CONNECTED DOMINATION PATH DECOMPOSITION OF TRIANGULAR SNAKE GRAPH

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Abstract: In this paper, we unite connected domination and decomposition to introduce a new concept called connected domination decomposition (CDD). We also establish connected domination path decomposition of a graph G . In particular, we investigate connected domination path decomposition of triangular snake T_m .

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1. Introduction

All basic terminologies from graph theory are used in this paper in the sense of Frank Harary [4]. By a graph considered here are simple undirected graph without loops or multiple edges. As usual p , q , m and s denote the vertices, edges, blocks and cut vertices of a graph G respectively.

The theory of domination is one of the fastest growing area in graph theory. Which has been investigated by S.T. Hedetniemi, C.J. Cockayne et al.

A set $D \subseteq V$ of vertices in a graph G is a dominating set if every vertex v in $V - D$ is adjacent to a vertex in D . The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. C.J. Cockayne and S.T. Hedetniemi was introduced the notation $\gamma(G)$ in 1977.

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A dominating set D of a graph G is a connected dominating set if the induced subgraph $\langle D \rangle$ is connected. The number of vertices in a minimum connected dominating set is defined as the Connected Domination number of a graph G and it is denoted by $\gamma_c(G)$. This concept was introduced by Sampathkumar in [8].

Another important area of a graph theory is decomposition of graphs.

Several authors studied various types of decompositions by imposing conditions on G_i in the decomposition.

Let $G = (V, E)$ be a simple connected graph with p vertices and q edges. If $G_1, G_2, \dots, G_n$ are connected edge disjoint subgraphs of G with $E(G) = E(G_1) \cup E(G_2) \cup \dots \cup E(G_n)$, then $(G_1, G_2, \dots, G_n)$ is said to be a Decomposition of G .

We introduce a new concept connected domination decomposition of a graph which is motivated by the concepts of Ascending Subgraph Decomposition [1], Continuous Monotonic Decomposition [3] and Ascending Domination Decomposition [6].

A vertex v is called a cut vertex if removing it from G increases the number of components of G . Here S is the set of cut vertices and $s(G) = |S|$. A connected graph that has no cut vertices is called a block (B) and $m(G) = |B|$. For a connected graph G with blocks $\{B_i\}$ and cut vertices $\{c_j\}$ the block cut point graph of G denoted by $bc(G)$ is defined as the graph having vertex set $\{B_i\} \cup \{c_j\}$ with two vertices adjacent if one corresponds to a block B_i and the other to a cut vertex c_j and c_j is in B_i . The neighborhood of a vertex u is the set $N(u)$ consisting of all vertices v which are adjacent with u .

An Umbrella graph $U(m, n)$ to be a graph obtained from a fan by joining a path of length m , P_m to a middle vertex of a path P_n . A Web graph (W_n) is a prism graph $Y_{m,n}$ with the edges of the outer cycle removed. The Helm graph H_n is obtained from a wheel by attaching a pendant vertex at each vertex of the n -cycle.

2. Connected Domination Decomposition (CDD)

Definition 2.1. A decomposition $(G_1, G_2, \dots, G_n)$ of G is said to be CDD if:

- (i) $E(G) = E(G_1) \cup E(G_2) \cup \dots \cup E(G_n)$
- (ii) Each G_i is connected

Figure 1

(iii) $\gamma_c(G_i) = i$, $1 \leq i \leq n$.

Example 2.2. The umbrella graph $U(7, 3)$ admits CDD (G_1, G_2, G_3) is illustrated in Figure 1.

3. Connected Domination Path Decomposition (CDPD)

Definition 3.1. A decomposition $(G_1, G_2, \dots, G_n)$ of G is said to be CDPD if:

(i) G admits CDD,

Figure 2

- (ii) $q(G_1) = 1$ or 2 ,
- (iii) $q(G_i) = i + 1, i = 2, 3, \dots, n$,
- (iv) each G_i ($1 \leq i \leq n$) is a path.

Example 3.2. The Web graph W_5 admits CDPD $(G_1, G_2, G_3, G_4, G_5)$ is illustrated in Figure 2.

Theorem 3.3. If a graph G admits CDPD $(G_1, G_2, \dots, G_n)$, then

$$q(G) = \begin{cases} \frac{n(n+3)}{2} - 1, & \text{if } q(G_1) = 1, \forall n \in N, \\ \frac{n(n+3)}{2}, & \text{if } q(G_1) = 2, \forall n \in N. \end{cases}$$

Proof. Assume that G admits CDPD. Therefore, $q(G_1) = 1$ or 2 and $q(G_i) = i + 1$, $i = 2, 3, \dots, n$.

Case 1. $q(G_1) = 1$.

$$\begin{aligned} q(G) &= q(G_1) + q(G_2) + \dots + q(G_n) \\ &= 1 + [3 + 4 + \dots + (n + 1)]. \end{aligned}$$

Therefore, $q(G) = \frac{n(n+3)}{2} - 1$.

Case 2. $q(G_1) = 2$.

$$\begin{aligned} q(G) &= q(G_1) + q(G_2) + \dots + q(G_n) \\ &= 2 + [3 + \dots + (n + 1)]. \end{aligned}$$

Therefore, $q(G) = \frac{n(n+3)}{2}$. □

Remark 3.4. In the above theorem: For case 1, $G_1 = P_2$, $G_2 = P_4$, $G_3 = P_5, \dots, G_n = P_{n+2}$ ($n \geq 2$) are the decompositions of G .

In case 2, $G_1 = P_3$, $G_2 = P_4$, $G_3 = P_5, \dots, G_n = P_{n+2}$ are the decompositions of G .

Remark 3.5. The converse of the above theorem need not be true. For the Helm graph (H_9),

$q(H_9) = 27$. 27 is the sum of 2,3,4,5,6 and 7. But H_9 does not admit CDPD because each G_i , $i = 1, 2, \dots, 6$ is not a path.

4. Connected Domination Path Decomposition of Triangular Snake Graph (T_m)

A triangular snake is a connected graph all of whose blocks are triangles. A triangular snake is a triangular cactus whose block cut point graph is a path. Equivalently it is obtained from a path $P = v_1, v_2, \dots, v_{m+1}$ by joining v_i and v_{i+1} to a new vertex u_i , $i = 1, 2, \dots, m$. A triangular snake has $2m + 1$ vertices and $3m$ edges, where m is the number of blocks in the triangular snake. We denote this triangular snake by T_m . Here $v_1, v_2, \dots, v_{m+1}$ is a set of base vertices of T_m . Since $q(T_m) = 3m$, $q(G_1)$ must be 2.

Theorem 4.1. *If a triangular snake T_m admits CDPD ($P_3, P_4, P_5, \dots, P_{3i+2}$), then $q(T_m) = \frac{(9i(i+1))}{2}$, $i \in \mathbb{N}$.*

Figure 3

Figure 4

Proof. Assume that T_m admits CDPD $(P_3, P_4, P_5, \dots, P_{3i+2})$. Here $G_1 = P_3$, $G_2 = P_4$, $G_3 = P_5, \dots, G_{3i} = P_{3i+2}$. By Theorem(3.3), $q(T_m) = \frac{9i(i+1)}{2}$, $i \in N$. $\square$

Remark 4.2. In a triangular snake T_m , $q(T_m) = 3m$ and $p(T_m) = 2m + 1$, where m is the number of blocks. By Theorem (4.1), T_m admits CDPD if $q(T_m) = \frac{9i(i+1)}{2}$, $i \in N$. Therefore, $3m = \frac{9i(i+1)}{2}$ That is, $m = \frac{3i(i+1)}{2}$, $i \in N$. $p(T_m) = 2m + 1 = 2[\frac{3i(i+1)}{2}] + 1$. That is, $p(T_m) = 3i^2 + 3i + 1$, $i \in N$.

Theorem 4.3. If T_m admits CDPD $(P_3, P_4, P_5, \dots, P_{3i+2})$, then $\gamma_c(T_m) = \frac{3i^2+3i-2}{2}$, $i \in N$.

Proof. Let $D \subseteq p(T_m)$ is a connected dominating set in T_m . By our as-

sumption, T_m admits CDPD. let $v_1, v_2, v_3, \dots, v_{\frac{3i(i+1)}{2}}, v_{\frac{3i(i+1)}{2}+1}$ be the base vertices of T_m . Then the cut vertices are $v_2, v_3, v_4, \dots, v_{\frac{3i(i+1)}{2}-1}, v_{\frac{3i(i+1)}{2}}$. let $S = \{v_2, v_3, v_4, \dots, v_{\frac{3i(i+1)}{2}-1}, v_{\frac{3i(i+1)}{2}}\}$. Then S is a minimum connected dominating set of T_m . Therefore, $|S| = |D| = \frac{3i(i+1)}{2} - 1 = \frac{3i^2+3i-2}{2}$. That is, $\gamma_c(T_m) = \frac{3i^2+3i-2}{2}$, $i \in N$. $\square$

Corollary 4.4. *If T_m admits CDPD $(P_3, P_4, P_5, \dots, P_{3i+2})$, $i \in N$, then $\gamma_c(T_m) = q(T_m) - p(T_m)$.*

Proof. We know that $p(T_m) = 3i^2 + 3i + 1$ and $q(T_m) = \frac{9i(i+1)}{2}$, $i \in N$. By Theorem(4.3), $\gamma_c(T_m) = \frac{3i^2+3i-2}{2} = \frac{9i(i+1)}{2} - (3i^2 + 3i + 1) = q(T_m) - p(T_m)$. $\square$

Theorem 4.5. *If T_m admits CDPD $(P_3, P_4, P_5, \dots, P_{3i+2})$, $i \in N$ and $\gamma(T_m) = |D|$, then*

$$\gamma_c(T_m) = \begin{cases} 2\gamma(T_m) - 2 & \text{if } m \text{ is odd,} \\ 2\gamma(T_m) - 1 & \text{if } m \text{ is even.} \end{cases}$$

Proof. Given that $\gamma(T_m) = |D|$. Therefore, D is a minimum dominating set of T_m .

Case 1: m is odd. D consists of $\frac{1}{2}[\frac{3i(i+1)}{2} + 1]$ vertices. Therefore, $\gamma(T_m) = \frac{3i(i+1)+2}{4}$. Let U and W be two vertex sets such that

$$U = \{u_1, u_2, u_3, \dots, u_{m-1}, u_m\}$$

and

$$W = \{v_1, v_2, v_3, \dots, v_m, v_{m+1}\}.$$

Now, $N(D) = (W - D) \cup U$. Then $N(D) - U$ is a minimum dominating set of T_m . Take $F = N(D) - U$. Therefore, $(D \cup F)$ is a set of base vertices of T_m and $|D \cup F| = \frac{3i(i+1)+2}{2} = 2\gamma(T_m)$. Let $T = D \cup F$. Neglect the first and last vertices in T . Therefore, $T = (D \cup F) - \{v_1, v_{m+1}\}$ forms a minimum connected dominating set of T_m and $|T| = 2\gamma(T_m) - 2$. That is, $\gamma_c(T_m) = 2\gamma(T_m) - 2$.

Case 2: m is even. D consists of $\frac{1}{2}[\frac{3i(i+1)}{2}]$ vertices. Therefore, $\gamma(T_m) = \frac{3i(i+1)}{4}$. Let X and Y be two vertex sets such that $X = \{u_1, u_2, u_3, \dots, u_{m-1}, u_m\}$ and $Y = \{v_1, v_2, v_3, \dots, v_m, v_{m+1}\}$.

Now, $N(D) = (Y - D) \cup X$. Then $N(D) - X$ is a dominating set of T_m . Take $F = N(D) - X$. Therefore, $(D \cup F)$ is a set of base vertices of T_m and $|D \cup F| = \frac{3i(i+1)}{2} + 1 = 2\gamma(T_m) + 1$. Let $T = D \cup F$. Neglect the first and last

vertices in T . Therefore, $T = (D \cup F) - \{v_1, v_{m+1}\}$ forms a minimum connected dominating set of T_m and $|T| = [2\gamma(T_m) + 1] - 2 = 2\gamma(T_m) - 1$. That is, $\gamma_c(T_m) = 2\gamma(T_m) - 1$. $\square$

Illustration 4.6. The following table illustrate (see Table 1) the CDPD, connected domination number (γ_c) and domination number (γ) of T_m .

m	p	q	CDPD	γ_c	γ
3	7	9	P_3, P_4, P_5	2	2
9	19	27	$P_3, P_4, \dots, P_8$	8	5
18	37	54	$P_3, P_4, \dots, P_{11}$	17	9
30	61	90	$P_3, P_4, \dots, P_{14}$	29	15
45	91	135	$P_3, P_4, \dots, P_{17}$	44	23
63	127	189	$P_3, P_4, \dots, P_{20}$	62	32
84	169	252	$P_3, P_4, \dots, P_{23}$	83	42
108	217	324	$P_3, P_4, \dots, P_{26}$	107	54
135	271	405	$P_3, P_4, \dots, P_{29}$	134	68
165	331	495	$P_3, P_4, \dots, P_{32}$	164	83
198	397	594	$P_3, P_4, \dots, P_{35}$	197	99
234	469	702	$P_3, P_4, \dots, P_{38}$	233	117
273	547	819	$P_3, P_4, \dots, P_{41}$	272	137
315	631	945	$P_3, P_4, \dots, P_{44}$	314	158
360	721	1080	$P_3, P_4, \dots, P_{47}$	359	180

Table 1

Theorem 4.7. If T_m admits CDPD $(P_3, P_4, P_5, \dots, P_{3i+2})$, then

$$\gamma_c(T_m) = \sum_{j=1}^{3i} \gamma_c(G_j) - (3i^2 + 1), \quad i \in N.$$

Proof. By our assumption, $\gamma_c(G_1) = 1, \gamma_c(G_2) = 2, \dots, \gamma_c(G_{3i}) = 3i, i \in N$.

Then

$$\begin{aligned}
 \sum_{j=1}^{3i} \gamma_c(G_j) - (3i^2 + 1) &= [\gamma_c(G_1) + \gamma_c(G_2) + \dots + \gamma_c(G_{3i})] - (3i^2 + 1) \\
 &= [1 + 2 + \dots + 3i] - (3i^2 + 1) \\
 &= \frac{3i(3i + 1)}{2} - (3i^2 + 1) \\
 &= \frac{3i^2 + 3i - 2}{2}.
 \end{aligned}$$

Then

$$\sum_{j=1}^{3i} \gamma_c(G_j) - (3i^2 + 1) = \gamma_c(T_m),$$

by Theorem 4.3. □

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