ON THE COMPOSITION OPERATORS
ON BANACH WEIGHTED HARDY SPACES

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Abstract: In this paper we consider composition operators on weighted Hardy spaces and the aim of the paper is to investigate the numerical range of composition operators.

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1. Introduction

Let \( \{\beta(n)\}_n \) be a sequence of positive numbers with \( \beta(0) = 1 \) and let \( 1 < p < \infty \). Let \( f = \{\hat{f}(n)\}_n \) be such that

\[
\|f\|_p = \|f\|_{H^p(\beta)} = \left( \sum_{n=0}^{+\infty} |\hat{f}(n)|^p \beta(n)^p \right)^{1/p} < \infty.
\]

The notation \( f(z) = \sum_{n=0}^{+\infty} \hat{f}(n)z^n \) shall be used whether or not the series converges for any value of \( z \). The space of such formal power series is called the weighted Hardy space, which is denoted by \( H^p(\beta) \). In the case \( p = 2 \), the classical Hardy
space, Bergman space and the Dirichlet space are weighted Hardy spaces with \( \beta(n) = 1, \beta(n) = (n + 1)^{-\frac{1}{2}} \) and \( \beta(n) = (n + 1)^{\frac{1}{2}} \), respectively. The spaces \( H^p(\beta) \) are reflexive Banach spaces with the norm \( \| \cdot \|_p \) and the dual of \( H^p(\beta) \) is \( H^q(\beta^{\frac{p}{q}}) \) where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \beta^{\frac{p}{q}} = \{ \beta(n)^{\frac{p}{q}} \}_n \).

Recall that for \( g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n \) in \( H^q(\beta^{p/q}) \), note that

\[
\| g \|_q^q = \| g \|_{H^q(\beta^{p/q})}^q = \sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta(n)^p < \infty.
\]

If \( \lim \frac{\beta(n+1)}{\beta(n)} = 1 \) or \( \lim inf \beta(n)^{\frac{1}{n}} = 1 \), then \( H^p(\beta) \) consists of functions analytic on the open unit disk \( U \).

A complex number \( \lambda \) is said to be a bounded point evaluation on \( H^p(\beta) \) if the functional of point evaluation at \( \lambda, e_{\lambda} \), is bounded. A complex number \( \lambda \) is a bounded point evaluation on \( H^p(\beta) \) if and only if \( \left\{ \frac{\lambda^n}{\beta(n)} \right\}_n \in l^q \).

Let \( \varphi \) be an analytic self map of \( U \). A composition operator \( C_{\varphi} \) maps an analytic function \( f \in H^p(\beta) \) into \( (C_{\varphi}f)(z) = f(\varphi(z)) \). The function \( \varphi \) is called the composition map. Some sources on formal power series and composition operators include [1–7].

### 2. Main Result

This work represents the necessary and sufficient conditions for the closedness of the numerical range of a compact composition operator acting on weighted Hardy spaces \( H^p(\beta) \).

In the following we define some definitions that will be used in the main theorem.

**Definition 2.1.** Let \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^p(\beta) \) and define \( f^*(z) = \sum_{n=0}^{\infty} |\hat{f}(n)|^{(p-q)/q} \hat{f}(n) z^n \).

Note that \( \| f^* \|_q^q = \| f^* \|_{H^q(\beta^{p/q})}^q = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p = \| f \|_p^p \) and obviously one can see that \( < f, f^* > = \| f \|_p^p \). Also
Definition 2.2. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n$ belongs to $H^q(\beta^{p/q})$ and define $^*g(z) = \sum_{n=0}^{\infty} |\hat{g}(n)|^{(q-p)/p} \hat{g}(n)z^n$.

Notice that $\|^*g\|_p^p = \sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta(n)^p = \|g\|_q^q < \infty$ and so $^*g \in H^p(\beta)$. Obviously, one can see that $^*(f^*) = f$ for all $f$ in $H^p(\beta)$ and $(^*g)^* = g$ for all $g$ in $(H^p(\beta))^*$. Also, clearly $< g, ^*g > = \|g\|_q^q$.

Definition 2.3. If $T$ is a bounded linear operator on $H^p(\beta)$, the numerical range of $T$ is denoted by $W(T)$ that is defined by

$$W(T) = \text{co}\{< Tf, f^* > : f \in H^p(\beta) \text{ and } \|f\|_p = 1\}.$$ 

Note that clearly $W(T) = \{< T(*g), g > : g \in (H^p(\beta))^* \text{ and } \|g\|_q = 1\}$.

Theorem 2.4. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\liminf \beta(n)^{\frac{1}{q}} = 1$, $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q} = \infty$, and $C_\varphi$ be compact on $H^p(\beta)$. Suppose that there exists $\xi_0$ in $\partial U$ such that the limit of $^*e_\lambda(\varphi(\lambda))$ exists and is finite as $\lambda \to \xi_0$. Then $0 \in W(C_\varphi)$ if and only if $W(C_\varphi)$ is closed.

Proof. First note that since $\liminf \beta(n)^{\frac{1}{q}} = 1$, thus for each $\lambda$ in the open unit disk, the functional of evaluation at $\lambda, e_\lambda$, is a bounded linear functional and we have

$$e_\lambda(z) = \sum_{n=0}^{\infty} \frac{\lambda^n}{\beta(n)^p} z^n,$$

and

$$\|e_\lambda\|^q = \sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q}.$$ 

Now suppose that $\{h_n\}$ is a sequence of unit vectors in $H^p(\beta)$. By weakly compactness of ball $H^p(\beta)$, a subsequence of $\{h_n\}$ is weakly convergent. For simplicity we suppose that $\{h_n\}$ converges weakly to a vector $h$ in $H^p(\beta)$. Then we have

$$| < C_\varphi h_n, h^*_n > - < C_\varphi h, h^*_n > | \leq | < C_\varphi h_n, h^*_n > - < C_\varphi h, h^*_n > |$$

$$+ | < C_\varphi h, h^*_n > - < C_\varphi h, h^*_n > |$$

$$= | < C_\varphi (h_n - h), h^*_n > |$$

$$+ | < C_\varphi h, (h^*_n - h^*_n) > |$$

$$\leq \|C_\varphi (h_n - h)\| \|h^*_n\|$$
\[ + | < C_\varphi h, (h_n^* - h^*) > |. \]

which converges to 0 since \( C_\varphi \) is completely continuous and \( h_n \to h \) weakly. Hence
\[
< C_\varphi h_n, h_n^* > \to ||h||_p^p < C_\varphi \frac{h}{||h||_p}, \frac{h^*}{||h^*||_q} >
\]
as \( n \to \infty \). Note that
\[
< C_\varphi \frac{h}{||h||_p}, \frac{h^*}{||h^*||_q} > \in W(C_\varphi)
\]
and from which we can conclude that \( \overline{W}(C_\varphi) \subseteq \bigcup_{0 \leq \alpha \leq 1} \alpha W(C_\varphi) \). Now we show that \( 0 \in \overline{W}(C_\varphi) \). To see this let \( \lambda \in U \) and note that \( ||e_\lambda||_p^p = ||e_\lambda||_q^q = \sum_{n=0}^{\infty} \frac{|\lambda|^n q}{\beta(n)^q} \). We have
\[
< C_\varphi (\frac{*e_\lambda}{||*e_\lambda||_p}), \frac{e_\lambda}{||e_\lambda||_q} > = \frac{1}{||*e_\lambda||_p ||e_\lambda||_q} < * e_\lambda, C_\varphi^* e_\lambda >
\]
\[
= \frac{1}{||e_\lambda||_q^q} < * e_\lambda, e_\varphi(\lambda) >
\]
\[
= \frac{1}{||e_\lambda||_q^q} * e_\lambda(\varphi(\lambda)).
\]
Letting \( \lambda \to \xi_0 \), we get \( 0 \in \overline{W}(C_\varphi) \). Now if \( W(C_\varphi) \) is closed, then \( 0 \in W(C_\varphi) \). Conversely, let \( 0 \in W(C_\varphi) \) and \( 0 \neq \alpha \in \overline{W}(C_\varphi) \). Then \( \alpha \in cW(C_\varphi) \) for some \( c \in (0, 1] \). Since \( W(C_\varphi) \) is convex and \( 0 \in W(C_\varphi) \), thus \( \alpha \in W(C_\varphi) \) and so \( \overline{W}(C_\varphi) = W(C_\varphi) \). Hence \( W(C_\varphi) \) is closed and this completes the proof. \( \Box \)

**Corollary 2.5.** Under the conditions of Theorem 2.4, \( \overline{W}(C_\varphi) = W(C_\varphi) \cup \{0\} \).

**References**


