STURM-LIOUVILLE DIFFERENTIAL EQUATION WITH NON-LOCAL BOUNDARY CONDITIONS

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Abstract: We investigate the existence and some general properties of eigenvalues and eigenfunctions of a nonlocal boundary value problem of the Sturm-Liouville differential equation.

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1. Introduction

Many interesting applications of differential equations arise in recent years with nonlocal condition often appear in Mathematics, mechanics, physics, geophysics and other branches of natural sciences (see [1]-[5] and [10]-[13]). Afterwards

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the number of differential problems with nonlocal boundary conditions had increased. Quite new area, related to problems of this type, deals with investigation of the spectrum of Sturm-Liouville with nonlocal conditions.

Recently, in ([6]-[8]) the authors studies the existence and some asymptotic properties of the eigenvalues and eigenfunctions of the boundary value problem of the Sturm- Liouville differential equation

$$-y'' + q(x)y = \lambda^2 y, \quad 0 \leq x \leq \pi, \quad (1)$$

with different kinds of nonlocal boundary conditions.

Consider the nonlocal boundary value problem of the Sturm-Liouville equation (1) with the non-local conditions

$$y'(0) - Hy(0) = 0, \quad y(\xi) = 0, \quad \xi \in (0, \pi], \quad (2)$$

where the non-negative real function $q(x)$ has a second piecewise integrable derivatives on $(0, \pi)$, H is real and λ is spectral parameter.

Here we study the existence and some general properties of the eigenvalues and eigenfunctions of the two non-local boundary value problems (1) and (2). Comparison with the local boundary value problem problem of equation (1) with the local boundary value problem

$$y'(0) - Hy(0) = 0, \quad y(\pi) = 0$$

will be given.

2. General Properties

Here we prove some results concerning the eigenvalues and eigenfunctions of the nonlocal problem (1)-(2).

Lemma 1. The eigenvalues of the nonlocal boundary value problem (1) and (2) are real.

Proof. Let $y_0(x)$ be the eigenfunction that corresponds to the eigenvalue λ_0 of the problem (1) and (2), then

$$-y_0'' + q(x)y_0 = \lambda_0^2 y_0 \quad (0 \leq x \leq \pi), \quad (3)$$

and

$$y_0(0) - Hy_0(0) = y_0(\xi) = 0. \quad (4)$$

Multiplying both sides of (3) by $\bar{y}_0$ and then integrating from 0 to ξ with respect to x , we have

$$-\bar{y}_0 y_0'|_0^\xi + \int_0^\xi |y_0'|^2 dx + \int_0^\xi q(x) |y_0|^2 dx = \lambda_0^2 \int_0^\xi |y_0|^2 dx.$$

Using the boundary condition (4), we have

$$\lambda_0^2 = \frac{\int_0^\xi [q(x) |y_0|^2 + |y_0'|^2] dx + H}{\int_0^\xi |y_0|^2 dx}.$$

From which it follows the reality of λ_0^2 .

Lemma 2. The eigenfunctions that corresponds to two different eigenvalues of the non-local boundary value problem (1) and (2) are orthogonal.

Proof. Let $\lambda_1 \neq \lambda_2$ be two different eigenvalues of the non-local boundary value problem (1) and (2). Let $y_1(x), y_2(x)$ be the corresponding eigenfunctions, then

$$-y_1'' + q(x)y_1 = \lambda_1^2 y_1 \quad (0 \leq x \leq \pi), \quad (5)$$

$$y_1(0) - Hy_1(0) = y_1(\xi) = 0 \quad (6)$$

and

$$-y_2'' + q(x)y_2 = \lambda_2^2 y_2 \quad (0 \leq x \leq \pi), \quad (7)$$

$$y_2(0) - Hy_2(0) = y_2(\xi) = 0 \quad (8)$$

Multiplying both sides of (5) by $\bar{y}_2$ and integrating with respect to x , we obtain

$$-\int_0^\xi y_1'' \bar{y}_2 dx + \int_0^\xi q(x) y_1 \bar{y}_2 dx = \lambda_1^2 \int_0^\xi y_1 \bar{y}_2 dx, \quad (9)$$

by taking the complex conjugate of (7) and multiply it by y_1 and integrate the resulting expression with respect to x , we have

$$-\int_0^\xi y_1 \bar{y}_2'' dx + \int_0^\xi q(x) y_1 \bar{y}_2 dx = \lambda_2^2 \int_0^\xi y_1 \bar{y}_2 dx. \quad (10)$$

Subtracting (9) from (10) and using the boundary conditions of (6) and (8) we obtain

$$(\lambda_1^2 - \lambda_2^2) \int_0^\xi y_1 \bar{y}_2 dx = 0, \quad \lambda_1^2 \neq \lambda_2^2.$$

which completes the proof.

3. The Asymptotic Formulas for the Solution

Here we study the asymptotic formulas for the solutions of problem (1) and (2).

Lemma 1.1 deals with the nature the eigenvalues. Let be $\phi(x, \lambda)$ the solution of equation (1) and (2) satisfying the initial conditions

$$\phi(0, \lambda) = 1, \quad \phi'(0, \lambda) = H \quad (11)$$

and by $\vartheta(x, \lambda)$ the solution of the same equation, satisfying the initial conditions

$$\vartheta(0, \lambda) = 0, \quad \vartheta'(0, \lambda) = 1. \quad (12)$$

We notes that $\phi(x, \lambda)$ and $\vartheta(x, \lambda)$ are linearly independent if and only if $\omega(\lambda) \neq 0$.

$$\omega(\lambda) = \phi(x, \lambda)\vartheta'(x, \lambda) - \phi'(x, \lambda)\vartheta(x, \lambda).$$

The solution

$$Y(x, \lambda) = \alpha\phi(x, \lambda) + \beta\vartheta(x, \lambda), \text{ atleast } \alpha \text{ or } \beta \neq 0.$$

$Y(x, \lambda)$ as eigenfunction must satisfy the first condition (2), we have

$$y'(0, \lambda) - Hy(0, \lambda) = 0,$$

and then,

$$\alpha\phi'(0, \lambda) + \beta\vartheta'(0, \lambda) - H(\alpha\phi(0, \lambda) + \beta\vartheta(0, \lambda)) = 0.$$

After using the condition (11), (12), we get

$$\alpha\phi(\xi, \lambda) = 0, \text{ where } \alpha \neq 0,$$

therefore, The characteristic equation will be

$$\omega(\lambda) = \phi(\xi, \lambda). \quad (13)$$

Lemma 3. The solution $\phi(x, \lambda)$ of problem (1) and (2) satisfy the integral equations

$$\phi(x, \lambda) = \cos \lambda x + \frac{H}{\lambda} \sin \lambda x + \int_0^x \frac{\sin \lambda(x - \tau)}{\lambda} q(\tau) \phi(\tau, \lambda) d\tau. \quad (14)$$

Proof. First we obtain formula (14) Indeed, with solution of the form $q(x) = 0$. (1) becomes $-y'' = \lambda^2 y$ by means of variation of parameter method, we have

$$\phi(x, \lambda) = C_1(x, \lambda) \cos \lambda x + C_2(x, \lambda) \sin \lambda x \quad (15)$$

and the direct calculation of $C_1(x, s)$ and $C_2(x, s)$, we have

$$\begin{aligned} C_1(x, \lambda) &= 1 - \int_0^x \frac{\sin \lambda \tau}{\lambda} q(\tau) \phi(\tau, \lambda) d\tau, \\ C_2(x, \lambda) &= \frac{H}{\lambda} + \int_0^x \frac{\cos \lambda \tau}{\lambda} q(\tau) \phi(\tau, \lambda) d\tau, \end{aligned} \quad (16)$$

substituting from (16) into (15) equation (14) follows. Second we show that the integral representation (14) satisfies the problem (1) and (11). Let $\varphi(x, \lambda)$ be the solution of (1), so that

$$q(x) \phi(x, \lambda) = \phi''(x, \lambda) + \lambda^2 \phi(x, \lambda).$$

We multiply both sides by

$$\frac{\sin \lambda(x - \tau)}{\lambda}$$

and integrating with respect to τ from 0 to x we obtain

$$\begin{aligned} \int_0^x \frac{\sin \lambda(x - \tau)}{\lambda} q(\tau) \phi(\tau, \lambda) d\tau &= \int_0^x \frac{\sin \lambda(x - \tau)}{\lambda} \phi''(\tau, \lambda) d\tau \\ &\quad + \lambda^2 \int_0^x \frac{\sin \lambda(x - \tau)}{\lambda} \phi(\tau, \lambda) d\tau. \end{aligned} \quad (17)$$

Integrating by parts twice and using the condition (11), we have

$$\begin{aligned} \int_0^x \frac{\sin \lambda(x - \tau)}{\lambda} \phi''(\tau, \lambda) d\tau \\ = \phi(x, \lambda) - \frac{H}{\lambda} \sin \lambda x - \cos \lambda x - \lambda \int_0^x \sin \lambda(x - \tau) \phi(\tau, \lambda) d\tau. \end{aligned} \quad (18)$$

By substituting from (18) into (17) we get the required formula (14).

Lemma 4. Let $\lambda = \sigma + it$, then there exists $\lambda_0 > 0$, such that $|\lambda| > \lambda_0$ the following relation of the non-local boundary value problem (1) and (2) holds true

$$\phi(x, \lambda) = \cos \lambda x + O\left(\frac{e^{|t|x}}{|\lambda|}\right). \quad (19)$$

Proof. We show first that

$$\phi(x, \lambda) = O\left(e^{|t|x}\right),$$

where the inequality is uniformly with respect to x . Form the integral equation (14) we have

$$|\phi(x, \lambda)| \leq e^{|t|(x)} + \frac{|H|}{|\lambda|} e^{|t|x} + \frac{1}{\lambda} \int_0^x e^{|t|\tau} |q(\tau)| |\phi(\tau, \lambda)| d\tau. \quad (20)$$

By using the notation $\phi(x, \lambda)e^{-|t|x} = F(x, \lambda)$, equation (20) takes the form

$$|F(x, \lambda)| \leq 1 + \frac{|H|}{|\lambda|} + \int_0^\pi \frac{|q(\tau)|}{|\lambda|} |F(\tau, \lambda)| d\tau. \quad (21)$$

Let $\mu = \max_{0 \leq x \leq \pi} F(x, \lambda)$, so that from (21) it follows that

$$\mu \leq \frac{1 + \frac{|H|}{|\lambda|}}{1 - \frac{1}{\lambda} \int_0^\pi |q(\tau)| d\tau}.$$

For $|\lambda| > \lambda_0 = \int_0^\pi |q(\tau)| d\tau$ it follows from the last inequality that $F(x, \lambda) \leq \text{constant} / |\lambda|$ and this implies that

$$\phi(x, \lambda) = O\left(e^{|t|x}\right). \quad (22)$$

By the aid of (21) we find that

$$\int_0^x \frac{\sin \lambda(x - \tau)}{\lambda} q(\tau) \phi(\tau, \lambda) d\tau = O\left(\frac{e^{|t|x}}{|\lambda|}\right). \quad (23)$$

From (14) and (21) it follows that, $\varphi(x, \lambda)$ has the asymptotic formula (19).

Theorem 5. *Let $\lambda = \sigma + it$ and suppose that $q(x)$ has a second order piecewise differentiable derivatives on $[0, \pi]$. Then the solution $\phi(x, \lambda)$ of nonlocal boundary value (1) and (2) have the following asymptotic formula*

$$\begin{aligned} \phi(x, \lambda) &= \cos \lambda x + \frac{\alpha_1(x)}{\lambda} \sin \lambda x + \frac{\alpha_2(x)}{\lambda^2} \cos \lambda x + \frac{\alpha_3(x)}{\lambda^3} \sin \lambda x \\ &+ O\left(\frac{e^{|t|x}}{|\lambda^4|}\right) \end{aligned} \quad (24)$$

where

$$\begin{aligned}
 \alpha_1(x) &= \frac{1}{2} \int_0^x q(t) dt + H, \\
 \alpha_2(x) &= -\frac{1}{4} \left(\int_0^x q(t) dt \right)^2 + \frac{H}{2} \int_0^x q(t) dt + \frac{1}{4} [q(x) - q(0)], \\
 \alpha_3(x) &= -\frac{1}{8} \left(\int_0^x q(t) dt \right)^3 + \frac{H}{8} \left(\int_0^x q(t) dt \right)^2 + \frac{H^2}{4} \int_0^x q(t) dt \\
 &\quad + \frac{H}{4} [q(x) - q(0)] + \frac{1}{8} [q'(x) - q'(0)].
 \end{aligned} \tag{25}$$

Proof. By substituting from (19) into the integral equation (14), we have

$$\begin{aligned}
 \phi(x, \lambda) &= \cos \lambda x + \frac{H}{\lambda} \sin \lambda x + \frac{\sin \lambda x}{2\lambda} \int_0^x q(t) dt \\
 &\quad + \frac{1}{2\lambda} \int_0^x \sin \lambda(x-2t) q(t) dt + O \left(\frac{e^{|Im\lambda|x}}{|\lambda|^2} \right).
 \end{aligned} \tag{26}$$

Integrating the last integration of (26) by parts and noticing that there exists $q'(x)$ such that $q' \in L_1[0, \pi]$

$$\begin{aligned}
 &\frac{1}{2\lambda} \int_0^x \sin \lambda(x-2t) q(t) dt \\
 &= \frac{1}{4} [q(x) - q(0)] \frac{\cos \lambda x}{\lambda^2} - \frac{1}{4\lambda^2} \int_0^x \cos \lambda(x-2t) q'(t) dt \\
 &= O \left(\frac{e^{|Im\lambda|x}}{|\lambda|^2} \right),
 \end{aligned} \tag{27}$$

Substituting (27) in (26), we get

$$\phi(x, \lambda) = \cos \lambda x + \frac{\alpha_1(x)}{\lambda} \sin \lambda x + O \left(\frac{e^{|Im\lambda|x}}{|\lambda|^2} \right), \tag{28}$$

where $\alpha_1(x)$ is defined by (25). In order to make $\phi(x, \lambda)$ more precise we repeat this procedure again by substituting from the last result (28) into the same integral equation (14), we have

$$\phi(x, \lambda) = \cos \lambda x + \frac{H}{\lambda} \sin \lambda x + \int_0^x \frac{\sin \lambda(x-t) \cos \lambda t}{\lambda} q(t) dt$$

$$\begin{aligned}
& + \int_0^x \frac{\sin \lambda(x-t) \sin \lambda x}{\lambda^2} q(t) \alpha_1(t) dt \\
& + \int_0^x \frac{\sin \lambda(x-t)}{\lambda} q(t) O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^2}\right) dt.
\end{aligned} \tag{29}$$

Now we estimate each term in (29). Integrating by parts twice the first term of (29), and noticing that $q'' \in L_1[0, \pi]$, we have

$$\begin{aligned}
& \int_0^x \frac{\sin \lambda(x-t) \cos \lambda t}{\lambda} q(t) dt \\
& = \frac{\sin \lambda x}{2\lambda} \int_0^x q(t) dt + \frac{[q(x) - q(0)]}{4\lambda^2} \cos \lambda x + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^3}\right).
\end{aligned} \tag{30}$$

Similary, we have

$$\begin{aligned}
& \int_0^x \frac{\sin \lambda(x-t) \sin \lambda t}{\lambda^2} q(t) \alpha_1(t) dt \\
& = -\frac{1}{2} \int_0^x \alpha_1(t) q(t) dt \frac{\cos \lambda x}{\lambda^2} + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^3}\right).
\end{aligned} \tag{31}$$

Substituting (30) and (31) in (29), we get

$$\phi(x, \lambda) = \cos \lambda x + \frac{\alpha_1(x)}{\lambda} \sin \lambda x + \frac{\alpha_2}{\lambda^2} \cos \lambda x + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^3}\right), \tag{32}$$

where $\alpha_1(x)$ and $\alpha_2(x)$ is defined by (25). In order to make $\phi(x, \lambda)$ more precise we repeat this procedure again by substituting from the last result (32) into the same integral equation (14), we have

$$\begin{aligned}
\phi(x, \lambda) & = \cos \lambda x + \frac{H}{\lambda} \sin \lambda x + \int_0^x \frac{\sin \lambda(x-t) \cos \lambda t}{\lambda} q(t) dt \\
& + \int_0^x \frac{\sin \lambda(x-t) \sin \lambda t}{\lambda^2} q(t) \alpha_1(t) dt \\
& + \int_0^x \frac{\sin \lambda(x-t) \cos \lambda t}{\lambda^3} q(t) \alpha_2(t) dt \\
& + \int_0^x \frac{\sin \lambda(x-t)}{\lambda} q(t) O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^4}\right) dt.
\end{aligned} \tag{33}$$

Now we estimate each term in (33). Integrating by parts twice the first term of (33), and noticing that $q'' \in L_1[0, \pi]$, we have

$$\begin{aligned} \int_0^x \frac{\sin \lambda(x-t) \cos \lambda t}{\lambda} q(t) dt &= \frac{\sin \lambda x}{2\lambda} \int_0^x q(t) dt \\ &+ \frac{[q(x) - q(0)]}{4\lambda^2} \cos \lambda x + \frac{\sin \lambda x}{8\lambda^3} [q'(x) - q'(0)] + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^4}\right). \end{aligned} \quad (34)$$

Further,

$$\begin{aligned} \int_0^x \frac{\sin \lambda(x-t) \sin \lambda t}{\lambda^2} q(t) \alpha_1(t) dt &= -\frac{\cos \lambda x}{2\lambda^2} \int_0^x \alpha_1(t) q(t) dt \\ &+ \frac{\sin \lambda x}{4\lambda^3} [q(0) \alpha_1(0) - q(x) \alpha_1(x)] + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^3}\right) \end{aligned} \quad (35)$$

and

$$\begin{aligned} \int_0^x \frac{\sin \lambda(x-t) \cos \lambda t}{\lambda^3} q(t) \alpha_2(t) dt \\ = \frac{\sin \lambda x}{2\lambda^3} \int_0^x \alpha_2(t) q(t) dt + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^3}\right). \end{aligned} \quad (36)$$

Substituting from (34)-(36) into (33) we get the required formula (24).

Now inserting the values of the functions $\varphi(x, \lambda)$ from the estimate (24) into the second of the boundary conditions in (2), we obtain the following equation for the determination of the eigenvalues equation (19) is the characteristic equation which gives roots of λ

$$\lambda_n^0 = \left(n + \frac{1}{2}\right) \frac{\pi}{\xi}, \quad n = 0, \pm 1, \pm 2, \dots$$

Then the $\omega(\lambda)$ has the same root of the function $\sin \lambda \xi$ (By Rouché's theorem)

$$\lambda_n = \lambda_n^0 + \varepsilon_n, \quad n = 0, 1, 2, \dots \quad (37)$$

Theorem 6. Let $q \in L_1(0, \pi)$, then we have the following asymptotic formulas for λ_n of non-local boundary value (1) and (2)

$$\lambda_n = \left(n + \frac{1}{2}\right) \frac{\pi}{\xi} + \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right), \quad (38)$$

where $\alpha_1(x)$ defined in (25).

Proof.

$$\omega(x, \lambda) = \cos \lambda \xi + \frac{\alpha_1}{\lambda} \sin \lambda \xi + \frac{\alpha_2}{\lambda^2} \cos \lambda \xi + \frac{\alpha_3}{\lambda^3} \sin \lambda \xi + O\left(\frac{e^{|Im \lambda| \xi}}{|\lambda^4|}\right) \quad (39)$$

It follows from (39) that

$$\cos \lambda \xi + \frac{\alpha_1}{\lambda} \sin \lambda \xi + \frac{\alpha_2}{\lambda^2} \cos \lambda \xi + \frac{\alpha_3}{\lambda^3} \sin \lambda \xi + O\left(\frac{e^{|Im \lambda| \xi}}{|\lambda^4|}\right) = 0. \quad (40)$$

From equation (40), we have

$$\left[1 + \frac{\alpha_2}{\lambda^2}\right] \cos \lambda \xi + \left[\frac{\alpha_1}{\lambda} + \frac{\alpha_3}{\lambda^3}\right] \sin \lambda \xi = 0. \quad (41)$$

Dividing (41) by $\sin \lambda \xi$ we obtain

$$\left[1 + \frac{\alpha_2}{\lambda^2}\right] \cot \lambda \xi = -\left[\frac{\alpha_1}{\lambda} + \frac{\alpha_3}{\lambda^3}\right]$$

since imaginary $\lambda = O\left(\frac{1}{n}\right)$, then

$$\cot \lambda_n \xi = -\frac{\alpha_1}{\lambda_n} + \frac{\alpha_1 \alpha_2}{\lambda_n^3} - \frac{\alpha_3}{\lambda_n^3} + O\left(\frac{1}{n^4}\right), \quad (42)$$

from (37), (42) after elementary calculation, we obtain

$$\varepsilon_n = \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right) \quad (43)$$

From (37) and (43), we have

$$\lambda_n = \left(n + \frac{1}{2}\right) \frac{\pi}{\xi} + \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right).$$

Corollary 7. *If $\xi = \pi$, then the eigenvalues of (38), we obtain*

$$\lambda_n = n + \frac{1}{2} + \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right).$$

Which meets with the result obtained in [9]

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