ON A CLASS OF PARA KENMOTSU MANIFOLDS

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Abstract: The object of this paper is to investigate Weyl-pseudosymmetric para-Kenmotsu manifolds and para-Kenmotsu manifolds satisfying the condition $C(X, Y).S = 0$ where $C(X, Y)$ is the Weyl conformal curvature tensor and S is the Ricci tensor of the manifold.

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1. Introduction

Sato [9] defined the notions of an almost para contact Riemannian manifold. After that, Adati and Matsumoto [1] defined and studied p -Sasakian and sp -Sasakian manifolds which are regarded as a special kind of an almost contact

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Riemannian manifolds. Before Sato, Kenmotsu [8] defined a class of almost contact Riemannian manifolds. In 1995, Sinha and Sai Prasad [11] defined a class of almost para contact metric manifolds namely para Kenmotsu (briefly p -Kenmotsu) and special para Kenmotsu (briefly sp -Kenmotsu) manifolds. In a recent paper, the authors Satyanarayana and Sai Prasad [10] studied Weyl semisymmetric p -Kenmotsu manifolds, and they prove that such a manifold is conformally flat and hence is an sp -Kenmotsu manifold.

Let (M_n, g) be an n -dimensional, $n \geq 3$, differentiable manifold of class C^∞ . Let ∇ be its Levi-Civita connection, R be the Riemannian Christoffel curvature tensor and C be the Weyl conformal curvature tensor and is defined by

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z \\ &- \frac{1}{n-2}[g(Y, Z)SX - g(X, Z)SY + S(Y, Z)X - S(X, Z)Y] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (1.1)$$

where S is the Ricci tensor, r is the scalar curvature of M_n [3]. The Ricci operator S and the $(0,2)$ -tensor S^2 are defined by

$$g(SX, Y) = S(X, Y) \quad (1.2)$$

and

$$S^2(X, Y) = S(SX, Y). \quad (1.3)$$

If $C = 0$ and $n \geq 4$, then M_n is conformally flat. If $\nabla C = 0$ then M_n is called conformally symmetric, where ∇ is Levi-Civita connection of the Riemannian metric [4]. It is obvious that a conformally symmetric manifold is Weyl-semisymmetric.

Furthermore we define the tensor $C(X, Y) \cdot S$ on M_n by

$$(C(X, Y) \cdot S)(Z, W) = -S(C(X, Y)Z, W) - S(Z, C(X, Y)W). \quad (1.4)$$

where $X, Y, Z, W \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields on M_n .

For a $(0, k)$ -tensor field T , $k \geq 1$, on (M_n, g) we define the tensors $R \cdot T$ and $Q(g, T)$ by

$$\begin{aligned} (R(X, Y) \cdot T)(X_1, X_2, \dots, X_k) &= -T(R(X, Y)X_1, X_2, \dots, X_k) \\ &- \dots - T(X_1, \dots, X_{k-1}, R(X, Y)X_k), \end{aligned} \quad (1.5)$$

$$\begin{aligned} Q(g, T)(X_1, X_2, \dots, X_k; X, Y) &= -T((X \wedge Y)X_1, X_2, \dots, X_k) \\ &- \dots - T(X_1, \dots, X_{k-1}, (X \wedge Y)X_k) \end{aligned} \quad (1.6)$$

respectively [7], where the endomorphism $(X \wedge Y)$ is defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y. \quad (1.7)$$

If the tensors R , C and $Q(g, C)$ are linearly dependent then the manifold is called Weyl-pseudosymmetric [7]. This is equivalent to

$$R.C = L_C Q(g, C), \quad (1.8)$$

holding on the set $U_C = \{x \in M/C \neq 0 \text{ at } x\}$, where L_C is some function of U_C .

As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. Locally symmetric, semisymmetric and Pseudosymmetric Para-Sasakian manifolds are widely studied by many geometers [2, 5, 6]. In this study, our aim is to obtain the characterisations of the Weyl-pseudosymmetric para-Kenmotsu manifolds which are the extended classes of Weyl-semisymmetric para-Kenmotsu manifolds and para-Kenmotsu manifolds satisfying the condition $C(X, Y).S = 0$ where $C(X, Y)$ is the Weyl conformal curvature tensor and S is the Ricci tensor of the manifold.

2. p -Kenmotsu Manifolds

Let M_n be an n -dimensional differentiable manifold equipped with structure tensors (Φ, ξ, η) where Φ is a tensor of type $(1, 1)$, ξ is a vector field, η is a 1-form such that

$$\eta(\xi) = 1 \quad (2.1)$$

$$\Phi^2(X) = X - \eta(X)\xi; \quad \overline{X} = \Phi X \quad (2.2)$$

Then M_n is called an almost para contact manifold.

Let g be the Riemannian metric such that, for all vector fields X and Y on M ,

$$g(X, \xi) = \eta(X) \quad (2.3)$$

$$\Phi\xi = 0, \quad \eta(\Phi X) = 0, \quad \text{rank } \Phi = n - 1 \quad (2.4)$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (2.5)$$

Then the manifold M_n [9] is said to admit an almost para contact Riemannian structure (Φ, ξ, η, g) .

A manifold of dimension n with Riemannian metric g admitting a tensor field Φ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying (2.1), (2.3) along with

$$(\nabla_X \eta)Y - (\nabla_Y \eta)X = 0 \quad (2.6)$$

$$(\nabla_X \nabla_Y \eta)Z = [-g(X, Z) + \eta(X)\eta(Z)]\eta(Y) + [-g(X, Y) + \eta(X)\eta(Y)]\eta(Z) \quad (2.7)$$

$$\nabla_X \xi = \Phi^2 X = X - \eta(X)\xi \quad (2.8)$$

$$(\nabla_X \Phi)Y = g(\Phi X, Y)\xi - \eta(Y)\Phi X \quad (2.9)$$

is called a para Kenmotsu manifold or briefly p -Kenmotsu manifold [11].

A p -Kenmotsu manifold admitting a 1-form η satisfying

$$(\nabla_X \eta)Y = g(X, Y)\xi - \eta(X)\eta(Y) \quad (2.10)$$

$$g(X, \xi) = \eta(X) \text{ and } (\nabla_X \eta)Y = \varphi(\overline{X}, Y), \text{ where } \varphi \text{ is an associate of } \Phi \quad (2.11)$$

is called a special p -Kenmotsu manifold or briefly sp -Kenmotsu manifold [11].

It is known that [11] in a p -Kenmotsu manifold the following relations hold:

$$S(X, \xi) = -(n-1)\eta(X) \quad (2.12)$$

$$g[R(X, Y)Z, \xi] = \eta[R(X, Y, Z)] = g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \quad (2.13)$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi \quad (2.14)$$

$$R(X, Y, \xi) = \eta(X)Y - \eta(Y)X; \text{ when } X \text{ is orthogonal to } \xi \quad (2.15)$$

where S is the Ricci tensor and R is the Riemannian curvature.

If the Ricci curvature tensor S is of the form

$$S = aI_d + b\eta \otimes \xi, \quad (2.16)$$

where a and b are smooth functions on M_n , then M_n is called as an η -Einstein manifold and if $b = 0$ then it is an Einstein manifold [2].

The above results will be used further in the next section.

3. Main Results

In this section our aim is to find the characterisation of para-Kenmotsu manifolds satisfying the condition $C(X, Y).S = 0$ and Weyl-pseudosymmetric para-Kenmotsu manifolds.

Firstly we give the following proposition.

Proposition 3.1. *Let M_n ($n \geq 4$) be an n -dimensional p -Kenmotsu manifold. If the condition $C(X, Y).S = 0$ holds on M_n then*

$$S^2(X, Y) = \left[\frac{r}{(n-1)} - n + 2 \right] S(X, Y) + [r + n - 1]g(X, Y). \quad (3.1)$$

Proof. Assume that M_n is an n -dimensional, $n \geq 4$, p -Kenmotsu manifold satisfying the condition $C(X, Y).S = 0$.

Then from (1.4) we have

$$S(C(U, X)Y, Z) + S(Y, C(U, X)Z) = 0. \quad (3.2)$$

Putting $U = \xi$ in (3.2) we have

$$S(C(\xi, X)Y, Z) + S(Y, C(\xi, X)Z) = 0. \quad (3.3)$$

By using (1.1), (2.12) and (2.13), (3.3) reduces to

$$\begin{aligned} & \eta(Y)S(X, Z) - g(X, Y)S(\xi, Z) + \eta(Z)S(X, Y) - g(X, Z)S(\xi, Y) \\ & - \frac{1}{(n-2)} \left[S(X, Y)S(\xi, Z) - S(X, Z)S(\xi, Y) + g(X, Y)S^2(\xi, Z) \right. \\ & - \eta(Y)S^2(X, Z) + S(X, Z)S(\xi, Y) - S(X, Y)S(\xi, Z) + g(X, Z)S^2(\xi, Y) \\ & \left. - \eta(Z)S^2(X, Y) \right] + \frac{r}{(n-1)(n-2)} \left[g(X, Y)S(\xi, Z) - \eta(Y)S(X, Z) \right. \\ & \left. + g(X, Z)S(\xi, Y) - \eta(Z)S(X, Y) \right] = 0. \end{aligned} \quad (3.4)$$

Hence by the use of (1.3) and (2.12) we get

$$\begin{aligned} & \eta(Y)S(X, Z) - (1-n)g(X, Y)\eta(Z) + \eta(Z)S(X, Y) - (1-n)g(X, Z)\eta(Y) \\ & - \frac{1}{(n-2)} \left[-\eta(Y)S^2(X, Z) - \eta(Z)S^2(X, Y) + (1-n)^2\eta(Z)g(X, Y) \right. \\ & \left. + (1-n)^2\eta(Y)g(X, Z) \right] + \frac{r}{(n-1)(n-2)} \left[-\eta(Y)S(X, Z) - \eta(Z)S(X, Y) \right] \\ & + (1-n)[\eta(Z)g(X, Y) + \eta(Y)g(X, Z)] = 0. \end{aligned} \quad (3.5)$$

Now by putting $Z = \xi$ in (3.5) and on using (1.3) and (2.12), we obtain

$$\begin{aligned} \frac{1}{(n-2)}S^2(X, Y) &= \left[\frac{r}{(n-1)(n-2)} - 1 \right] S(X, Y) \\ &+ \left[\frac{r}{(n-2)} + \frac{(n-1)^2}{(n-2)} - (n-1) \right] g(X, Y), \end{aligned} \quad (3.6)$$

which is the required eqn (3.1), since $n \geq 4$.

Now, let us consider an η -Einstein p -Kenmotsu manifold. Then we can write the Ricci tensor S as

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \quad (3.7)$$

where X and Y are any vector fields and a and b are smooth functions on M_n .

On contracting (3.7), we have

$$r = n a + b, \quad (3.8)$$

where r is the scalar curvature.

On the other hand, putting $X = Y = \xi$ in (3.7) and on using (2.12), we also have

$$1 - n = a + b. \quad (3.9)$$

Hence it follows from (3.7), (3.8) and (3.9) that the Ricci tensor S of an η -Einstein p -Kenmotsu manifold is given by

$$S(Y, Z) = \left(1 + \frac{r}{n-1} \right) g(Y, Z) + \left(\frac{r}{1-n} - n \right) \eta(Y)\eta(Z) \quad (3.10)$$

Proposition 3.2. *Let M_n ($n \geq 4$) be an n -dimensional η -Einstein p -Kenmotsu manifold. Then the condition $C(X, Y).S = 0$ holds on M_n .*

Proof. Assume that M_n is an η -Einstein p -Kenmotsu manifold. Since the Weyl tensor C has all symmetries of a curvature tensor, then from (1.4) it is easy to show that

$$(C(U, X).S)(Y, Z) = \left(\frac{r}{1-n} - n \right) [\eta(C(U, X)Y)\eta(Z) + \eta(C(U, X)Z)\eta(Y)], \quad (3.11)$$

for all vector fields U, X, Y, Z on M_n .

On using (1.1), (2.12), (2.13) and (3.10), (3.11) reduces to $(C(U, X).S)(Y, Z) = 0$, which proves the proposition.

Theorem 3.1. *Let M_n be an n -dimensional ($n \geq 4$) p -Kenmotsu manifold. If M_n is Weyl-pseudosymmetric then M_n is either conformally flat, in which case M_n is a SP-Kenmotsu manifold, or $L_C = -1$ holds on M_n .*

Proof. Assume that M_n is an n -dimensional ($n \geq 4$) Weyl-pseudosymmetric p -Kenmotsu manifold and $X, Y, U, V, W \in \chi(M_n)$. Then from (1.8) we have

$$(R(X, Y).C)(U, V, W) = L_C Q(g, C)(U, V, W; X, Y). \quad (3.12)$$

On using (1.5) and (1.6), we write (3.12) as:

$$\begin{aligned} R(X, Y) C(U, V)W - C(R(X, Y)U, V)W - C(U, R(X, Y)V)W \\ - C(U, V)R(X, Y)W = L_C[(X \wedge Y)C(U, V)W - C((X \wedge Y)U, V)W \\ - C(U, (X \wedge Y)V)W - C(U, V)(X \wedge Y)W]. \end{aligned} \quad (3.13)$$

Now by replacing X with ξ in (3.13) we have

$$\begin{aligned} R(\xi, Y) C(U, V)W - C(R(\xi, Y)U, V)W - C(U, R(\xi, Y)V)W \\ - C(U, V)R(\xi, Y)W = L_C[(\xi \wedge Y)C(U, V)W - C((\xi \wedge Y)U, V)W \\ - C(U, (\xi \wedge Y)V)W - C(U, V)(\xi \wedge Y)W]. \end{aligned} \quad (3.14)$$

So on using (2.13), (1.7) and by taking the inner product of (3.14) with ξ , we get

$$\begin{aligned} [1 + L_C][-\eta(Y)\eta(C(U, V)W) + C(U, V, W, Y) + \eta(U)\eta(C(Y, V)W) \\ - g(Y, U)\eta(C(\xi, V)W) + \eta(V)\eta(C(U, Y)W) - g(Y, V)\eta(C(U, \xi)W) \\ + \eta(W)\eta(C(U, V)Y) - g(Y, W)\eta(C(U, V)\xi)] = 0. \end{aligned} \quad (3.15)$$

Putting $Y = U$ in (3.15) and on contraction with respect to U , (3.15) gives us

$$(1 + L_C)[\eta(C(\xi, V)W)] = 0. \quad (3.16)$$

If $L_C = 0$ then M_n is Weyl-semisymmetric and so the equation reduces to

$$\eta(C(\xi, V)W) = 0 \quad (3.17)$$

which gives (3.10), shows that M_n is an η -Einstein manifold.

Now on using (3.17) and (3.10), (3.15) takes the form $C(U, V, W, Y) = 0$, means that the para-Kenmotsu manifold M_n is conformally flat and hence by [10], M_n is a SP-Kenmotsu manifold.

If $L_C \neq 0$ and $\eta(C(\xi, V)W) \neq 0$ then $(1 + L_C) = 0$, which gives $L_C = -1$. This completes the proof of the theorem.

So we have the following corollary.

Corollary 3.1. *Every n -dimensional, $n \geq 4$, para-Kenmotsu manifold is a Weyl-pseudosymmetric manifold of the form $R \cdot C = -Q(g, C)$.*

Remark. The above findings are quite in similar to the results obtained for Weyl-pseudosymmetric para-Sasakian manifolds [5].

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