HOMOTOPY SECTIONS OF ONE DIMENSIONAL IDEALS OVER A LAURENT POLYNOMIAL RING

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Abstract: Let R be a commutative Noetherian ring and I be a one dimensional ideal of the Laurent polynomial ring $R[X, X^{-1}]$ that contains a doubly monic polynomial. Define $I(1) := \langle \{f(1) : f \in I\} \rangle$. Let I/I^2 be generated by $n \geq 2$ elements over $R[X, X^{-1}]/I$. Then any set of n generators of $I(1)$ over R may not be lifted to a set of n generators of I over $R[X, X^{-1}]$ by describing an example of one dimensional ideal.

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1. Introduction

Let R be a commutative Noetherian ring of finite Krull dimension with identity and I be an ideal of the Laurent polynomial ring $R[X, X^{-1}]$. Let M be a finitely generated R -module. We denote $\mu(M)$ to be the least number of elements required to generate M as an R -module. The comparison of the minimal numbers of generators of I and of its module I/I^2 , denoted by $\mu(I)$ and $\mu(I/I^2)$, has been an enliven area of interest (see [1], [5], [6], [11], [14], [15] and [17]). The co-normal bundle I/I^2 of an ideal I in a ring $R[X, X^{-1}]$ is an $R[X, X^{-1}]/I$ -module. Many algebraic properties of this module are intertwined with those

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of the ideal I . Define $I(1) = \langle \{f(1) : f \in I\} \rangle$. In general, $\mu(I(1)) \leq \mu(I)$ and $\mu(I/I^2) \leq \mu(I) \leq \mu(I/I^2) + 1$.

Any set of n generators of the maximal ideal $(X_1, X_2, \dots, X_{n-1})$ of $K[X_1, X_2, \dots, X_{n-1}]$ can be lifted (via the map setting X_n to 0) to a set of n generators of the maximal ideal $(X_1, X_2, \dots, X_n)$ of $K[X_1, X_2, \dots, X_n]$. For $n = 2$, let $(F_1(X_1), F_2(X_1))$ generate the ideal (X_1) of $K[X_1]$. We wish to lift these to a set of two generators of maximal ideals (X_1, X_2) of $K[X_1, X_2]$. It is easy to see that we can perform Euclidean transformations on F_1, F_2 and using the Euclidean transformations we may assume that $F_1 = X_1$ and $F_2 = 0$. Then for these generators, the lift (X_1, X_2) works.

Since lifting of generators is a natural problem which have been studied by many mathematicians in various fashion (see [4], [7], [12], [10], [18]). Here we are giving some examples on lifting problem. Let X be an integral, projective variety of co-dimension two, degree d and dimension r . Suppose Y is its general hyperplane section. The problem of lifting generators of minimal degree σ from the homogeneous ideal of X is studied in ([12]). Answers in terms of relations between d and σ are known for $r = 1, 2$. For $r = 1$, the Laudal's "Generalized trisecant lemma" is given in ([2], [9]). For $r = 2$, there is the analogous result in ([13]). In ([18]), let X be an integral, projective variety of dimension n and degree d in P^N , defined over an algebraically closed field K of characteristic zero. Consider the hyperplane section $Y = X \cap K$ of X , where $K \cong P^{N-1}$ is a general hyperplane in P^N . The "Lifting problem" is the problem of finding conditions on d, N, n and s such that any degree s hypersurface in P^{n-1} containing Y can be lifted to a hypersurface in P^N containing X . Motivated by this problem, in [10] Mandal proved that the following:

Problem 1.1. [10] If R is a commutative Noetherian ring, I is an ideal of Laurent polynomial ring $R[X, X^{-1}]$ containing a doubly monic polynomial, and I/I^2 is generated by n elements over $R[X, X^{-1}]/I$, where

$$n \geq \dim(R[X, X^{-1}]/I) + 2,$$

then I is generated by n elements over $R[X, X^{-1}]$. Note that $I(1)$ is also generated by n elements, where $I(1)$ is an ideal of R by putting $X = 1$.

In ([4]), lifting problem in various cases has been discussed. If I is an ideal of $R[X]$, then $I(0) := \langle \{f(0) : f \in I\} \rangle$ is also an ideal of R so it is very natural to ask whether a set of generators for $I(0)$ can be lifted to a set of generators for I . One interesting consequence of this is in the Laurent case. We consider the problem for one dimensional ideals of Laurent polynomial rings. A natural question is: can we lift any set of n generators of $I(1)$ to a set of

n generators of I ? The objective of the paper is to come out with a negative answer of the above problem in case of the one dimensional ideals, i.e., a set of n generators of $I(1)$ may not be lifted to a set of n generators of I .

2. Preliminaries

In this section, we define some terms used in this paper and provide certain standard results without proof. We hope that this will improve the readability and understanding of proof of the paper.

An ideal I of a ring R is called zero dimensional if every prime ideal of R containing I is a maximal ideal, that is, $\dim(R/I) = 0$. For example every maximal ideal is a zero dimensional ideal. Further if I is the intersection of finitely many maximal ideals, then $\dim(R/I) = 0$ and the converse is also true if I is a reduced ideal. An ideal I of a ring R is called one dimensional if $\dim(R/I) = 1$. A polynomial f in Laurent polynomial ring $R[X, X^{-1}]$ is said to be doubly monic polynomial if coefficient of the highest degree term and the lowest degree term are units. We say that an ideal I is efficiently generated if $\mu(I) = \mu(I/I^2)$. We have a theorem of Mandal [10] about efficient generation of ideals.

Theorem 2.1. [10] *Let I be an ideal of $R[X, X^{-1}]$ over a commutative Noetherian ring R . Suppose that I contains a doubly monic polynomial and $\mu(I/I^2) \geq \dim(R[X, X^{-1}]/I) + 2$. Then ideal I is efficiently generated.*

The following theorem proved by Kumar and Mandal in [7] shows the positive result in the case of zero dimensional ideal.

Theorem 2.2. [7] *Let R be a commutative Noetherian ring and I be an ideal of the Laurent polynomial ring $R[X, X^{-1}]$ that contains a doubly monic polynomial. Suppose P is a projective R -module of rank*

$$n \geq \dim(R[X, X^{-1}]/I) + 2.$$

Let $S : P \rightarrow I(1)$ and $\phi : R[X, X^{-1}] \rightarrow I/I^2$ be two surjective homomorphisms such that $\phi(1) \equiv S \bmod (X - 1, I(1)^2)$. Then there exists a surjective homomorphism $\psi : R[X, X^{-1}] \rightarrow I$ such that $\psi(1) = S$ and ψ lifts ϕ .

3. Main Result

Now we give an example to show that a set of n generators of $I(1)$ may not be lifted to a set of n generators of I in the case of one dimensional ideals.

Example 3.1. Let $R = \mathbb{Q}[U, V, W]$, where $\mathbb{Q}$ is the field of rational numbers and $I = (U, V, X + 1 + \phi_p(W))$ be an ideal of $R[X, X^{-1}]$ generated by $(U, V, X + 1 + \phi_p(W))$, where $\phi_p(W)$ is a cyclotomic polynomial in W of degree $p-1$ and $\phi_p(W)$ is irreducible over $\mathbb{Q}$ and $X + 1 + \phi_p(W) \in \mathbb{Q}[W][X, X^{-1}]$ has a root in $\mathbb{Q}[W]$. When we set $X = 1$ the ring acquires a unit W which can not be lifted to original ring. We have $I(1) = (U, V, \phi_p(W))$. Note that $R[X, X^{-1}]/I = \mathbb{Q}[U, V, W, X, X^{-1}]/(U, V, X + 1 + \phi_p(W)) \cong \mathbb{Q}[W, X, X^{-1}]/(X + 1 + \phi_p(W))$ and $R/I(1) = \mathbb{Q}[U, V, W]/(U, V, \phi_p(W)) \cong \mathbb{Q}[W]/\phi_p(W)$ is a field, because $\phi_p(W)$ is irreducible polynomial over $\mathbb{Q}$. Clearly,

$$\dim(R[X, X^{-1}]/I) = \dim(\mathbb{Q}[W, X, X^{-1}]/(X + 1 + \phi_p(W))) = 1$$

and $\dim(R/I(1)) = 0$, $\mu(I) = ht(I) = 3$ and $\mu(I(1)) = ht(I(1)) = 3$. This implies that I and $I(1)$ are complete intersection ideals of $R[X, X^{-1}]$ and R respectively. Hence I/I^2 and $I(1)/I(1)^2$ are free modules of rank 3 over $R[X, X^{-1}]/I$ and $R/I(1)$ respectively. It follows that $\{U + I(1)^2, V + I(1)^2, \phi_p(W) + I(1)^2\}$ is a free basis of $I(1)/I(1)^2$ over $R/I(1)$. Since $\{F_1 + I(1)^2, F_2 + I(1)^2, F_3 + I(1)^2\}$ is also a free basis of $I(1)/I(1)^2$ over $R/I(1)$, where each $F_i = f_i(U, V, W)$ for $i = 1, 2, 3$. So there exists an invertible matrix $T_1 \in GL_3(R/I(1))$ such that

$$\begin{bmatrix} F_1 + I(1)^2 \\ F_2 + I(1)^2 \\ F_3 + I(1)^2 \end{bmatrix} = T_1 \begin{bmatrix} U + I(1)^2 \\ V + I(1)^2 \\ \phi_p(W) + I(1)^2 \end{bmatrix}$$

where $\{F_1, F_2, F_3\}$ generates $I(1)$ and $T_1 = [a_{ij}(U, V, W) + I(1)]$, $1 \leq i, j \leq 3$. Suppose that $\{F_1, F_2, F_3\}$ has a lifting $\{G_1, G_2, G_3\}$ to a generating set of I , where each $g_i(U, V, W, X + 1) = G_i$ and each $g_i(U, V, W, 0) = F_i$, for $i = 1, 2, 3$. It follows that $\{G_1 + I^2, G_2 + I^2, G_3 + I^2\}$ is a free basis of I/I^2 over $R[X, X^{-1}]/I$. Since $\{U + I^2, V + I^2, X + 1 + \phi_p(W) + I^2\}$ is also a free basis of I/I^2 over $R[X, X^{-1}]/I$, there exists an invertible matrix $T_2 \in GL_3(R[X, X^{-1}]/I)$ such that

$$\begin{bmatrix} G_1 + I^2 \\ G_2 + I^2 \\ G_3 + I^2 \end{bmatrix} = T_2 \begin{bmatrix} U + I^2 \\ V + I^2 \\ X + 1 + \phi_p(W) + I^2 \end{bmatrix}.$$

Suppose $T_2 = [b_{ij}(U, V, W, X + 1) + I]$, where $b_{ij}(U, V, W, X + 1) \in R[X, X^{-1}]$, $1 \leq i, j \leq 3$. Let $f(W)$ be a root of $X + 1 + \phi_p(W)$ in $\mathbb{Q}[W]$. Since $R[X, X^{-1}]/I \cong \mathbb{Q}[W, X, X^{-1}]/(X + 1 + \phi_p(W))$, $\text{Det}[b_{ij}(0, 0, W, f(W))]$ is free from W and is a non-zero member of $\mathbb{Q}$. Substituting $X = 1$, we get

$$\begin{bmatrix} F_1 + I(1)^2 \\ F_2 + I(1)^2 \\ F_3 + I(1)^2 \end{bmatrix} = [b_{ij}(U, V, W, 0) + I(1)] \begin{bmatrix} U + I(1)^2 \\ V + I(1)^2 \\ \phi_p(W) + I(1)^2 \end{bmatrix},$$

Let α be a root of $\phi_p(W)$. Then this implies that

$$\text{Det}[a_{ij}(0, 0, \alpha)] = \text{Det}[b_{ij}(0, 0, f(0))],$$

which is a member of $\mathbb{Q}$. Thus for an example, if we take

$$T_1 = \begin{bmatrix} W + I(1) & 0 + I(1) & 0 + I(1) \\ 0 + I(1) & 1 + I(1) & 0 + I(1) \\ 0 + I(1) & 0 + I(1) & 1 + I(1) \end{bmatrix},$$

then

$$\text{Det} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \alpha,$$

which is not a member of $\mathbb{Q}$, for $\phi_p(W)$ is irreducible over $\mathbb{Q}$ of degree greater than 1. Therefore,

$$\begin{bmatrix} F_1 + I(1)^2 \\ F_2 + I(1)^2 \\ F_3 + I(1)^2 \end{bmatrix} = \begin{bmatrix} W + I(1) & 0 + I(1) & 0 + I(1) \\ 0 + I(1) & 1 + I(1) & 0 + I(1) \\ 0 + I(1) & 0 + I(1) & 1 + I(1) \end{bmatrix} \begin{bmatrix} U + I(1)^2 \\ V + I(1)^2 \\ \phi_p(W) + I(1)^2 \end{bmatrix}.$$

Hence, $\{F_1, F_2, F_3\}$ does not have a lifting to a generating set of I , that is, the generating set $\{U, V, \phi_p(W)\}$ of $I(1)$ can not be lifted to a generating set of I .

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