < C_\varphi \frac{h}{||h||_p}, \frac{h^*}{||h^*||_q} >$$

as $n \to \infty$. Note that

$$< C_\varphi \frac{h}{||h||_p}, \frac{h^*}{||h^*||_q} > \in W(C_\varphi)$$

and from which we can conclude that $\overline{W}(C_\varphi) \subseteq \bigcup_{0 \le \alpha \le 1} \alpha W(C_\varphi)$. Now we show that $0 \in \overline{W}(C_\varphi)$. To see this let $\lambda \in U$ and note that $||^* e_\lambda||_p^p = ||e_\lambda||_q^q = \sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q}$. We have

$$\begin{aligned} < C_\varphi(\frac{^* e_\lambda}{||^* e_\lambda||_p}), \frac{e_\lambda}{||e_\lambda||_q} > &= \frac{1}{||^* e_\lambda||_p \, ||e_\lambda||_q} <^* e_\lambda, C_\varphi^* e_\lambda > \\ &= \frac{1}{||e_\lambda||_q^q} <^* e_\lambda, e_{\varphi(\lambda)} > \\ &= \frac{1}{||e_\lambda||_q^q} \, ^* e_\lambda(\varphi(\lambda)). \end{aligned}$$

Letting $\lambda \to \xi_0$, we get $0 \in \overline{W}(C_\varphi)$. Now if $W(C_\varphi)$ is closed, then $0 \in W(C_\varphi)$. Conversely, let $0 \in W(C_\varphi)$ and $0 \neq \alpha \in \overline{W}(C_\varphi)$. Then $\alpha \in cW(C_\varphi)$ for some $c \in (0, 1]$. Since $W(C_\varphi)$ is convex and $0 \in W(C_\varphi)$, thus $\alpha \in W(C_\varphi)$ and so $\overline{W}(C_\varphi) = W(C_\varphi)$. Hence $W(C_\varphi)$ is closed and this completes the proof. □

Corollary 2.5. Under the conditions of Theorem 2.4, $\overline{W}(C_\varphi) = W(C_\varphi) \cup \{0\}$.

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