**ON GENERALIZED HERMITE-HADAMARD TYPE
INEQUALITIES FOR GENERALIZED
HARMONIC INVEX FUNCTIONS**

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1. Introduction

Let X be a topological vector space. Let $K \subset X \setminus \{0\}$ be a set satisfying the following conditions. For $x, y \in K$, let $I[y, x]$ be a path joining y and x contained in K and the map $\gamma_{xy} : [0, 1] \rightarrow I[y, x]$ be continuous. Let $\eta : K \times K \rightarrow X$ be any vector valued mapping. The set K has the harmonic invex combination (HIC) property with respect to the direction η if the followings are satisfied:

(P1) $y + t\eta(x, y) \in K$ for all $t \in [0, 1]$, $x, y \in K$,

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$$(P2) \quad y + t\eta(x, y) = \begin{cases} y, & \text{if } t = 0; \\ x, & \text{if } t = 1 \end{cases} \quad \text{iff } y + t\eta(x, y) = \frac{x+y}{2} \text{ for } t = \frac{1}{2},$$

(P3) for any $z \in I[y, x] \subset K$, we have

$$z = y + t\eta(x, y) = x + (1 - t)\eta(x, y),$$

$$(P4) \quad \frac{xy}{y+t\eta(x, y)} \in I[y, x] \text{ for all } x, y \in K.$$

Let $X = \mathbb{R}$ and for any $\eta : K \times K \rightarrow X$, $h : K \times K \times [0, 1] \rightarrow \mathbb{R}_+$ $K = K_\eta = [a, b] \subset X \setminus \{0\}$ be a set with HIC properties P_1 to P_4 in the direction η . Then the inequality

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab\bar{h}(x, y)}{\eta(a, b)} \int_a^b \frac{f(x)}{x^2} dx \leq [\bar{h}(x, y)]^2 \frac{f(a) + f(b)}{2} \quad (1)$$

hold if the mapping $f : K \rightarrow \mathbb{R}$ is a harmonically invex function with respect to η associated with h on K_η defined in equation (4).

Remark 1. The inequality (1) coincides with the inequality (2) if $\eta(a, b) = b - a$ and $\bar{h}(x, y) = 1$.

2. Preliminaries

Hermite and Hadamard have studied an integral inequality associated with a convex function as follows: Let $K \subset \mathbb{R}$ be a convex set and $f : K \rightarrow \mathbb{R}$ be a convex function. Then for any interval $[a, b] \subset K$, the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

hold. This integral inequality is known as Hermite and Hadamard inequality. In 2014, I. Insan [3] has introduced the concept of harmonically convex set and harmonically convex functions and has studied the Hermite-Hadamard type inequalities for harmonically convex functions as an extension work of Hermite-Hadamard inequalities. We recall the concepts of harmonic sets and harmonic convex function introduced by Insan [3].

Definition 2. Let $K \subset \mathbb{R} \setminus \{0\}$ be any set and $f : K \rightarrow \mathbb{R}$ be any map.

1. K is said to be harmonic convex set if

$$\frac{xy}{tx + (1-t)y} \in K$$

for all $x, y \in K$ and $t \in [0, 1]$,

2. f is said to be harmonically convex function on the harmonically convex set K if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in K$.

We recall the Hermite-Hadamard type inequalities for harmonically convex functions studied by Insan [3]. Let $L[a, b]$ be the set of integrable functions in the interval $[a, b]$.

Theorem 3. Let $f : K \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a harmonically convex function and $a, b \in K$ with $a < b$. If $f \in L[a, b]$, then the following inequalities hold

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}. \quad (2)$$

The above inequalities are sharp.

Theorem 4. Let $f : K \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on $\text{Int}(K)$, the interior of K and $a, b \in K$ with $a < b$. If $f \in L[a, b]$, then the following inequalities hold

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \\ &= \frac{ab(b-a)}{2} \int_0^1 \frac{1-2t}{(tb + (1-t)a)^2} f\left(\frac{ab}{tb + (1-t)a}\right) dt. \end{aligned} \quad (3)$$

3. Main Results

In this section, we have extended the results (2) and (3) under certain conditions. For our need, we make the definition of invariant harmonically convex function in the IHC set K .

Definition 5. Let $K \subset \mathbb{R} \setminus \{0\}$ be any set and $f : K \rightarrow \mathbb{R}$ be any map.

1. K is said to be harmonic invex (HI) set given in the direction $\eta \in \mathbb{R} \setminus 0$ if K has the HIC properties P_1 to P_4 .
2. f is said to be harmonically invex (HI) function associated with h on the HI set K if there exists a vector function $\eta : K \times K \rightarrow X$ and $h : K \times K \times [0, 1] \rightarrow \mathbb{R}_+ = [0, \infty]$ satisfying

$$h(x, y, t) = t\bar{h}(x, y)$$

such that

$$f\left(\frac{xy}{y+t\eta(x,y)}\right) \leq h(x,y,t)f(y) + (1-h(x,y,t))f(x) \quad (4)$$

for all $x, y \in K$, $t \in [0, 1]$.

For existence of the result (1), we show the following result.

Lemma 6. For $\eta : K \times K \rightarrow X$, let $K = K_\eta \subset \mathbb{R} \setminus \{0\}$ be a HI set. Let $f : K \rightarrow \mathbb{R}$ be a HI function on $\text{Int}(K)$ with respect to the direction $\eta \in \mathbb{R} \setminus \{0\}$ associated with h , then for all $x, y \in K$, the following inequality hold

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{\bar{h}(x,y)}{2}(f(x) + f(y)).$$

Proof. By invexity properties of f , we have

$$f\left(\frac{xy}{y+t\eta(x,y)}\right) \leq h(x,y,t)f(y) + (1-h(x,y,t))f(x)$$

for all $x, y \in K$, $t \in [0, 1]$. At $t = \frac{1}{2}$, we have

$$f\left(\frac{2xy}{2y+\eta(x,y)}\right) \leq \bar{h}(x,y)\frac{f(x) + f(y)}{2}$$

for all $x, y \in K$. Since $x = y + \eta(x, y)$, we have

$$f\left(\frac{2xy}{x+y}\right) \leq \bar{h}(x,y)\frac{f(x) + f(y)}{2}$$

for all $x, y \in K$. This completes the proof. $\square$

Theorem 7. Let $\eta : K \times K \rightarrow X$ be a vector valued antisymmetric (i.e., $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$) map and $K = K_\eta \subset \mathbb{R} \setminus \{0\}$ be a HI set. Let $a < b \in K$. Let $f : K \rightarrow \mathbb{R}$ be a HI function on $\text{Int}(K)$ with respect to the direction $v \in \mathbb{R} \setminus \{0\}$ associated with h . Let $f \in L[a, b]$, then the following inequality hold:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab\bar{h}(x,y)}{\eta(a,b)} \int_a^b \frac{f(x)}{x^2} dx \leq [\bar{h}(x,y)]^2 \frac{f(a) + f(b)}{2}.$$

Proof. In view of Lemma 6, we have

$$f\left(\frac{2xy}{x+y}\right) \leq \bar{h}(x, y) \frac{f(x) + f(y)}{2} \quad (5)$$

for all $x, y \in K$, implying

$$f\left(\frac{2ab}{a+b}\right) \leq \bar{h}(a, b) \frac{f(a) + f(b)}{2}$$

for $a, b \in K$ with $a < b$. Assuming $x = \frac{ab}{b+t\eta(a, b)}$ and $y = \frac{ab}{a+t\eta(b, a)}$, we have

$$\frac{2xy}{x+y} = \frac{ab}{a+b+t(\eta(a, b) + \eta(b, a))} = \frac{ab}{a+b}.$$

If $s = \frac{\alpha\beta}{\beta+tz}$ where $t \in [0, 1]$ and $\beta = \alpha + z$. As $t : 0 \rightarrow 1$,

$$s : \alpha \rightarrow \frac{\alpha\beta}{\beta+z} = \frac{\alpha\beta}{\alpha} = \beta.$$

Again

$$t = \frac{1}{z} \left(\frac{\alpha\beta}{s} - \beta \right) \Rightarrow dt = -\frac{\alpha\beta}{zs^2} ds.$$

Thus

$$\int_0^1 f\left(\frac{\alpha\beta}{\beta+tz}\right) dt = \frac{\alpha\beta}{z} \int_\alpha^\beta \frac{f(s)}{s^2} ds. \quad (6)$$

Replacing x by $\frac{ab}{b+t\eta(a, b)}$ and y by $\frac{ab}{a+t\eta(b, a)}$ in (5), we obtain

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\bar{h}(x, y)}{2} \left(f\left[\frac{ab}{b+t\eta(a, b)}\right] + f\left(\frac{ab}{a+t\eta(b, a)}\right) \right)$$

for all $t \in [0, 1]$, implying

$$\begin{aligned} & \int_0^1 f\left(\frac{2ab}{a+b}\right) dt \\ & \leq \frac{\bar{h}(x, y)}{2} \left[\int_0^1 f\left(\frac{ab}{b+t\eta(a, b)}\right) dt + \int_0^1 f\left(\frac{ab}{a+t\eta(b, a)}\right) dt \right]. \end{aligned}$$

Using (6), we get

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\bar{h}(x, y)}{2} \left[\frac{ab}{\eta(a, b)} \int_a^b \frac{f(x)}{x^2} dx - \frac{ab}{\eta(b, a)} \int_a^b \frac{f(x)}{x^2} dx \right]$$

$$= \frac{ab\bar{h}(x, y)}{\eta(a, b)} \int_a^b \frac{f(x)}{x^2} dx. \quad (7)$$

For fixed $a, b \in K$, f satisfies

$$f\left(\frac{ab}{b + t\eta(a, b)}\right) \leq h(a, b, t)f(b) + (1 - h(a, b, t))f(a),$$

for all $t \in [0, 1]$. Applying (6), we have

$$\begin{aligned} \int_0^1 f\left(\frac{ab}{b + t\eta(a, b)}\right) dt &\leq \int_0^1 h(x, y, t)f(b) dt \\ &\quad + \int_0^1 (1 - h(x, y, t))f(a) dt \\ &= f(a) \int_0^1 h(x, y, t) dt \\ &\quad + f(b) \int_0^1 (1 - h(x, y, t)) dt \\ &= \bar{h}(x, y) \frac{f(a) + f(b)}{2}, \end{aligned}$$

i.e.,

$$\frac{ab}{\eta(a, b)} \int_a^b \frac{f(x)}{x^2} dx \leq \bar{h}(x, y) \frac{f(a) + f(b)}{2}. \quad (8)$$

Hence (7) and (8), we get

$$f\left(\frac{2ab}{a + b}\right) \leq \frac{ab\bar{h}(x, y)}{\eta(a, b)} \int_a^b \frac{f(x)}{x^2} dx \leq [\bar{h}(x, y)]^2 \frac{f(a) + f(b)}{2}.$$

This completes the proof. $\square$

Theorem 8. For any set $K \subset \mathbb{R} \setminus \{0\}$, let $\eta : K \times K \rightarrow \mathbb{R}$ be a vector valued function. Let $K_\eta \subset \mathbb{R} \setminus \{0\}$ be a HI set associated with h defined by

$$y + h(x, y, t)\eta(x, y) = \begin{cases} y, & \text{if } t = 0; \\ x, & \text{if } t = 1 \end{cases}$$

for all $x, y \in K$. Let η be antisymmetric on K . Let $f : K \rightarrow \mathbb{R}$ be a HI function associated with h on $\text{Int}(K)$ with respect to the direction $\eta \in \mathbb{R} \setminus \{0\}$. Let $f \in L[a, b]$, then the following inequality hold:

$$\bar{h}(a, b) \frac{f(a) + f(b)}{2} - \frac{ab}{\eta(b, a)} \int_a^b \frac{f(x)}{x^2} dx$$

$$= \frac{ab\bar{h}(a,b)\eta(b,a)}{2} \int_0^1 \frac{1-2t}{(a+t\eta(b,a))^2} f\left(\frac{ab}{a+t\eta(b,a)}\right) dt.$$

Proof. Assuming $x = \frac{ab}{a+h(a,b,t)\eta(b,a)} = \frac{ab}{a+t\bar{h}(a,b)\eta(b,a)}$, we get

$$t = \frac{ab - ax}{\bar{h}(a,b)\eta(b,a)x} \Rightarrow 1 - 2t = \frac{\bar{h}(a,b)\eta(b,a) + 2a}{\bar{h}(a,b)\eta(b,a)} - \frac{2ab}{\bar{h}(a,b)\eta(b,a)x}$$

and

$$dt = -\frac{(a+t\bar{h}(a,b)\eta(b,a))^2}{ab\bar{h}(a,b)\eta(b,a)} dx.$$

Since $a + h(a,b,1)\eta(b,a) = a + \bar{h}(a,b)\eta(b,a) = b$ and $t : 0 \rightarrow 1$, we have $x : b \rightarrow \frac{ab}{a+\bar{h}(a,b)\eta(b,a)} = a$. Thus

$$\begin{aligned} & \frac{ab\eta(b,a)}{2} \int_0^1 \frac{1-2t}{(a+h(a,b,t)\eta(b,a))^2} f\left(\frac{ab}{a+h(a,b,t)\eta(b,a)}\right) dt \\ &= \frac{1}{2} \int_a^b \left(\frac{\bar{h}(a,b)\eta(b,a) + 2a}{\bar{h}(a,b)\eta(b,a)} - \frac{2ab}{\bar{h}(a,b)\eta(b,a)x} \right) f(x) dx \\ &= \frac{b-a}{2\bar{h}(a,b)\eta(b,a)} [f(a) + f(b)] - \frac{ab}{\bar{h}(a,b)\eta(b,a)} \int_a^b \frac{f(x)}{x^2} dx \\ &= \frac{f(a) + f(b)}{2} - \frac{ab}{\bar{h}(a,b)\eta(b,a)} \int_a^b \frac{f(x)}{x^2} dx, \end{aligned}$$

since $b - a = \bar{h}(a,b)\eta(b,a)$. Thus

$$\begin{aligned} & \frac{ab\bar{h}(a,b)\eta(b,a)}{2} \int_0^1 \frac{1-2t}{(a+h(a,b,t)\eta(b,a))^2} f\left(\frac{ab}{a+h(a,b,t)\eta(b,a)}\right) dt \\ &= \bar{h}(a,b) \frac{f(a) + f(b)}{2} - \frac{ab}{\eta(b,a)} \int_a^b \frac{f(x)}{x^2} dx. \end{aligned}$$

This completes the proof. $\square$

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