WEAKLY α -CONTINUOUS FUZZY MULTIFUNCTIONS

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Abstract: In this paper we have defined three new types of fuzzy multifunctions and several characterizations are done. Also a mutual relationship between these fuzzy multifunctions are established. In the last section some applications of these multifunctions are shown.

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Key Words: fuzzy upper (lower) weakly α -continuous multifunctions, fuzzy α -preopen multifunction, fuzzy regular space, fuzzy almost compact space, fuzzy upper (lower) C -nbd

1. Introduction

Fuzzy multifunction, a function from an ordinary topological space X into a fuzzy topological space Y was introduced by Papageorgiou [19] in 1985 which has paved a new direction to many researchers to carry on their interest in this area. He also defined fuzzy upper and lower inverses and fuzzy upper and lower semi continuity. Afterwards, it was noticed in [14] that the definition of fuzzy lower inverse and fuzzy lower semi continuity were not natural so that Mukherjee and Malakar [14] redefined these two notions via q -coincidence and q -neighbourhoods of Pu and Liu [21]. In this paper, the definition of fuzzy upper inverse and fuzzy upper semi continuity of Papageorgiou [19] and fuzzy lower inverse and fuzzy lower semi continuity of Mukherjee and Malakar[14] have been used.

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2. Preliminaries

Throughout the paper, (X, τ) or simply X will stand for an ordinary topological space, while by (Y, τ_Y) or simply by Y will always mean a fuzzy topological space (fts, for short) in the sense of Chang [6]. A fuzzy set [23] A is a mapping from a nonempty set Y into the closed interval $I = [0, 1]$, i.e., $A \in I^Y$. The support of a fuzzy set A in Y will be denoted as $\text{supp}A$ [23] and is defined by $\text{supp}A = \{y \in Y : A(y) \neq 0\}$. A fuzzy point [21] with the singleton support $y \in Y$ and the value α ($0 < \alpha \leq 1$) at y will be denoted by y_α . 0_Y and 1_Y are the constant fuzzy sets taking respectively the constant values 0 and 1 on Y . The complement of a fuzzy set A in Y will be denoted by $1_Y \setminus A$ [23] and is defined by $(1_Y \setminus A)(y) = 1 - A(y)$, for all $y \in Y$. For two fuzzy sets A and B in Y , we write $A \leq B$ iff $A(y) \leq B(y)$, for each $y \in Y$, while we write AqB to mean A is quasi-coincident (q-coincident, for short) with B [21] if there exists $y \in Y$ such that $A(y) + B(y) > 1$; the negation of AqB is written as $A \not q B$. clA and $\text{int}A$ of a set A in X (respectively, a fuzzy set A in Y) respectively stand for the closure and interior (resp., fuzzy closure and fuzzy interior [6]) of A in X (respectively, in Y).

A fuzzy set A in Y is called fuzzy regular closed [2] if $A = cl\text{int}A$. A (fuzzy) set A in X (in Y) is said to be semiopen (fuzzy semiopen, fuzzy α -open) in X (in Y) if $A \subseteq cl\text{int}A$ (resp., $A \leq cl\text{int}A$ [2], $A \leq \text{int}cl\text{int}A$ [5]) [10]. The complement of a semiopen (fuzzy semiopen, fuzzy α -open) set in X (in Y) is called semiclosed (fuzzy semiclosed [2], fuzzy α -closed [5]) [10] set in X (in Y). The intersection of all (fuzzy) semiclosed (fuzzy α -closed) sets in X (in Y) containing a (fuzzy) set A in X (in Y) is called (fuzzy) semiclosure (fuzzy α -closure [5]) [10] ([2]) of A and is denoted by $sclA$ (αclA). The set of all semiopen sets containing a fuzzy point x in X is denoted by $SO(X, x)$. A set A in X is called preopen (resp., α -open) if $A \subseteq \text{int}clA$ (resp., $A \subseteq \text{int}cl\text{int}A$ [12]) [11]. The set of all preopen sets in X is denoted by $PO(X)$. The complement of an α -open set in X is called α -closed set. The union of all α -open sets contained in a set A in X is called α -interior of A , to be denoted by $\alpha - \text{int}A$ [12]. The intersection of all α -closed sets containing a set A in X is called α -closure of A , to be denoted by αclA [12]. A set U in X is said to be an α -nbd of a point $x \in X$ if there exists an α -open set V in X such that $x \in V \subseteq U$ [12]. The set of all α -open sets containing a point $x \in X$ is denoted by $\alpha O(X, x)$. A fuzzy point y_t is called a fuzzy θ -cluster point of a fuzzy set A in Y if $clUqA$ for every fuzzy open set U in Y with y_tqU [16]. The union of all fuzzy θ -cluster points of a fuzzy set A is called fuzzy θ -closure [16] of A , to be denoted by θclA . A fuzzy set A in Y is called fuzzy θ -closed if $A = \theta clA$ [16]. The complement of a

fuzzy θ -closed set is called fuzzy θ -open [16]. The union of all fuzzy θ -open sets contained in a fuzzy set A in Y is called fuzzy θ -interior of A , to be denoted by $\theta - \text{int}A$ [16]. A fuzzy set A in Y is fuzzy θ -open if $A = \theta - \text{int}A$ [16]. It is known that θclA is fuzzy closed for any fuzzy set A in Y and $clA = \theta clA$ for any fuzzy open set A in Y [16].

3. Some Well Known Definitions, Theorems and Lemmas

Let us recall the following definition from [19] for ready references.

Definition 3.1 [19]. Let (X, τ) and (Y, τ_Y) be respectively an ordinary topological space and an fts. We say that $F : X \rightarrow Y$ is a fuzzy multifunction if corresponding to each $x \in X$, $F(x)$ is a unique fuzzy set in Y .

Henceforth by $F : X \rightarrow Y$ we shall mean a fuzzy multifunction in the above sense.

Definition 3.2 [19, 14]. For a fuzzy multifunction $F : X \rightarrow Y$, the upper inverse F^+ and lower inverse F^- are defined as follows:

For any fuzzy set A in Y , $F^+(A) = \{x \in X : F(x) \leq A\}$ and $F^-(A) = \{x \in X : F(x) q A\}$.

There is a following relationship between the upper and the lower inverses of a fuzzy multifunction.

Theorem 3.3 [14]. For a fuzzy multifunction $F : X \rightarrow Y$, we have $F^-(1_Y \setminus A) = X \setminus F^+(A)$, for any fuzzy set A in Y .

Definition 3.4 [6]. Let A be a fuzzy set in an fts Y . A collection $\mathcal{U}$ of fuzzy sets in Y is said to be a fuzzy cover of A if $\sup\{U(x) : U \in \mathcal{U}\} = 1$ for each $x \in \text{supp}A$. If, in addition, each member of $\mathcal{U}$ is fuzzy open, then $\mathcal{U}$ is called a fuzzy open cover of A . In particular, if $A = 1_Y$, we get the definition of fuzzy cover (fuzzy open cover) of the fts. Y .

Definition 3.5 [7]. A fuzzy cover $\mathcal{U}$ of a fuzzy set A in an fts Y is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 \geq A$. Clearly, if $A = 1_Y$, in particular, then the requirements on $\mathcal{U}_0$ is $\bigcup \mathcal{U}_0 = 1_Y$.

Definition 3.6. An fts Y is called fuzzy compact [6] (resp., fuzzy almost compact [15]) space if every fuzzy open cover $\mathcal{U}$ of Y has a finite subcover (resp., finite proximate subcover, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\bigcup_{U \in \mathcal{U}_0} clU = 1_Y$).

Lemma 3.7 [20]. The following properties hold for a subset A of a topological space X :

- (1) If A is open in X , then $sclA = intclA$.
- (2) A is α -open in X iff there is some open set U in X such that $U \subset A \subset sclU$.
- (3) $\alpha clA = A \cup cl(int(clA))$.

Lemma 3.8 [3]. For a fuzzy set A in an fts (Y, τ_Y) , $sclA = intclA$.

4. Fuzzy Upper and Lower Weakly α -Continuous Multifunctions

In this section we first introduce three types of fuzzy multifunctions and characterized them in several ways.

Definition 4.1. A fuzzy multifunction $F : X \rightarrow Y$ is called fuzzy:

- (i) upper α -continuous (f.u. α .c., for short) at a point $x \in X$ if for each fuzzy open set V of Y containing $F(x)$, there exists $U \in \alpha O(X, x)$ such that $F(U) \leq V$,
- (ii) lower α -continuous (f.l. α .c., for short) at a point $x \in X$ if for each fuzzy open set V of Y with $F(x)qV$, there exists $U \in \alpha O(X, x)$ such that $F(u)qV$, for all $u \in U$,
- (iii) upper (lower) α -continuous on X if F has this property at each point of X .

Definition 4.2. A fuzzy multifunction $F : X \rightarrow Y$ is said to be fuzzy:

- (i) upper almost α -continuous (f.u.a. α .c., for short) at a point $x \in X$ for each $U \in SO(X, x)$ and each fuzzy open set V containing $F(x)$, there exists a non empty open set G in X with $G \subset U$ such that $F(G) \leq sclV$,
- (ii) lower almost α -continuous (f.l.a. α .c., for short) at a point $x \in X$ if for each $U \in SO(X, x)$ and each fuzzy open set V with $F(x)qV$, there exists a non empty open set G in X with $G \subset U$ such that $F(g)qsclV$, for all $g \in G$,
- (iii) upper (lower) almost α -continuous if it is fuzzy upper (lower) almost α -continuous at every point of X .

Definition 4.3. A fuzzy multifunction $F : X \rightarrow Y$ is said to be fuzzy:

- (i) upper weakly α -continuous (f.u.w. α .c., for short) at a point $x \in X$ for each $U \in SO(X, x)$ and each fuzzy open set V containing $F(x)$, there exists a non empty open set G in X with $G \subset U$ such that $F(G) \leq clV$,
- (ii) lower weakly α -continuous (f.l.w. α .c., for short) at a point $x \in X$ if for each $U \in SO(X, x)$ and each fuzzy open set V with $F(x)qV$, there exists a non empty open set G in X with $G \subset U$ such that $F(g)qclV$, for all $g \in G$,

(iii) upper (lower) weakly α -continuous if it is fuzzy upper (lower) weakly α -continuous at every point of X .

Theorem 4.4. For a fuzzy multifunction $F : X \rightarrow Y$, the following statements are equivalent:

- (i) F is f.u.w. α .c. at a point $x \in X$,
- (ii) for any fuzzy open set V of Y with $F(x) \leq V$, there exists $S \in \alpha O(X, x)$ such that $F(S) \leq clV$,
- (iii) $x \in \alpha int(F^+(clV))$ for every fuzzy open set V with $F(x) \leq V$,
- (iv) $x \in int(cl(int(F^+(clV))))$ for every fuzzy open set V with $F(x) \leq V$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in X$ and V be any fuzzy open set in Y with $F(x) \leq V$. By (i), for each $U \in SO(X, x)$, there exists a non empty open set $G_U \subset U$ such that $F(G_U) \leq clV$. Let $W = \bigcup \{G_U : U \in SO(X, x)\}$. Then W is open in X . Put $S = W \bigcup \{x\}$. Then $x \in sclW$ and $F(W) \leq clV$. Since $W \subseteq S \subseteq sclW$ where W is open in X , by Lemma 3.7(2), $S \in \alpha O(X, x)$. Also $F(S) \leq clV$.

(ii) $\Rightarrow$ (iii) Let V be any fuzzy open set in Y with $F(x) \leq V$. By (ii), there exists $S \in \alpha O(X, x)$ such that $F(S) \leq clV$. Then $x \in S \subseteq F^+(clV) \Rightarrow x \in S = \alpha int S \subseteq \alpha int(F^+(clV))$.

(iii) $\Rightarrow$ (iv) Let V be any fuzzy open set in Y with $F(x) \leq V$. Then by (iii), $x \in \alpha int(F^+(clV))$. Put $U = \alpha int(F^+(clV))$. Then $U \in \alpha O(X, x)$ and so $x \in U \subseteq F^+(clV)$. Therefore, $x \in U \subseteq int(cl(intU)) \subseteq int(cl(int(F^+(clV))))$.

(iv) $\Rightarrow$ (i) Let $U \in SO(X, x)$ and V be any fuzzy open set in Y such that $F(x) \leq V$. By (iv), $x \in int(cl(int(F^+(clV)))) = scl(int(F^+(clV)))$ (by Lemma 3.7 (1)). By definition, $U \cap int(F^+(clV)) \neq \phi$ and $U \cap int(F^+(clV)) \in SO(X)$ (by Lemma 3 [18] and Lemma 1 [17]). Put $G = int(U \cap int(F^+(clV)))$. Then $G \neq \phi$ (by Lemma 4 [17]), $G \subset U$ and $F(G) \leq clV$.

Theorem 4.5. For a fuzzy multifunction $F : X \rightarrow$, the following statements are equivalent:

- (i) F is f.l.w. α .c. at a point $x \in X$,
- (ii) for any fuzzy open set V with $F(x)qV$, there exists $U \in \alpha O(X, x)$ such that $F(u)qclV$, for every $u \in U$,
- (iii) $x \in \alpha int(F^-(clV))$ for every fuzzy open set V with $F(x)qV$,
- (iv) $x \in int(cl(int(F^-(clV))))$ for every fuzzy open set V with $F(x)qV$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in X$ and V be any fuzzy open set in Y with $F(x)qV$. By (i), for each $U \in SO(X, x)$, there exists a non empty open set $G_U \subset U$ such that $F(g)qclV$, for all $g \in G_U$. Put $W = \bigcup \{G_U : U \in SO(X, x)\}$. Then W is open in X . Let $S = W \bigcup \{x\}$. Then $x \in sclW$ and $F(w)qclV$, for all $w \in W$. By Lemma 3.7(2), $S \in \alpha O(X, x)$ (as $W \subseteq S \subseteq sclW$) and $F(s)qclV$, for all

$s \in S$.

(ii) $\Rightarrow$ (iii) Let V be any fuzzy open set in Y with $F(x)qV$. By (ii), there exists $U \in \alpha O(X, x)$ such that $F(u)qclV$, for all $u \in U$. Then $x \in U \subseteq F^-(clV)$ and hence $x \in U = \alpha int U \subseteq \alpha int(F^-(clV))$.

(iii) $\Rightarrow$ (iv) Let V be any fuzzy open set in Y with $F(x)qV$. Then by (iii), $x \in \alpha int(F^-(clV))$. Put $U = \alpha int(F^-(clV))$. Then $U \in \alpha O(X, x)$ and so $x \in U \subseteq F^-(clV)$. Thus $x \in U \subseteq int(cl(intU)) \subseteq int(cl(int(F^-(clV))))$.

(iv) $\Rightarrow$ (i) Let $U \in SO(X, x)$ and V be any fuzzy open set in Y such that $F(x)qV$. By (iv), $x \in int(cl(int(F^-(clV)))) = scl(int(F^-(clV)))$ (by Lemma 3.7 (1)) and so $U \cap int(F^-(clV)) \neq \phi$ and $U \cap int(F^-(clV)) \in SO(X)$. Put $G = int(U \cap int(F^-(clV)))$. Then $G(\neq \phi)$ is open in X with $G \subset U$ and $F(g)qclV$, for all $g \in G$.

Theorem 4.6. For a fuzzy multifunction $F : X \rightarrow Y$, the following statements are equivalent:

- (i) F is f.u.a. α .c. (resp., f.u. α .c.) at a point $x \in X$,
- (ii) for any fuzzy open set V of Y with $F(x) \leq V$, there exists $S \in \alpha O(X, x)$ such that $F(S) \leq sclV$ (resp., $F(S) \leq V$),
- (iii) $x \in \alpha int(F^+(sclV))$ (resp., $x \in \alpha int(F^+(V))$) for every fuzzy open set V with $F(x) \leq V$,
- (iv) $x \in int(cl(int(F^+(sclV))))$ (resp., $x \in int(cl(int(F^+(V))))$) for every fuzzy open set V with $F(x) \leq V$.

Proof. The proof is same as that of Theorem 4.4.

Theorem 4.7. For a fuzzy multifunction $F : X \rightarrow Y$, the following statements are equivalent:

- (i) F is f.l.a. α .c. (resp. f.l. α .c.) at a point $x \in X$,
- (ii) for any fuzzy open set V with $F(x)qV$, there exists $U \in \alpha O(X, x)$ such that $F(u)qsclV$ (resp., $F(u)qV$), for every $u \in U$,
- (iii) $x \in \alpha int(F^-(sclV))$ (resp., $x \in \alpha int(F^-(V))$) for every fuzzy open set V with $F(x)qV$,
- (iv) $x \in int(cl(int(F^-(clV))))$ (resp., $x \in int(cl(int(F^-(V))))$) for every fuzzy open set V with $F(x)qV$.

Proof. The proof is same as that of Theorem 4.5.

Theorem 4.8. For a fuzzy multifunction $F : X \rightarrow Y$, the following statements are equivalent:

- (i) F is f.u.w. α .c.,
- (ii) for each $x \in X$ and each fuzzy open set V of Y with $F(x) \leq V$, there exists $U \in \alpha O(X, x)$ such that $F(U) \leq clV$,
- (iii) $F^+(V) \subseteq int(cl(int(F^+(clV))))$ for every fuzzy open set V of Y ,

- (iv) $cl(int(cl(F^-(intK)))) \subseteq F^-(K)$ for every fuzzy closed set V of Y ,
- (v) $\alpha cl(F^-(intK)) \subseteq F^-(K)$ for every fuzzy closed set K of Y ,
- (vi) $\alpha cl(F^-(int(clB))) \subseteq F^-(clB)$ for every fuzzy set B of Y ,
- (vii) $F^+(intB) \subseteq \alpha int(F^+(cl(intB)))$ for every fuzzy set B of Y ,
- (viii) $F^+(V) \subseteq \alpha int(F^+(clV))$ for every fuzzy open set V of Y ,
- (ix) $\alpha cl(F^-(intK)) \subseteq F^-(K)$ for every fuzzy regular closed set K of Y ,
- (x) $\alpha cl(F^-(V)) \subseteq F^-(clV)$ for every fuzzy open set V of Y ,
- (xi) $\alpha cl(F^-(int(\theta clB))) \subseteq F^-(\theta clB)$ for every fuzzy set B of Y .

Proof. (i) $\Rightarrow$ (ii) Follows from Theorem 4.4.

(ii) $\Rightarrow$ (iii) Let $x \in X$ and V be any fuzzy open set in Y with $x \in F^+(V)$. Then $F(x) \leq V$. By (ii), there exists $U \in \alpha O(X, x)$ such that $F(U) \leq clV$. Therefore, $x \in U \subseteq F^+(clV)$. Since $U \in \alpha O(X, x)$, $x \in U \subseteq int(cl(intU)) \subseteq int(cl(int(F^+(clV)))) \Rightarrow F^+(V) \subseteq int(cl(int(F^+(clV))))$.

(iii) $\Rightarrow$ (iv) Let K be a fuzzy closed set of Y . Then $1_Y \setminus K$ is fuzzy open in Y . By (iii),

$$\begin{aligned} F^+(1_Y \setminus K) &\subseteq int(cl(int(F^+(cl(1_Y \setminus K))))) \\ &\Rightarrow X \setminus F^-(K) \subseteq int(cl(int(X \setminus F^-(intK)))) = X \setminus cl(int(cl(F^-(intK)))) \\ &\Rightarrow cl(int(cl(F^-(intK)))) \subseteq F^-(K). \end{aligned}$$

(iv) $\Rightarrow$ (v) Let K be a fuzzy closed set of Y . By (iv), $cl(int(cl(F^-(intK)))) \subseteq F^-(K)$ and hence by Lemma 3.7 (3), $\alpha cl(F^-(intK)) \subseteq F^-(K)$.

(v) $\Rightarrow$ (vi) Let B be a fuzzy set of Y . Then clB is fuzzy closed in Y and so by (v), $\alpha cl(F^-(int(clB))) \subseteq F^-(clB)$.

(vi) $\Rightarrow$ (vii) Let B be a fuzzy set of Y . Then $X \setminus F^+(intB) = F^-(1_Y \setminus intB) = F^-(cl(1_Y \setminus B)) \supseteq \alpha cl(F^-(int(cl(1_Y \setminus B)))) = \alpha cl(F^-(1_Y \setminus cl(intB))) = \alpha cl(X \setminus F^+(cl(intB))) = X \setminus \alpha int F^+(cl(intB)) \Rightarrow \alpha int(F^+(cl(intB))) \supseteq F^+(intB)$.

(vii) $\Rightarrow$ (viii) Obvious.

(viii) $\Rightarrow$ (i) Let $x \in X$ and V be any fuzzy open set in Y such that $F(x) \leq V$. Then $x \in F^+(V) \subseteq \alpha int(F^+(clV))$ (by (viii)) $\subseteq int(cl(int(F^+(clV))))$ (by Theorem 23 [1]) and then by Theorem 4.4, F is f.u.w. α c. at x .

(v) $\Rightarrow$ (ix) Obvious.

(ix) $\Rightarrow$ (x) Let V be any fuzzy open set of Y . Then clV is fuzzy regular closed in Y . Indeed, $cl(int(clV)) \leq clV$ and $clV = cl(intV) \leq cl(int(clV))$. By (ix), $\alpha cl(F^-(int(clV))) \subseteq F^-(clV) \Rightarrow \alpha cl(F^-(V)) \subseteq \alpha cl(F^-(int(clV))) \subseteq F^-(clV)$.

(x) $\Rightarrow$ (viii) Let V be any fuzzy open set of Y . By (x), $\alpha cl(F^-(V)) \subseteq F^-(clV)$. Therefore, $X \setminus \alpha int(F^+(clV)) = \alpha cl(X \setminus F^+(clV)) = \alpha cl(F^-(1_Y \setminus$

$clV)) \subseteq F^-(cl(1_Y \setminus clV)) = F^-(1_Y \setminus int(clV)) = X \setminus F^+(int(clV))$ and hence $F^+(V) \subseteq F^+(int(clV)) \subseteq \alpha int(F^+(clV))$.

(x) $\Rightarrow$ (xi) Let B be any fuzzy set of Y . Put $V = int(\theta clB)$. Then V is fuzzy open in Y . By (x), $\alpha cl(F^-(int(\theta clB))) \subseteq F^-(cl(int(\theta clB))) \subseteq F^-(cl(\theta clB)) = F^-(\theta clB)$ (as θclB is fuzzy closed in Y [16]).

(xi) $\Rightarrow$ (ix) Let K be any fuzzy regular closed set of Y . Then by (xi),

$$\begin{aligned} \alpha cl(F^-(intK)) &= \alpha cl(F^-(int(clK))) = \alpha cl(F^-(int(cl(cl(intK)))))) \\ &= \alpha cl(F^-(int(cl(intK)))) = \alpha cl(F^-(int(\theta cl(intK)))) \end{aligned}$$

(since $intK$ is fuzzy open in Y) $\subseteq \alpha cl(F^-(\theta cl(intK))) \subseteq F^-(\theta cl(intK))$ (by (xi)) $= F^-(cl(intK))$ (as $intK$ is fuzzy open in Y) $= F^-(K)$.

Theorem 4.9. For a fuzzy multifunction $F : X \rightarrow Y$, the following statements are equivalent:

- (i) F is f.l.w. α c.,
- (ii) for each $x \in X$ and each fuzzy open set V of Y with $F(x)qV$, there exists $U \in \alpha O(X, x)$ such that $U \subseteq F^-(clV)$, i.e., $F(u)qclV$, for all $u \in U$,
- (iii) $F^-(V) \subseteq int(cl(int(F^-(V))))$ for every fuzzy open set V of Y ,
- (iv) $cl(int(cl(F^+(intK)))) \subseteq F^+(K)$ for every fuzzy closed set V of Y ,
- (v) $\alpha cl(F^+(intK)) \subseteq F^+(K)$ for every fuzzy closed set K of Y ,
- (vi) $\alpha cl(F^+(int(clB))) \subseteq F^+(clB)$ for every fuzzy set B of Y ,
- (vii) $F^-(intB) \subseteq \alpha int(F^-(cl(intB)))$ for every fuzzy set B of Y ,
- (viii) $F^-(V) \subseteq \alpha int(F^-(clV))$ for every fuzzy open set V of Y ,
- (ix) $\alpha cl(F^+(intK)) \subseteq F^+(K)$ for every fuzzy regular closed set K of Y ,
- (x) $\alpha cl(F^+(V)) \subseteq F^+(clV)$ for every fuzzy open set V of Y ,
- (xi) $\alpha cl(F^+(int(\theta clB))) \subseteq F^+(\theta clB)$ for every fuzzy set B of Y .

PROOF. (i) $\Rightarrow$ (ii) Follows from Theorem 4.5.

(ii) $\Rightarrow$ (iii) Let $x \in X$ and V be any fuzzy open set in Y with $x \in F^-(V)$. Then $F(x) \leq V$. By (ii), there exists $U \in \alpha O(X, x)$ such that $U \subseteq F^-(clV)$. Therefore, $x \in U \subseteq int(cl(intU)) \subseteq int(cl(int(F^-(clV)))) \Rightarrow F^-(V) \subseteq int(cl(int(F^-(clV))))$.

(iii) $\Rightarrow$ (iv) Let K be a fuzzy closed set of Y . Then $1_Y \setminus K$ is fuzzy open in Y . By (iii),

$$\begin{aligned} F^-(1_Y \setminus K) &\subseteq int(cl(int(F^-(cl(1_Y \setminus K))))) \\ &\Rightarrow X \setminus F^+(K) \subseteq int(cl(int(X \setminus F^+(intK)))) = X \setminus cl(int(cl(F^+(intK)))) \\ &\Rightarrow cl(int(cl(F^+(intK)))) \subseteq F^+(K). \end{aligned}$$

(iv) $\Rightarrow$ (v) Let K be a fuzzy closed set of Y . By (iv), $cl(int(cl(F^+(intK)))) \subseteq F^+(K)$ and hence by Lemma 3.7 (3),

$$\begin{aligned}\alpha cl(F^+(intK)) &= F^+((intK) \bigcup cl(int(cl(F^+(intK)))) \\ &\subseteq F^+(intK) \bigcup F^+(K) \subseteq F^+(K)\end{aligned}$$

(as $F^+(intK) \subseteq F^+(K)$).

(v) $\Rightarrow$ (vi) Let B be a fuzzy set of Y . Then clB is fuzzy closed in Y and so by (v), $\alpha cl(F^+(int(clB))) \subseteq F^+(clB)$.

(vi) $\Rightarrow$ (vii) Let B be any fuzzy set of Y . Then $X \setminus F^-(intB) = F^+(1_Y \setminus intB) = F^+(cl(1_Y \setminus B)) \supseteq \alpha cl(F^+(int(cl(1_Y \setminus B)))) = \alpha cl(F^+(1_Y \setminus cl(intB))) = \alpha cl(X \setminus F^-(cl(intB))) = X \setminus \alpha int F^-(cl(intB)) \Rightarrow \alpha int(F^-(cl(intB))) \supseteq F^-(intB)$.

(vii) $\Rightarrow$ (viii) Obvious.

(viii) $\Rightarrow$ (i) Let $x \in X$ and V be any fuzzy open set in Y such that $F(x)qV$. Then $x \in F^-(V) \subseteq \alpha int(F^-(clV))$ (by (viii)) $\subseteq int(cl(int(F^-(clV))))$ (by Theorem 23 [1]) and then by Theorem 4.5, F is f.l.w. α .c. at x .

(v) $\Rightarrow$ (ix) The proof follows from the fact that every fuzzy regular closed set is fuzzy closed.

(ix) $\Rightarrow$ (x) Let V be any fuzzy open set of Y . Then clV is fuzzy regular closed in Y . By (ix), $\alpha cl(F^+(int(clV))) \subseteq F^+(clV) \Rightarrow \alpha cl(F^+(V)) \subseteq \alpha cl(F^+(int(clV))) \subseteq F^+(clV)$.

(x) $\Rightarrow$ (viii) Let V be any fuzzy open set of Y . By (x), $\alpha cl(F^+(V)) \subseteq F^+(clV)$. Therefore, $X \setminus \alpha int(F^-(clV)) = \alpha cl(X \setminus F^-(clV)) = \alpha cl(F^+(1_Y \setminus clV)) \subseteq F^+(cl(1_Y \setminus clV)) = F^+(1_Y \setminus int(clV)) = X \setminus F^-(int(clV))$ and hence $F^-(V) \subseteq F^-(int(clV)) \subseteq \alpha int(F^-(clV))$.

(x) $\Rightarrow$ (xi) Let B be any fuzzy set of Y . Put $V = int(\theta clB)$. Then V is fuzzy open in Y . By (x), $\alpha cl(F^+(V)) \subseteq F^+(clV) \Rightarrow \alpha cl(F^+(int(\theta clB))) \subseteq F^+(cl(int(\theta clB))) \subseteq F^+(cl(\theta clB)) = F^+(\theta clB)$ (as θclB is fuzzy closed in Y [16]).

(xi) $\Rightarrow$ (ix) Let K be any fuzzy regular closed set of Y . Then, $\alpha cl(F^+(intK)) = \alpha cl(F^+(int(clK))) = \alpha cl(F^+(int(cl(cl(intK)))) = \alpha cl(F^+(int(cl(intK)))) = \alpha cl(F^+(int(\theta cl(intK))))$ (since $intK$ is fuzzy open in $Y \subseteq \alpha cl(F^+(\theta cl(intK))) \subseteq F^+(\theta cl(intK))$ (by (xi)) $= F^+(cl(intK))$ (as $intK$ is fuzzy open in Y) $= F^+(K)$.

Lemma 4.10. If $F : X \rightarrow Y$ is f.l.w. α .c., then for each $x \in X$ and each fuzzy set B of Y with $F(x)q(\theta intB)$, there exists $U \in \alpha O(X, x)$ such that $U \subseteq F^-(B)$.

Proof. Let F be f.l.w. α .c. and $x \in X$ and B be fuzzy open in Y with $F(x)q(\theta intB)$. Then there exists $y \in Y$ such that $[F(x)](y) + (\theta intB)(y) > 1$.

Let $[F(x)](y) = t$. Then $(\theta \text{int} B)(y) + t > 1 \Rightarrow y_t q(\theta \text{int} B)$ and $t > 1 - (\theta \text{int} B)(y) \Rightarrow y_t \notin 1_Y \setminus \theta \text{int} B = \theta cl(1_Y \setminus B)$. Then there exists a fuzzy open set V in Y with $y_t qV$, $clV \not/q(1_Y \setminus B) \Rightarrow clV \leq B$. Therefore, $F(x)qV$ (as $V(y) + t = V(y) + [F(x)](y) > 1$) and $V \leq clV \leq B$. Since F is f.l.w. α .c., there exists $U \in \alpha O(X, x)$ such that $F(u)qclV$, for all $u \in U \Rightarrow F(u)qB$, for all $u \in U \Rightarrow U \subseteq F^-(B)$.

Theorem 4.11. For a fuzzy multifunction $F : X \rightarrow Y$, the following statements are equivalent:

- (i) F is f.l.w. α .c.,
- (ii) $\alpha cl(F^+(B)) \subseteq F^+(\theta cl B)$, for every fuzzy set B of Y ,
- (iii) $F(\alpha cl A) \leq \theta cl(F(A))$, for every subset A of X .

Proof. (i) $\Rightarrow$ (ii) Let B be a fuzzy set of Y and $x \notin F^+(\theta cl B)$. Then $x \in X \setminus F^+(\theta cl B) = F^-(1_Y \setminus \theta cl B) = F^-(\theta \text{int}(1_Y \setminus B))$. By Lemma 4.10, there exists $U \in \alpha O(X, x)$ such that $U \subseteq F^-(1_Y \setminus B) = X \setminus F^+(B)$ and so $U \cap F^+(B) = \phi$. Again, $F^+(B) \subseteq \alpha cl(F^+(B)) \Rightarrow X \setminus F^+(B) \supseteq X \setminus \alpha cl(F^+(B))$. Now $F^+(B) \subseteq X \setminus U \Rightarrow \alpha cl(F^+(B)) \subseteq \alpha cl(X \setminus U) = X \setminus U \Rightarrow U \subseteq X \setminus \alpha cl(F^+(B)) \Rightarrow x \in U \subseteq X \setminus \alpha cl(F^+(B)) \Rightarrow x \notin \alpha cl(F^+(B))$.

(ii) $\Rightarrow$ (i) Let V be a fuzzy open set of Y . Then $clV = \theta clV$. By (ii), $\alpha cl(F^+(V)) \subseteq F^+(\theta cl V) = F^+(clV)$ and so by Theorem 4.9 (x) $\Rightarrow$ (i), F is f.l.w. α .c.

(ii) $\Rightarrow$ (iii) Let $A \subseteq X$. By (ii), $\alpha cl A \subseteq \alpha cl(F^+(F(A))) \subseteq F^+(\theta cl(F(A)))$ and so $F(\alpha cl A) \leq \theta cl(F(A))$.

(iii) $\Rightarrow$ (ii) Let B be a fuzzy set of Y . Then $F^+(B) \subseteq X$. By (iii), $F(\alpha cl(F^+(B))) \leq \theta cl(F(F^+(B))) \leq \theta cl B \Rightarrow \alpha cl(F^+(B)) \subseteq F^+(\theta cl B)$.

Let us now recall the following definition and lemmas from [4] for ready references.

Definition 4.12. For a fuzzy multifunction $F : X \rightarrow Y$, $clF : X \rightarrow Y$ and $\alpha clF : X \rightarrow Y$ are defined by $(clF)(x) = clF(x)$, for all $x \in X$ [3] and $(\alpha clF)(x) = \alpha clF(x)$, for all $x \in X$ [4].

Lemma 4.13 [4]. For a fuzzy multifunction $F : X \rightarrow Y$, $(clF)^-(V) = F^-(V)$, $(\alpha clF)^-(V) = F^-(V)$, for all fuzzy open set V in Y .

Lemma 4.14 [4]. For a fuzzy multifunction $F : X \rightarrow Y$, $(\alpha clF)^+(V) = F^+(V)$, for all fuzzy set V in Y .

Theorem 4.15. A fuzzy multifunction $F : X \rightarrow Y$ is f.l.w. α .c. implies $clF : X \rightarrow Y$ (resp., $\alpha clF : X \rightarrow Y$) is f.l.w. α .c.

Proof. Let F be f.l.w. α .c. Let $x \in X$ and G be any fuzzy open set in Y such that $(clF)(x)qG$ (resp., $(\alpha clF)(x)qG$). Then $x \in (clF)^-(G)$ (resp.,

$x \in (\alpha cl F)^-(G) = F^-(G)$ (by Lemma 4.13). As F is f.l.w. α .c., there exists $U \in \alpha O(X, x)$ such that $F(u)qclG$, for all $u \in U \Rightarrow (cl F)(u)qclG$ (resp., $(\alpha cl F)(u)qclG$), for all $u \in U \Rightarrow u \in (cl F)^-(clG)$ (resp., $u \in (\alpha cl F)^-(clG)$), for all $u \in U \Rightarrow U \subseteq (cl F)^-(clG)$ (resp., $U \subseteq (\alpha cl F)^-(clG) \Rightarrow cl F$ (resp., $\alpha cl F$) is f.l.w. α .c.

Theorem 4.16. The fuzzy multifunction $F : X \rightarrow Y$ is f.u.w. α .c. implies $\alpha cl F : X \rightarrow Y$ is f.u.w. α .c.

Proof. Let $x \in X$ and G be any fuzzy open set in Y such that $(\alpha cl F)(x) \leq G$. Then $x \in (\alpha cl F)^+(G) = F^+(G)$, by Lemma 4.14. Since F is f.u.w. α .c., there exists $U \in \alpha O(X, x)$ such that $F(U) \leq clG \Rightarrow (\alpha cl F)(U) \leq \alpha cl(clG) \leq cl(clG) = clG \Rightarrow \alpha cl F$ is f.u.w. α .c.

Definition 4.17 [14]. For a fuzzy multifunction $F : X \rightarrow Y$, the fuzzy graph multifunction $G_F : X \rightarrow X \times Y$ of F is defined as $G_F(x)$ = the fuzzy set $x_1 \times F(x)$ of $X \times Y$, where x_1 is the fuzzy set in X , whose value is 1 at $x \in X$ and 0 at other points of X . We shall write $\{x\} \times F(x)$ for $x_1 \times F(x)$.

Lemma 4.18 [3]. The following hold for a fuzzy multifunction $F : X \rightarrow Y$:

- (a) $(G_F)^+(A \times B) = A \cap F^+(B)$ and
- (b) $(G_F)^-(A \times B) = A \cap F^-(B)$ for every subset A of X and every fuzzy set B of Y .

Theorem 4.19. A fuzzy multifunction $F : X \rightarrow Y$ is f.l.w. α .c. iff $G_F : X \rightarrow X \times Y$ is f.l.w. α .c.

Proof. Let G_F be f.l.w. α .c. on X . Let $x \in X$ and V be any fuzzy open set in Y with $F(x)qV$. Then there exists $y \in Y$ such that $[F(x)](y) + V(y) > 1$. Now $1_X \times V$ is fuzzy open in $X \times Y$ such that $[\{x\} \times F(x)](x, y) + (1_X \times V)(x, y) > 1$ and so $G_F(x)q(1_X \times V)$. Since G_F is f.l.w. α .c., there exists $U \in \alpha O(X, x)$ such that $(G_F)(u)qcl(1_X \times V)$, for all $u \in U \Rightarrow U \subseteq (G_F)^-(1_X \times clV) = 1_X \cap F^-(clV) = F^-(clV) \Rightarrow F$ is f.l.w. α .c. on X .

Conversely, let F be f.l.w. α .c. Let $x \in X$ and W be any fuzzy open set of $X \times Y$ such that $G_F(x)qW$. Then there exists $(x, y) \in X \times Y$ such that $[\{x\} \times F(x)](y) + W(x, y) > 1$. Then $[F(x)](y) + W(x, y) > 1$. Let $[F(x)](y) = \alpha$. Then $W(x, y) + \alpha > 1 \Rightarrow (x, y)_\alpha qW$ and hence $(x, y)_\alpha q(U \times V) \leq W$ for some open set U of X and a fuzzy open set V of Y with $y_\alpha qV$. Then $V(y) + \alpha > 1 \Rightarrow V(y) + [F(x)](y) > 1 \Rightarrow VqF(x)$. Since F is f.l.w. α .c., there exists $G \in \alpha O(X, x)$ such that $F(g)qclV$, for all $g \in G \Rightarrow G \subseteq F^-(clV)$. Then by Lemma 2.13(b), $U \cap G \subseteq U \cap F^-(clV) = (G_F)^-(U \times clV) \subseteq (G_F)^-(clW)$. Also $x \in U \cap G \in \alpha O(X, x)$ which proves that G_F is f.l.w. α .c.

Theorem 4.20. When X is product related to Y , a fuzzy multifunction $F : X \rightarrow Y$ is f.u.w. α .c. if its fuzzy graph multifunction $G_F : X \rightarrow X \times Y$ is

f.u.w. α .c.

Proof. Let G_F be f.u.w. α .c. on X . Let $x \in X$ and V be any fuzzy open set in Y with $F(x) \leq V$. Then $G_F(x) \leq 1_X \times V$ and $1_X \times V$ is open in $X \times V$. Then there exists $U \in \alpha O(X, x)$ such that $G_F(U) \leq cl(1_X \times V) = 1_X \times clV$. Now for any $z \in U$ and for any $y \in Y$, $[F(z)](y) = [G_F(z)](z, y) \leq (1_X \times clV)(z, y) = (clV)(y)$, i.e., $[F(z)](y) \leq (clV)(y)$, for all $y \in Y \Rightarrow F(z) \leq clV$, for any $z \in U \Rightarrow F(U) \leq clV \Rightarrow U \subseteq F^+(clV) \Rightarrow F$ is f.u.w. α .c.

Lemma 4.21 [13]. Let $U, X_0 \subseteq X$. Then the following properties hold:

- (i) if $U \in \alpha O(X)$ and $X_0 \in SO(X) \cup PO(X)$, then $U \cap X_0 \in \alpha O(X_0)$,
- (ii) if $U \subseteq X_0 \subseteq X$, $U \in \alpha O(X_0)$ and $X_0 \in \alpha O(X)$, then $U \in \alpha O(X)$.

Theorem 4.22. A fuzzy multifunction $F : X \rightarrow Y$ is f.u.w. α .c. (resp., f.l.w. α .c.) and $X_0 \in SO(X) \cup PO(X)$ iff $F/X_0 : X_0 \rightarrow Y$ is f.u.w. α .c. (resp., f.l.w. α .c.).

Proof. Let F be f.u.w. α .c. (resp., f.l.w. α .c.). Let $x \in X_0$ and V be any fuzzy open set in Y such that $(F/X_0)(x) \leq V$ (resp., $(F/X_0)(x)qV$). Then $F(x) \leq V$ (resp., $F(x)qV$). By hypothesis, there exists $U \in \alpha O(X, x)$ such that $F(U) \leq clV$ (resp., $F(u)qclV$, for all $u \in U$). Let $U_0 = U \cap X_0$. Then by Lemma 4.21(i), $U_0 \in \alpha O(X_0, x)$ and $(F/X_0)(U_0) = F(U_0) \leq clV$ (resp., $(F/X_0)(u) = F(u)qclV$, for all $u \in U_0$) $\Rightarrow F/X_0$ is f.u.w. α .c. (resp., f.l.w. α .c.).

Conversely, let F/X_0 be f.u.w. α .c. (resp., f.l.w. α .c.). Let $x \in X$ and V be any fuzzy open set in Y such that $F(x) \leq V$ (resp., $F(x)qV$). Then there exists $X_0 \in \alpha O(X_0, x)$ such that $F/X_0 : X_0 \rightarrow Y$ is f.u.w. α .c. (resp., f.l.w. α .c.). Then there exists $U_0 \in \alpha O(X_0, x)$ such that $(F/X_0)(U_0) \leq clV$ (resp., $(F/X_0)(u)qclV$, for all $u \in U_0$). By Lemma 4.21(ii), $U_0 \in \alpha O(X, x)$ and $F(U_0) \leq clV$ (resp., $F(u)qclV$, for all $u \in U_0$) $\Rightarrow F$ is f.u.w. α .c. (resp., f.l.w. α .c.).

5. Mutual Relationships

It is clear from Definition 4.1, Definition 4.2 and Definition 4.3 that f.u. α .c. $\Rightarrow$ f.u.a. α .c. $\Rightarrow$ f.u.w. α .c. and similarly f.l. α .c. $\Rightarrow$ f.l.a. α .c. $\Rightarrow$ f.l.w. α .c. but none of the above implications are reversible as seen from the following examples.

Example 5.1. Let $X = \{a, b\}$, $\tau_X = \{\phi, X\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.35, B(y) = 0.4$, for all $y \in Y$. Then (X, τ_X) and (Y, τ_Y) are a topological space and an fts respectively. Let $F : (X, \tau_X) \rightarrow (Y, \tau_Y)$ be a fuzzy multifunction defined by $F(a) = B, F(b) = C$ where $C(y) = 0.6$, for all $y \in Y$. Now $F^+(A) = \phi, F^+(B) = \{a\}$ and $F^+(clB) = F^+(1_Y \setminus B) = X$.

Again $\text{int}(\text{cl}(\text{int}(F^+(\text{cl}B)))) = X \Rightarrow F^+(B) \subseteq \text{int}(\text{cl}(\text{int}(F^+(\text{cl}B)))) \Rightarrow F$ is f.u.w. α .c. by Theorem 4.8(iii) $\Rightarrow$ (i).

But $\text{int}(\text{cl}(\text{int}(F^+(B)))) = \text{int}(\text{cl}(\text{int}(\{a\}))) = \text{int}(\text{cl}(\text{int}(\phi))) = \phi \Rightarrow F^+(B) \not\subseteq \text{int}(\text{cl}(\text{int}(F^+(B)))) \Rightarrow F$ is not f.u. α .c. Now B is fuzzy semiclosed in Y as $\text{int}(\text{cl}B) = \text{int}(1_Y \setminus B) = B$ and so

$$\begin{aligned} \text{int}(\text{cl}(\text{int}(F^+(\text{scl}B)))) &= \text{int}(\text{cl}(\text{int}(F^+(B)))) = \text{int}(\text{cl}(\text{int}(\{a\}))) = \phi \\ &\Rightarrow F^+(B) \not\subseteq \text{int}(\text{cl}(\text{int}(F^+(\text{scl}B)))) \Rightarrow F \end{aligned}$$

is not f.u.a. α .c.

Example 5.2. Let $X = \{a, b\}$, $\tau_X = \{\phi, X\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.4$, $B(y) = 0.55$, for all $y \in Y$. Then (X, τ_X) and (Y, τ_Y) are a topological space and an fts respectively. Let us consider a fuzzy multifunction $F : (X, \tau_X) \rightarrow (Y, \tau_Y)$ defined by $F(a) = B$, $F(b) = C$ where $C(y) = 0.45$, for all $y \in Y$. Here $F^-(A) = \phi$ and so $F^-(A) \subseteq \text{int}(\text{cl}(\text{int}(F^-(\text{cl}A))))$, $F^-(B) = \{a\}$, $F^-(\text{cl}B) = F^-(1_Y \setminus A) = X$ and so $F^-(B) \subseteq \text{int}(\text{cl}(\text{int}(F^-(\text{cl}B)))) \Rightarrow F$ is f.l.w. α .c. But $\text{int}(\text{cl}(\text{int}(F^-(B)))) = \phi$ and so

$$F^-(B) \not\subseteq \text{int}(\text{cl}(\text{int}(F^-(B)))) \Rightarrow F$$

is not f.l. α .c. Again, $\text{int}(\text{cl}B) = \text{int}(1_Y \setminus A) = B \Rightarrow B$ is fuzzy semiclosed. So

$$\begin{aligned} \text{int}(\text{cl}(\text{int}(F^-(\text{scl}B)))) &= \text{int}(\text{cl}(\text{int}(F^-(B)))) = \phi \\ &\Rightarrow F^-(B) \not\subseteq \text{int}(\text{cl}(\text{int}(F^-(\text{scl}B)))) \Rightarrow F \end{aligned}$$

is not f.l.a. α .c.

Example 5.3. f.u.a. α .c. $\not\Rightarrow$ f.u. α .c.

Let $X = \{a, b\}$, $\tau_X = \{\phi, X\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, A, B\}$ where $A(y) = 0.5$, $B(y) = 0.4$, for all $y \in Y$. Then (X, τ_X) and (Y, τ_Y) are a topological space and an fts respectively. Let us consider a fuzzy multifunction $F : (X, \tau_X) \rightarrow (Y, \tau_Y)$ defined by $F(a) = B$, $F(b) = C$ where $C(y) = 0.45$, for all $y \in Y$. Now $F^+(A) = X$ and so $F^+(A) = \text{int}(\text{cl}(\text{int}(F^+(\text{scl}A)))) = \text{int}(\text{cl}(\text{int}(F^+(A)))) = X$ (as A is fuzzy semiopen in Y). $F^+(B) = \{a\}$, $F^+(\text{scl}B) = F^+(A) = X$ (as the collection of all fuzzy semiopen sets in Y is $\{0_Y, 1_Y, U\}$ where $B \leq U \leq A$ and so that the collection of all fuzzy semiclosed sets in Y is $\{0_Y, 1_Y, 1_Y \setminus U\}$ where $1_Y \setminus A \leq 1_Y \setminus U \leq 1_Y \setminus B$). Therefore $F^+(B) \subseteq \text{int}(\text{cl}(\text{int}(F^+(\text{scl}B))))$. Therefore F is f.u.a. α .c. But $\text{int}(\text{cl}(\text{int}(F^+(B)))) = \text{int}(\text{cl}(\text{int}(\{a\}))) = \phi$ and so $F^+(B) \not\subseteq \text{int}(\text{cl}(\text{int}(F^+(B)))) \Rightarrow F$ is not f.u. α .c.

Example 5.4. f.l.a. α .c. $\not\Rightarrow$ f.l. α .c.

Let $X = \{a, b\}$, $\tau_X = \{\phi, X\}$, $Y = [0, 1]$, $\tau_Y = \{0_Y, 1_Y, B\}$ where $B(y) = 0.57$, for all $y \in Y$. Then (X, τ_X) and (Y, τ_Y) are topological space and an fts respectively. Let us consider a fuzzy multifunction $F : (X, \tau_X) \rightarrow (Y, \tau_Y)$ defined by $F(a) = B, F(b) = C$ where $C(y) = 0.4$, for all $y \in Y$. Clearly $F^-(B) = \{a\} \Rightarrow \text{int}(cl(\text{int}(F^-(B)))) = \text{int}(cl(\text{int}(\{a\}))) = \phi \not\supseteq F^-(B) \Rightarrow F$ is not f.l. α .c. But the collection of all fuzzy semiopen sets in Y is $\{0_Y, 1_Y, U\}$ where $U \geq B$ and that of fuzzy semiclosed sets in Y is $\{0_Y, 1_Y, 1_Y \setminus U\}$ where $1_Y \setminus U \leq 1_Y \setminus B$. So $F^-(scl B) = F^-(1_Y) = X$ and so $\text{int}(cl(\text{int}(F^-(scl B)))) = \text{int}(cl(\text{int}(X))) = X \supseteq F^-(B) \Rightarrow F$ is f.l.a. α .c.

Let us now recall a definition from [8] for ready reference.

Definition 5.5. An fts (Y, τ_Y) is said to be fuzzy normal if for any two fuzzy closed sets A, B of Y with $A \not\leq B$, there are fuzzy open sets U, V in Y such that $A \leq U, B \leq V$ and $U \not\leq V$.

Theorem 5.6. If $F : X \rightarrow Y$ is a fuzzy multifunction such that $F(x)$ is fuzzy closed in Y for each $x \in X$ and Y is fuzzy normal space, then the following statements are equivalent:

- (i) F is f.u. α .c.
- (ii) F is f.u.a. α .c.
- (iii) F is f.u.w. α .c.

Proof. Obviously (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). So we have to prove only (iii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (i) Let $x \in X$ and V be any fuzzy open set in Y with $F(x) \leq V$. Then $F(x) \not\leq (1_Y \setminus V)$ where $F(x)$ and $1_Y \setminus V$ are fuzzy closed sets in Y . Since Y is fuzzy normal, there exist fuzzy open sets U, W in Y such that $F(x) \leq U, 1_Y \setminus V \leq W$ and $U \not\leq W$. Then $U \leq 1_Y \setminus W \Rightarrow cl U \leq cl(1_Y \setminus W) = 1_Y \setminus W$ and so $F(x) \leq U \leq cl U \leq 1_Y \setminus W \leq V \Rightarrow F(x) \leq U \leq cl U \leq V$. Since F is f.u.w. α .c., there exists $U_1 \in \alpha O(X, x)$ such that $F(U_1) \leq cl U \leq V \Rightarrow F$ is f.u. α .c.

Definition 5.7. A fuzzy multifunction $F : X \rightarrow Y$ is said to be fuzzy α -preopen if $F(U) \leq \text{int}(cl(F(U)))$, for all $U \in \alpha O(X, x)$.

Theorem 5.8. If a fuzzy multifunction $F : X \rightarrow Y$ is f.u.w. α .c. and fuzzy α -preopen, then F is f.u.a. α .c.

Proof. Let $x \in X$ and V be any fuzzy open set in Y with $F(x) \leq V$. Since F is f.u.w. α .c., there exists $U \in \alpha O(X, x)$ such that $F(U) \leq cl V$. Since F is fuzzy α -preopen, $F(U) \leq \text{int}(cl(F(U))) \leq \text{int}(cl(cl V)) = \text{int}(cl V) = scl V$, as V is fuzzy open in Y (by Lemma 3.8) $\Rightarrow F$ is f.u.a. α .c. (by Theorem 4.6 (i) $\Rightarrow$ (ii)).

Theorem 5.9. Let $F : X \rightarrow Y$ be a fuzzy multifunction such that $F(x)$ is fuzzy open in Y for each $x \in X$. Then the following statements are equivalent:

- (i) F is f.l. α .c.
- (ii) F is f.l.a. α .c.
- (iii) F is f.l.w. α .c.

Proof. Obviously (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). So we prove only (iii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (i) Let $x \in X$ and V be any fuzzy open set in Y with $F(x)qV$. Then by (iii), there exists $U \in \alpha O(X, x)$ such that $F(u)qclV$, for all $u \in U$. Since by hypothesis, $F(u)$ is fuzzy open in Y , $F(u)qV$, for all $u \in U$ (if not, then $F(u) \not qV$ for some $u \in U \Rightarrow F(u) \not qclV$, as $F(u)$ is fuzzy open in Y , a contradiction) $\Rightarrow F$ is f.l. α .c.

Let us now recall a definition from [9] for ready reference.

Definition 5.10. An fts (Y, τ_Y) is called fuzzy regular if for any fuzzy set A in Y and any fuzzy open set U in Y with AqU , there exists a fuzzy open set V in Y such that $AqV \leq clV \leq U$.

Theorem 5.11. Let $F : X \rightarrow Y$ be f.l.w. α .c. where Y be fuzzy regular. Then F is f.l.a. α .c.

Proof. Let $x \in X$ and V be fuzzy open set in Y such that $F(x)qV$. Since Y is fuzzy regular, there exists a fuzzy open set U in Y such that $F(x)qU \leq clU \leq V$. As F is f.l.w. α .c., there exists $W \in \alpha O(X, x)$ such that $F(w)qclU$, for all $w \in W \Rightarrow F(w)qV$, for all $w \in W \Rightarrow F(w)qsclV$, for all $w \in W \Rightarrow F$ is f.l.a. α .c.

We now recall the following lemma from [20].

Lemma 5.12 [20]. If a subset A of a topological space X is α -open and α -closed, then A is clopen in X .

Lemma 5.13. If a fuzzy multifunction $F : X \rightarrow Y$ is f.u.w. α .c. and f.l.w. α .c., then $F^+(V)$ is clopen in X for every fuzzy clopen set V of Y ,

Proof. Let V be fuzzy clopen in Y . As F is f.u.w. α .c., then by Theorem 4.8 (i) $\Rightarrow$ (viii), $F^+(V) \subseteq \alpha int(F^+(clV)) = \alpha int(F^+(V)) \Rightarrow F^+(V)$ is α -open in X . Again, since F is f.l.w. α .c., by Theorem 4.9 (i) $\Rightarrow$ (x), $\alpha cl(F^+(V)) \subseteq F^+(clV) = F^+(V) \Rightarrow F^+(V)$ is α -closed in X . So $F^+(V)$ is α -clopen in X . By Lemma 5.12, $F^+(V)$ is clopen in X .

6. Applications

We now recall the definition of connectedness of an fts from [22].

Definition 6.1. Two fuzzy sets A and B in an fts (Y, τ_Y) such that $A \not qB$ and $A(x) + B(x) < 1$ for $x \in A_0 \cap B_0$ are said to be weakly separated iff

$clA \not\leq B$ and $A \not\leq clB$. (Y, τ_Y) is said to be fuzzy disconnected iff there are two non-zero weakly separated fuzzy sets A and B ($A_0 \neq B_0$) in (Y, τ_Y) such that $A \cup B = 1_Y$. An fts (Y, τ_Y) is said to be fuzzy connected iff Y is not fuzzy disconnected.

Remark 6.2. From Definition 6.1, we claim that (Y, τ_Y) is fuzzy disconnected if there exist two non-zero fuzzy open sets A and B with $A \not\leq B$ such that $A \cup B = 1_Y$ and $A \wedge B = 0_Y$. Indeed, $A \cup B = 1_Y$ and $A(x) + B(x) < 1$, for $x \in A_0 \cap B_0 \Rightarrow A_0 \cap B_0 = \phi \Rightarrow A \wedge B = 0_Y$ and so $A = 1_Y \setminus B$ and so $B = 1_Y \setminus A$. Now $clA \not\leq B \Rightarrow clA \leq 1_Y \setminus B$. Again, $clA = cl(1_Y \setminus B)$ and so $cl(1_Y \setminus B) \leq 1_Y \setminus B \Rightarrow 1_Y \setminus B$ is fuzzy closed in Y and so B is fuzzy open in Y . Similarly we can show that A is fuzzy open in Y . As $B = 1_Y \setminus A$, B is fuzzy closed in Y and similarly A is fuzzy closed in Y .

So we can rewrite the definitions of fuzzy connectedness of a fuzzy set and an fts as follows.

Definition 6.3. A fuzzy set A in an fts (Y, τ_Y) is called disconnected if there exist two non-zero open sets U, V in Y such that $A = U \vee V$ and $U \wedge V = 0_Y$. A is called connected if it is not disconnected.

Definition 6.4. An fts (Y, τ_Y) is called disconnected if there exist two non-zero open sets U, V in Y such that $U \vee V = 1_Y$ and $U \wedge V = 0_Y$. Y is called connected if it is not disconnected.

Definition 6.5. A topological space is said to be disconnected if there exist two non-empty open sets U, V in X such that $U \cup V = X$ and $U \cap V = \phi$. X is called connected if it is not disconnected.

Result 6.6. For a fuzzy multifunction $F : X \rightarrow Y$,

$$F^+(U \wedge V) = F^+(U) \cap F^+(V)$$

for any two fuzzy sets U, V in Y .

Proof. Let $x \in F^+(U) \cap F^+(V)$. Then $F(x) \leq U$ and $F(x) \leq V \Rightarrow F(x) \leq U \wedge V \Rightarrow x \in F^+(U \wedge V)$ and so $F^+(U) \cap F^+(V) \subseteq F^+(U \wedge V)$. Retracing the above steps, we get $F^+(U \wedge V) \subseteq F^+(U) \cap F^+(V)$.

Result 6.7. For a fuzzy multifunction $F : X \rightarrow Y$,

$$F^+(U \vee V) = F^+(U) \cup F^+(V)$$

for any two fuzzy sets U, V in Y .

Proofs. Let $x \in F^+(U) \cup F^+(V)$. Then $F(x) \leq U \vee V \Rightarrow F(x) \leq U$ or $F(x) \leq V \Rightarrow x \in F^+(U)$ or $x \in F^+(V) \Rightarrow x \in F^+(U) \cup F^+(V)$

and so $F^+(U \cup V) \subseteq F^+(U) \cup F^+(V)$. Retracing the above steps, we get $F^+(U) \cup F^+(V) \subseteq F^+(U \cup V)$.

Theorem 6.8. Let $F : X \rightarrow Y$ be an f.u.w. α .c. and f.l.w. α .c. surjective multifunction. If X is connected and $F(x)$ is fuzzy connected for each $x \in X$, then Y is fuzzy connected.

Proof. If possible, let Y be fuzzy disconnected. Then there exist two fuzzy non-zero open sets U, V in Y such that $U \vee V = 1_Y$ and $U \wedge V = 0_Y$. Then $F^+(U \vee V) = F^+(1_Y) = X$ and $F^+(U \wedge V) = F^+(0_Y) = \phi$. Since $F(x)$ is fuzzy connected for each $x \in X$, then either $F(x) \leq U$ or $F(x) \leq V$. Then either $x \in F^+(U)$ or $x \in F^+(V) \Rightarrow x \in F^+(U) \cup F^+(V)$, for each $x \in X \Rightarrow X \subseteq F^+(U) \cup F^+(V) = F^+(U \vee V)$ by Result 6.7. Since F is surjective and U, V are non-zero fuzzy sets in Y , there exist $x, y \in X$ such that $F(x) \leq U, F(y) \leq V \Rightarrow x \in F^+(U), y \in F^+(V)$ and so $F^+(U) \neq \phi \neq F^+(V)$. Therefore, we have

- (i) $X = F^+(U \vee V) = F^+(U) \cup F^+(V)$
- (ii) $F^+(U) \cap F^+(V) = F^+(U \cap V) = \phi$ (by Lemma 6.6),
- (iii) $F^+(U) \neq \phi, F^+(V) \neq \phi$.

Again U, V are fuzzy open in Y with $U \vee V = 1_Y, U \wedge V = 0_Y \Rightarrow U = 1_Y \setminus V, V = 1_Y \setminus U$ which shows that U and V are fuzzy clopen in Y . By Lemma 5.13, $F^+(U)$ and $F^+(V)$ are clopen in X . Then (i), (ii) and (iii) together proves that X is disconnected, a contradiction.

Definition 6.9 [20]. The α -frontier of a subset A of a topological space X , denoted by $\alpha Fr(A)$, is defined by $\alpha Fr(A) = \alpha cl A \cap \alpha cl(X \setminus A) = \alpha cl A \setminus \alpha int A$.

Theorem 6.10. Let $F : X \rightarrow Y$ be a fuzzy multifunction. Let $A = \{x \in X : F \text{ is not f.u.w.}\alpha.c. \text{ at } x\}$, $B = \bigcup \{\alpha Fr(F^+(clV)) : F(x) \leq V \text{ and } V \text{ is fuzzy open in } Y\}$. Then $A = B$.

Proof. Let $x \in X$ be such that F be not f.u.w. α .c. at x . Then there exists a fuzzy open set V in Y with $F(x) \leq V$ but $U \not\subseteq F^+(clV)$, for all $U \in \alpha O(X, x)$. Then $U \cap (X \setminus F^+(clV)) \neq \phi \Rightarrow x \in \alpha cl(X \setminus F^+(clV)) = X \setminus \alpha int(F^+(clV)) \Rightarrow x \notin \alpha int(F^+(clV))$. As $x \in F^+(V) \subseteq F^+(clV) \subseteq \alpha cl(F^+(clV)) \Rightarrow x \in \alpha Fr(F^+(clV)) \Rightarrow A \subseteq B$.

Conversely, let $x \in X$ and V be fuzzy open set in Y with $F(x) \leq V$ such that $x \in \alpha Fr(F^+(clV))$. If possible, let F be f.u.w. α .c. at x . Then there exists $U \in \alpha O(X, x)$ such that $U \subseteq F^+(clV)$. Then $x \in U = \alpha int U \subseteq \alpha int(F^+(clV)) \Rightarrow x \in \alpha int(F^+(clV)) \Rightarrow x \notin \alpha Fr(F^+(clV))$, a contradiction and hence F is not f.u.w. α .c. at x . Therefore, $B \subseteq A$. As a result, $A = B$.

Theorem 6.11. Let $F : X \rightarrow Y$ be a fuzzy multifunction. Let $A = \{x \in X : F \text{ is not f.l.w.}\alpha.c. \text{ at } x\}$, $B = \bigcup \{\alpha Fr(F^-(clV)) : F(x) q V \text{ and } V \text{ is fuzzy open in } Y\}$. Then $A = B$.

Proof. Let $x \in X$ be such that F be not f.l.w. α .c. at x . Then there exists a fuzzy open set V in Y with $F(x)qV$ but $U \not\subseteq F^-(clV)$, for all $U \in \alpha O(X, x)$. Then $U \cap (X \setminus F^-(clV)) \neq \emptyset \Rightarrow x \in \alpha cl(X \setminus F^-(clV)) = X \setminus \alpha int(F^-(clV)) \Rightarrow x \notin \alpha int(F^-(clV))$. As $x \in F^-(V) \subseteq F^-(clV) \subseteq \alpha cl(F^-(clV)) \Rightarrow x \in \alpha Fr(F^-(clV)) \Rightarrow A \leq B$.

Conversely, let $x \in X$ and V be fuzzy open set in Y with $F(x)qV$ such that $x \in \alpha Fr(F^-(clV))$. If possible, let F be f.l.w. α .c. at x . Then there exists $U \in \alpha O(X, x)$ such that $U \subseteq F^-(clV)$. Then $x \in U = \alpha int U \subseteq \alpha int(F^-(clV)) \Rightarrow x \in \alpha int(F^-(clV)) \Rightarrow x \notin \alpha Fr(F^-(clV))$, a contradiction and hence F is not f.l.w. α .c. at x . Therefore, $B \leq A$. As a result, $A = B$.

Definition 6.12. A fuzzy set A in an fts Y is said to be a fuzzy upper (lower) C -nbd of a fuzzy set B in Y if there exists a fuzzy open set V in Y such that BqV (resp., $B \leq V$) and $clV \not\subseteq (1_Y \setminus A)$.

Theorem 6.13. A fuzzy multifunction $F : X \rightarrow Y$ is f.u.w. α .c. on X iff for each point $x_0 \in X$ and each fuzzy upper (lower) C -nbd M of $F(x_0)$, $F^+(M)$ (resp., $F^-(M)$) is an α -nbd of x_0 .

Proof. Let F be f.u.w. α .c. (resp., f.l.w. α .c.) on X . Then for any $x_0 \in X$ and for any fuzzy upper (lower) C -nbd M of $F(x_0)$, there exists a fuzzy open set V in Y such that $F(x_0) \leq V$ (resp., $F(x_0)qV$) and $clV \not\subseteq (1_Y \setminus M) \Rightarrow clV \leq M$. Since F is f.u.w. α .c. (resp., f.l.w. α .c.) at x_0 , there exists $U \in \alpha O(X, x_0)$ such that $U \subseteq F^+(clV)$ (resp., $U \subseteq F^-(clV)$) $\subseteq F^+(M)$ (resp., $U \subseteq F^-(M)$) $\Rightarrow F^+(M)$ (resp., $F^-(M)$) is an α -nbd of x_0 .

Conversely, let for any $x_0 \in X$ and any fuzzy open set V of Y with $F(x_0) \leq V$ (resp., $F(x_0)qV$), we have $clV \not\subseteq (1_Y \setminus clV) \Rightarrow clV$ is a fuzzy upper (lower) C -nbd of $F(x_0)$. Then by hypothesis, $F^+(clV)$ (resp., $F^-(clV)$) is an α -nbd of $x_0 \Rightarrow$ there exists $U \in \alpha O(X, x_0)$ such that $x_0 \in U \subseteq F^+(clV)$ (resp., $x_0 \in U \subseteq F^-(clV)$) $\Rightarrow F$ is f.u.w. α .c. (resp., f.l.w. α .c.) at $x_0 \Rightarrow F$ is f.u.w. α .c. (resp., f.l.w. α .c.) on X , as x_0 is an arbitrary point of X .

Definition 6.14 [11]. A topological space X is said to be α -compact if every cover of X by α -open sets of X has a finite subcover.

Definition 6.15 [7]. A fuzzy set A in an fts (Y, τ_Y) is said to be fuzzy compact set if every fuzzy open cover of A has a finite subcover for A .

Definition 6.16. An fts (Y, τ_Y) is called fuzzy s^* -compact if every fuzzy open cover $\mathcal{U}$ of Y has a finite s -proximate subcover (i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{sclU : U \in \mathcal{U}_0\}$ covers Y).

Theorem 6.17. Let $F : X \rightarrow Y$ be an f.u.w. α .c. surjective multifunction and $F(x)$ be a fuzzy compact set in Y for each $x \in X$. If X is α -compact, then

Y is fuzzy almost compact.

Proof. Let $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of Y . Now for each $x \in X$, $F(x)$ is fuzzy compact in Y and so there is a finite subset Λ_x of Λ such that $F(x) \leq \bigcup\{A_\alpha : \alpha \in \Lambda_x\}$. Let $A_x = \bigcup\{A_\alpha : \alpha \in \Lambda_x\}$. Then A_x is fuzzy open in Y such that $F(x) \leq A_x$. Since F is f.u.w. α .c. on X , there exists $U_x \in \alpha O(X, x)$ such that $U_x \subseteq F^+(clA_x)$. Then $\mathcal{U} = \{U_x : x \in X\}$ is a cover of X by α -open sets of X . As X is α -compact, there are finitely many points $x_1, x_2, \dots, x_n$ in X such that $X = \bigcup_{i=1}^n U_{x_i}$. As F is surjective, $1_Y = F(X) = F(\bigcup_{i=1}^n U_{x_i}) = \bigcup_{i=1}^n F(U_{x_i}) \leq \bigcup_{i=1}^n clA_{x_i} = \bigcup_{i=1}^n cl(\bigcup_{\alpha \in \Lambda_{x_i}} A_\alpha) = \bigcup_{i=1}^n (\bigcup_{\alpha \in \Lambda_{x_i}} clA_\alpha) \Rightarrow Y$ is fuzzy almost compact.

Now we state the following two theorems, the proofs of these two are analogue to the proof of Theorem 6.17.

Theorem 6.18. Let $F : X \rightarrow Y$ be an f.u.a. α .c. surjective multifunction and $F(x)$ be a fuzzy compact set in Y for each $x \in X$. If X is α -compact, then Y is fuzzy s^* -compact.

Theorem 6.19. Let $F : X \rightarrow Y$ be an f.u. α .c. surjective multifunction and $F(x)$ be a fuzzy compact set in Y for each $x \in X$. If X is α -compact, then Y is fuzzy compact.

References

- [1] D. Andrijević, Some properties of the topology of α -sets, *Mat. Vesnik*, **36** (1984), 1-10.
- [2] K.K. Azad, On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity, *J.Math. Anal. Appl.*, **82** (1981), 14-32.
- [3] Anjana Bhattacharyya, Concerning almost quasi continuous fuzzy multifunctions, *Universitatea Din Bacău Studii Şi Cercetări Ştiinţifice, Seria: Matematică*, Nr. **11** (2001), 35-48.
- [4] Anjana Bhattacharyya, Fuzzy γ -continuous multifunction, *Int. Journal of Advanced Research in Science and Engineering*, **4** Issue No. **02** (2015), 195-209.
- [5] A.S. Bin Shahna, On fuzzy strong semicontinuity and fuzzy precontinuity, *Fuzzy Sets and Systems*, **44** (1991), 303-308.
- [6] C.L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, **24** (1968), 182-190.
- [7] S. Ganguly, S. Saha, A note on compactness in fuzzy setting, *Fuzzy Sets and Systems*, **34** (1990), 117-124.
- [8] B. Hutton, Normality in fuzzy topological spaces, *J. Math. Anal. Appl.*, **50** (1975), 74-79.

- [9] B. Hutton, I. Reilly, Separation axioms in fuzzy topological spaces, *Fuzzy Sets and Systems*, **31** (1980), 93-104.
- [10] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, **70** (1963), 36-41.
- [11] S.N. Maheshwari, S.S. Thakur, On α -compact spaces, *Bull. Inst. Math. Acad. Sinica*, **13** (1985), 341-347.
- [12] A.S. Mashhour, M.E. Abd El-Monsef, S.N. El-Deeb, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt*, **53** (1982), 47-53.
- [13] A.S. Mashhour, I.A. Hasanein, S.N. El-Deeb, α -continuous and α -open mappings, *Acta Math. Hungar.*, **41** (1983), 213-218.
- [14] M.N. Mukherjee, S. Malakar, On almost continuous and weakly continuous fuzzy multifunctions, *Fuzzy Sets and Systems*, **41** (1991), 113-125.
- [15] M.N. Mukherjee, S.P. Sinha, Almost compact fuzzy sets in fuzzy topological spaces, *Fuzzy Sets and Systems*, **38** (1990), 389-396.
- [16] M.N. Mukherjee, S.P. Sinha, Fuzzy θ -closure operator on fuzzy topological spaces, *Internat. J. Math. and Math. Sci.*, **14**, No. 2 (1991), 309-314.
- [17] T. Noiri, On semi-continuous mappings, *Atti Acad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur.* **54**, No. 8 (1973), 210-214.
- [18] T. Noiri, A note on semi-continuous mappings, *Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **55**, No. 8 (1973), 400-403.
- [19] N.S. Papageorgiou, Fuzzy topology and fuzzy multifunctions, *Jour. Math. Anal. Appl.*, **109** (1985), 397-425.
- [20] V. Popa, T. Noiri, On upper and lower weakly α -continuous multifunctions, *Novi. Sad. J. Math.*, **32**, No. 1 (2002), 7-24.
- [21] Pao Ming Pu, Ying Ming Liu, Fuzzy topology I. Neighbourhood structure of a fuzzy point and Moore-Smith convergence, *Jour. Math. Anal. Appl.*, **76** (1980), 571-599.
- [22] S. Saha, Local connectedness in a fuzzy setting, *Simon Stevin*, **61** (1987), 3-13.
- [23] L.A. Zadeh, Fuzzy Sets, *Inform. Control*, **8** (1965), 338-353.