ON THE WEIGHTED COMPOSITION OPERATORS
ON HILBERT SPACES OF FORMAL POWER SERIES

Bahmann Yousefi\textsuperscript{1,}\textsuperscript{§}, Fatemeh Zangeneh\textsuperscript{2}

\textsuperscript{1,2}Department of Mathematics
Payame Noor University
P.O. Box: 19395-3697, Tehran, IRAN

Abstract: In this paper we consider composition operators on the weighted Hardy spaces and we investigate that when the numerical range of a compact composition operator is closed.

AMS Subject Classification: 47B37, 47A16
Key Words: Hilbert spaces of formal power series associated with a sequence $\beta$, weighted composition operator, numerical range, completely continuous operator

1. Introduction

Let $\{\beta(n)\}_n$ be a sequence of positive numbers with $\beta(0) = 1$. Let $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ be such that

$$
\|f\|^2 = \|f\|^2_{H^2(\beta)} = \sum_{n=0}^{+\infty} |\hat{f}(n)|^2 \beta(n)^2 < \infty.
$$

The notation $f(z) = \sum_{n=0}^{+\infty} \hat{f}(n)z^n$ shall be used whether or not the series converges for any value of $z$. The space of such formal power series is called the weighted Hardy space, which is denoted by $H^2(\beta)$. The classical Hardy space, Bergman space and the Dirichlet space are weighted Hardy spaces with $\beta(n) = 1$, $\beta(n) = (n + 1)^{-\frac{1}{2}}$ and $\beta(n) = (n + 1)^{\frac{1}{2}}$, respectively. The space $H^2(\beta)$ becomes a
Hilbert space with inner product

$$\langle f, g \rangle = \sum_{n=0}^{+\infty} a_n \overline{b_n} \beta(n)^2$$

where $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{+\infty} b_n z^n$ are the elements of $H^2(\beta)$. If

$$\lim \frac{\beta(n+1)}{\beta(n)} = 1$$

or

$$\lim \inf \frac{1}{\beta(n)} = 1,$$

then $H^2(\beta)$ consists of functions analytic on the open unit disk $U$.

A complex number $\lambda$ is said to be a bounded point evaluation on $H^2(\beta)$ if the functional of point evaluation at $\lambda, e_\lambda$, is bounded. A complex number $\lambda$ is a bounded point evaluation on $H^2(\beta)$ if and only if

$$\left\{ \frac{\lambda^n}{\beta(n)} \right\}_{n \in \mathbb{N}} \in l^2.$$

We denote the set of multipliers

$$\{ \varphi \in H^2(\beta) : \varphi H^2(\beta) \subseteq H^2(\beta) \}$$

by $H^2_\infty(\beta)$ and the operator of multiplication by $\varphi$ on $H^2(\beta)$ by $M_\varphi$ with

$$\| \varphi \|_\infty = \| M_\varphi \|.$$

Let $\varphi$ be an analytic self map of $U$ and $\psi$ be a multiplier of $H^2(\beta)$. A weighted composition operator $C_{\psi, \varphi}$ maps an analytic function $f \in H^2(\beta)$ into

$$(C_{\psi, \varphi} f)(z) = \psi(z) f(\varphi(z)).$$

The function $\varphi$ is called the composition map and the function $\psi$ is called the multiplier map. We will use the notations $H(U)$ and $C(\overline{U})$ to denote the set of analytic functions on $U$ and the set of continuous functions on $\overline{U}$, the closure of $U$. Some sources on formal power series and composition operators include [1–7].

2. Main Result

This work represents the necessary and sufficient conditions for the closedness of the numerical range of a compact composition operator acting on Banach spaces $H^2(\beta)$.

Definition 2.1. The numerical range of $C_{\psi, \varphi}$ acting on $H^2(\beta)$, is denoted by $W(C_{\psi, \varphi})$ that is defined by

$$W(C_{\psi, \varphi}) = \{ \langle C_{\psi, \varphi} f, f \rangle : f \in H^p(\beta) \text{ and } \| f \|_p = 1 \}. $$
In the following we suppose that lim inf $\beta(n)^{1/n} = 1$ and this implies that $H^2(\beta)$ consists of functions analytic on the open unit disk $U$.

**Theorem 2.2.** Let $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} = \infty$, $\varphi : U \to U$ be analytic and $\psi \in C(\overline{U}) \cap H^2(\beta)$. Also, suppose that $C_{\psi, \varphi}$ is bounded on $H^2(\beta)$ and there exists $\xi_0 \in \partial U$ such that $\varphi$ is defined on $\xi_0$ and $\sum_{n \geq 0} \frac{\xi_0^n \varphi(\xi_0)^n}{\beta(n)^2}$ is finite. Then $0 \in \overline{W(C_{\psi, \varphi})}$.

**Proof.** First note that $\xi_0$ can not be a fixed point of $\varphi$. Let $\lambda \in U$, since $e_\lambda$ is bounded on the Hilbert space $H^2(\beta)$, there exists a function $k_\lambda \in H^2(\beta)$ such that $e_\lambda(f) = \langle f, k_\lambda \rangle$ for all $f \in H^2(\beta)$, and $\|e_\lambda\| = \|k_\lambda\|$. Put

$$K_\lambda = \frac{k_\lambda}{\|k_\lambda\|}, \quad E_\lambda = \frac{e_\lambda}{\|e_\lambda\|},$$

where $\lambda \in U$. Let

$$\alpha_\lambda = \langle C_{\psi, \varphi} K_\lambda, E_\lambda \rangle,$$

so $\alpha_\lambda \in W(C_{\psi, \varphi})$. Now we have

$$\begin{align*}
\alpha_\lambda &= \langle C_{\psi, \varphi} K_\lambda, E_\lambda \rangle \\
&= \langle K_\lambda, C_{\psi, \varphi}^* E_\lambda \rangle \\
&= \frac{\psi(\lambda)}{\|k_\lambda\|^2} \langle k_\lambda, \overline{\psi(\lambda)} e_{\varphi(\lambda)} \rangle \\
&= \frac{\|k_\lambda\|^2}{\|e_\lambda\|^2} \sum_{n \geq 0} \frac{\varphi(\lambda)^n}{\beta(n)^2} \lambda^n.
\end{align*}$$

Let $\lambda \to \xi_0$ and note that $\psi(\lambda) \to \psi(\xi_0)$, so we can see that $\alpha_\lambda \to 0$. Hence $0 \in \overline{W(C_{\psi, \varphi})}$ and the proof is complete.

From the proof of Theorem 2.2, we have the following corollary.

**Corollary 2.3.** Suppose that $C_{\psi, \varphi}$ is bounded on $H^2(\beta)$. If $\varphi$ has a fixed point $\xi$ in $U$, then $\psi(\xi)$ belongs to $W(C_{\psi, \varphi})$.

**Proof.** Put

$$K_\xi = \frac{k_\xi}{\|k_\xi\|}, \quad E_\xi = \frac{e_\xi}{\|e_\xi\|}.$$

Then

$$\langle C_{\psi, \varphi} K_\xi, E_\xi \rangle = \langle K_\xi, C_{\psi, \varphi}^* E_\xi \rangle$$
\[
\begin{align*}
&= \frac{1}{||k_\xi||^2} < k_\xi, \psi(\xi)e_\varphi(\xi) > \\
&= ||k_\xi||^{-2} \psi(\xi) < k_\xi, e_\xi > \\
&= ||k_\xi||^{-2} \psi(\xi)k_\xi(\xi) \\
&= \psi(\xi)
\end{align*}
\]

belongs to \(W(C_{\psi,\varphi})\) and so the proof is complete. \(\square\)

Recall that we say an operator \(T\) is completely continuous on a Banach space \(X\), if weakly convergence of \(x_i \to x\) implies the convergence \(Tx_i \to Tx\) in norm.

**Proposition 2.4.** Let \(C_{\psi,\varphi}\) be completely continuous on \(H^2(\beta)\). Then under the conditions of Theorem 2.2, \(W(C_{\psi,\varphi})\) is closed if and only if \(0 \in W(C_{\psi,\varphi})\).

**Proof.** If \(W(C_{\psi,\varphi})\) is closed, then clearly \(0 \in W(C_{\psi,\varphi})\). Conversely, let \(w \in \overline{W}(C_{\psi,\varphi})\), then there exists a sequence \(\{h_n\}_n\) in ball\(H^2(\beta)\) and \(h \in H^2(\beta)\) such that \(h_n \to h\) weakly and \(\{<C_{\psi,\varphi}h_n,h_n>\}_n\) converges to \(w\). This implies that \(w = <C_{\psi,\varphi}h,h>\) and so

\[
w \in ||h||^2 W(C_{\psi,\varphi})
\]

where \(h \in ballH^2(\beta)\). Suppose that \(h \neq 0\), hence

\[
\frac{w}{||h||^2} \in W(C_{\psi,\varphi}).
\]

But \(W(C_{\psi,\varphi})\) is convex and \(0 \in W(C_{\psi,\varphi})\), hence \(w \in W(C_{\psi,\varphi})\). So \(W(C_{\psi,\varphi})\) is closed whenever \(0 \in W(C_{\psi,\varphi})\). \(\square\)

**References**


