ON UNIQUENESS GENERALIZED PROBLEM OF TRICOMI FOR THE CHAPLYGIN EQUATION

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Abstract: In this paper we consider the generalized Tricomi problem in a mixed domain for the Chaplygin equation. Frankl was first which showed that the problem of the expiry of a supersonic jet from a vessel with plane walls in the hodograph plane is reduced to the Tricomi problem for Chaplygin equation. By method of auxiliary functions we received a new theorem of uniqueness of the solution of this problem without any restrictions, except the smoothness on the elliptical part of the border domain.

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Key Words: Chaplygin equation, Tricomi problem, equation of mixed type

1. Introduction

Consider the equation

$$Lz = K(y)z_{xx} + z_{yy} = 0, \quad (1)$$

in an open domain D , where $yK(y) > 0$ for $y \neq 0$. The domain D is bounded by curves: a piecewise smooth curve Γ in the half-plane $y > 0$, which intersects the line $y = 0$ at the points $A(0, 0)$ and $B(l, 0)$, $l > 0$; in $y < 0$, D is bounded by two curves: monotonic curve $\gamma_1 : y = \alpha(x)$, $(\alpha'(x) < 0)$ issuing from A and meeting at the point C with characteristic $\gamma_2 : \eta = x - \int_0^y \sqrt{-K(t)} dt = l$, issuing from B .

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Let $K(y) \in C[y_c, 0] \cap C^2[y_c, 0)$, y_c – the ordinate of point C and let D_+ be subdomain of D with $y > 0$ and D_- be subdomain of D with $y < 0$.

In this paper using a variation of the energy-integral method (abc method) we obtain sufficient conditions for the uniqueness of solution of generalized problem of Tricomi for the Chaplygin equation. It arises in the study of transonic flow, and the proof of uniqueness in this case leads to a proof that continuous transonic flows past smooth profiles do not exist in general [1].

The generalized problem of Tricomi. Find function $z(x, y)$ satisfying the following conditions:

$$\begin{aligned} Lz(x, y) &\equiv 0, (x, y) \in D_- \cup D_+; \\ z(x, y) &\in C(\overline{D}) \cap C^1(D) \cap C^2(D_- \cup D_+); \\ z|_{\Gamma} &= \varphi(s), \quad 0 \leq s \leq L; \\ z|_{\gamma_1} &= \psi(x), \quad 0 \leq x \leq x_C, \end{aligned}$$

where φ and ψ are given functions, L – length of curve Γ .

Definition 1. We call a function regular solution of (1) if the following hold

- i) $z(x, y) \in C(\overline{D}) \cap C^1(D) \cap C^2(D_- \cup D_+)$;
- ii) we can to applicate Greens theorem to the integrals

$$\iint_D zLz dx dy \quad \iint_D zLz_x dx dy \quad \iint_D z_y Lz dx dy;$$

iii) the boundary integrals which arise exist in the sense that: the limits taken over corresponding interior curves exist as these interior curves approach the boundary.

The question uniqueness of solution of generalized problem of Tricomi for equation of mixed type has been dealt with in the literature by many authors. For an extensive bibliography we refer the reader to [1], [2], [3].

2. Theorem of Uniqueness

We introduce Francé's function

$$F(y) = 2 \left(\frac{K}{K'} \right)' + 1.$$

The following statement is a more general result than Theorem 6, given in [2].

Theorem 2. If 1) $K(y) \in C^2[y_c, 0)$, $K(0) = 0$, $K'(y) \neq 0$ for $y < 0$, $F(0) > 0$; 2) there is constant $d > 0$ such that $F(y) > -d$ in D_- ; 3) $z(x, y)$ regular solution of (1) in D , 4) $z|_{\Gamma \cup \gamma_1} = 0$, then $z(x, y) \equiv 0$ in D .

Proof. Consider the area integral I over domain D

$$I = \iint_D (az + bz_x + cz_y) (K(y)z_{xx} + z_{yy}) dx dy, \quad (2)$$

where $a(x, y)$, $b(x, y)$, $c(x, y)$ are given functions. By (1), the integral I vanishes. We shall show that over D integral I can be made non-positive by proper choice of functions $a(x, y)$, $b(x, y)$ and $c(x, y)$.

Applying Green's formula to the integral (2), similar to the work [3], we get:

$$\begin{aligned} 0 &= \iint_D \left[\frac{1}{2} (Ka_{xx} + a_{yy}) z^2 - a (Kz_x^2 + z_y^2) - \frac{1}{2} b_x (Kz_x^2 - z_y^2) \right. \\ &\quad \left. - b_y z_x z_y + \frac{1}{2} (Kc)_y z_x^2 - c_x K z_x z_y - \frac{1}{2} c_y z_y^2 \right] dx dy \\ &\quad + \int_{\Gamma + \gamma_1 + \gamma_2} \left[-a z z_y + \frac{1}{2} a_y z^2 - b z_x z_y + \frac{1}{2} c (Kz_x^2 - z_y^2) \right] dx \\ &\quad + \left[a K z z_x - \frac{1}{2} a_x K z^2 + c K z_x z_y + \frac{1}{2} b (Kz_x^2 - z_y^2) \right] dy = J_1 + J_2. \end{aligned}$$

Choose $b = c \equiv 0$ in D_+ . From $z(x, y) = 0$ on $\Gamma \cup \gamma_1$ and the fact that $dx = \sqrt{-K} dy$ (on γ_2) and $z_x dx + z_y dy = 0$ γ_1 we get

$$\begin{aligned} J_2 &= \frac{1}{2} \int_{\gamma_1} \left(b \frac{dy}{dx} - c \right) \left(\left(\frac{dx}{dy} \right)^2 + K \right) z_x^2 dx \\ &\quad - \frac{1}{2} \int_{\gamma_2} (b + c\sqrt{-K}) (\sqrt{-K} z_x^2 + 2z_x z_y + \frac{1}{\sqrt{-K}} z_y^2) dx \\ &\quad - \int_{\gamma_2} a \sqrt{-K} z dz - \frac{1}{2} \sqrt{-K} z^2 (a_x dx + a_y dy) = I_1 + I_2 + I_3, \\ J_1 &= - \iint_{D_+} a (Kz_x^2 + z_y^2) dx dy + \iint_D (Ka_{xx} + a_{yy}) z^2 dx dy \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \iint_{D_-} \left[(2aK + Kb_x - (Kc)_y) z_x^2 + 2z_x z_y (b_y + Kc_x) \right. \\
& \quad \left. + (2a - b_x + c_y) z_y^2 \right] dx dy \\
& = I_4 + I_5 + I_6.
\end{aligned}$$

We must choose functions $a(x, y)$, $b(x, y)$ and $c(x, y)$ so that all the integrals $I_1, I_2, \dots, I_6$ or at least their partial combinations were non-positive. If this occurs then $z(x, y) \equiv 0$ follows immediately.

Follow [6] for $y < 0$ choose

$$c = \frac{4aK(y)}{K'(y)}, \quad b = -c\sqrt{-K(y)}. \quad (3)$$

Then employing identities (3) we get

$$I_1 = - \int_{\gamma_1} \frac{2aK}{K'} \left(\frac{dy}{dx} - \sqrt{-K} \right) \left(\frac{dx}{dy} + \sqrt{-K} \right)^2 z_x^2 dy.$$

Obviously $I_2 = 0$. An integration by parts I_3 we get

$$I_3 = \int_{\gamma_2} \left(\sqrt{-K} a_x + a_y + \frac{aK'}{4K} \right) z^2 dx$$

The integral I_6 is non-positive if the following two conditions hold in D_- :

$$(Kc_x + b_y)^2 \leq (2a - b_x + c_y)(2aK + Kb_x - (Kc)_y), \quad (4)$$

$$2a + b_x - c_y \geq 0. \quad (5)$$

Obviously, condition (4) holds for all $a(x, y)$. If, now we substitute functions $a(x, y)$, $b(x, y)$ and $c(x, y)$ into (5) we obtain

$$\sqrt{-K} a_x + a_y + a \frac{K'}{2K} F(y) \leq 0. \quad (6)$$

Similarly as to the work [2] let

$$a = \begin{cases} e^{-\beta x}, & y \leq 0, \\ e^{-\beta x} \cos \gamma y, & y \geq 0, \end{cases}$$

where γ, β are positive constants.

Obviously I_4 is non-positive.

Substituting function $a(x, y)$ into I_1 , I_3 and (6) we get

$$\begin{aligned} I_1 + I_3 = & e^{-\beta x} \int_{\gamma_2} \left(-\sqrt{-K}\beta + \frac{K'}{4K} \right) z^2 dx \\ & e^{-\beta x} - \int_{\gamma_1} \frac{2K}{K'} \left(\frac{dy}{dx} - \sqrt{-K} \right) \left(\frac{dx}{dy} + \sqrt{-K} \right)^2 z_x^2 dy \\ & - \sqrt{-K}\beta + \frac{K'}{4K} F(y) \leq 0. \end{aligned} \quad (7)$$

We choose $\gamma = \frac{\pi}{2y_m}$ (where y_m is maximum ordinate in the elliptical domain) and β so large that sum of integrals $I_1 + I_3$ is non positive and hold inequality (7). If y_m sufficiently small number then the inequality $I_5 \leq 0$ holds.

Thus, we can conclude that since the sum of integrals $I_1 + I_3$, I_2 , I_4 , $I_5, \dots, I_6$ is nonnegative, and each integral is nonpositive, then all the integrals are zero and, in particular, the integral $I_4 = 0$, whence we get that $z(x, y) = 0$ in D^+ , and, in particular, $z(x, 0) = 0$ and $\frac{\partial z(x, 0)}{\partial y} = 0$. Then the uniqueness of the solution of the Cauchy problem implies $z(x, y) \equiv 0$ in D_- . As a result, we obtain $z(x, y) \equiv 0$ in the domain D . $\square$

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