SOLVING STRONGLY NONLINEAR OSCILLATORS BY NEW NUMERICAL METHOD

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Abstract: In this article, we present the solution of nonlinear oscillators by new numerical method called the Daftardar-Gejji and Jafari method (DJM). The results derived by DJM are compared with variational iteration method (VIM), Homotopy perturbation method (HPM) and Runge-Kutta method in order to prove the accuracy of DJM.

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Key Words: nonlinear oscillators, Daftardar-Gejji and Jafari method, Runge-Kutta method, variational iteration method, homotopy perturbation method

1. Introduction

In this paper, our attention focuses on the general nonlinear oscillator in the following form:

$$u''(x) + c_1 u(x) + c_2 u^2(x) + c_3 u^3(x) = \epsilon f(u(x), u'(x)), \quad u(0) = a, \quad u'(0) = b \quad (1.1)$$

where c_1, c_2 and c_3 are positive real numbers, ϵ is a parameter and $f(u(x), u'(x))$ are an arbitrary nonlinear functions of its arguments. Many studies focus on the strongly nonlinear oscillators [1, 2, 3, 4, 5, 6, 7, 8], for example, Nayfeh [1] used

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the harmonic balance method (HB) to solve the nonlinear oscillators. In another study [1, 2], Nayfeh applied the elliptic Lindstedt-Poincare method (LP) to solve Eq. (1.1) and applied the multiple scales method (MSM) for strongly oscillators systems, Krylov and Bogoliubov [3] used Krylov-Bogoliubov-Mitropolsky method (KBM) for Eq. (1.1), Lan Xu [4] employed His parameter-expanding methods for the solution of oscillators equations, Ghosh *et al.* [5] applied the adaptation of adomian decomposition for numeric-analytic integration of strongly nonlinear and chaotic oscillators, J.H. He [6] used the homotopy perturbation method (HPM) for nonlinear oscillators with discontinuities, Momani *et al.* [7] presented the modified homotopy perturbation method for solving strongly nonlinear oscillators, K. Batiha and B. Batiha [8] applied the variational iteration method (VIM) for solving nonlinear oscillators.

Also, we will introduce the DJM to find the analytical solution for the strongly nonlinear oscillators (1.1).

2. Main Results

Daftardar-Gejji and Jafari method (DJM) was first introduced by Daftardar-Gejji and Jafari [9] in 2006, it has been proved that this method is a better technique for solving different kinds of nonlinear equations [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. DJM has been used to create a new predictor-corrector method [21, 22]. Noor et al. [23, 24, 25, 26, 27] used DJM to create numerical methods to handle algebraic equations. Here the Daftardar-Gejji and Jafari method will be discussed, which successfully used to solve differential equations and nonlinear equations in the form:

$$y = f + L(y) + N(y), \quad (2.1)$$

Where L , N are linear and non-linear operators respectively and f is a giving function. The solution of eq. (2.1) have the form:

$$y = \sum_{i=0} y_i. \quad (2.2)$$

Suppose we have,

$$H_0 = N(y_0), \quad (2.3)$$

$$H_m = N\left(\sum_{i=0}^m y_i\right) - N\left(\sum_{i=0}^{m-1} y_i\right). \quad (2.4)$$

then we get,

$$H_0 = N(y_0), \quad (2.5)$$

$$H_1 = N(y_0 + y_1) - N(y_0), \quad (2.6)$$

$$H_2 = N(y_0 + y_1 + y_2) - N(y_0 + y_1), \quad (2.7)$$

$$H_3 = N(y_0 + y_1 + y_2 + y_3) - N(y_0 + y_1 + y_2) + \cdots . \quad (2.8)$$

Thus $N(y)$ is decomposed as:

$$\begin{aligned} N\left(\sum_{i=0}^m y_i\right) &= N(y_0) + N(y_0 + y_1) - N(y_0) + N(y_0 + y_1 + y_2) - N(y_0 + y_1) \\ &\quad + N(y_0 + y_1 + y_2 + y_3) - N(y_0 + y_1 + y_2) + \cdots . \end{aligned} \quad (2.9)$$

So, the recurrence relation is given as follows:

$$\begin{aligned} y_0 &= f \\ y_1 &= L(y_0) + H_0 \\ y_{m+1} &= L(y_m) + H_m, \quad m = 1, 2, \cdots . \end{aligned} \quad (2.10)$$

Since L is linear, then:

$$\sum_{i=0}^m L(y_i) = L\left(\sum_{i=0}^m y_i\right). \quad (2.11)$$

Hence,

$$\begin{aligned} \sum_{i=0}^{m+1} y_i &= \sum_{i=0}^m L(y_i) + N\left(\sum_{i=0}^m y_i\right) \\ &= L\left(\sum_{i=0}^m y_i\right) + N\left(\sum_{i=0}^m y_i\right), \quad m = 1, 2, \cdots \end{aligned} \quad (2.12)$$

Thus,

$$\sum_{i=0}^m y_i = f + L\left(\sum_{i=0}^m y_i\right) + N\left(\sum_{i=0}^m y_i\right). \quad (2.13)$$

The k - term solution is given by the following form:

$$y = \sum_{i=0}^{k-1} y_i. \quad (2.14)$$

3. Convergence of the DJM

First, we will introduce the condition of convergence of DJM.

Theorem 1. *If N is $C^{(1)}$ in a neighborhood of u_0 and $\|N^{(n)}(u_0)\| \leq L$, for any n and for some real $L > 0$ and $\|u_i\| \leq M < \frac{1}{e}, i = 1, 2, \dots$, then the series $\sum_{n=0}^{\infty} H_n$ is absolutely convergent and $\|H_n\| \leq LM^n e^{n-1}(e-1), n = 1, 2, \dots$*

Proof. For the proof of Theorem 1, please check [28]. $\square$

Theorem 2. *If N is $C^{(1)}$ and $\|N^{(n)}(u_0)\| \leq M \leq e^{-1}, \forall n$, then the series $\sum_{n=0}^{\infty} H_n$ is absolutely convergent.*

Proof. For the proof of Theorem 2, please see [28]. $\square$

4. Numerical Implementation and Discussion

In this section, we shall apply the DJM to solve some nonlinear oscillators that arise in nonlinear dynamics.

4.1. Examples

Example 4.1. In this example, we will study the following Helmholtz equation:

$$u''(x) + 2u(x) + u^2(x) = 0, \quad (4.1)$$

with initial conditions:

$$u(0) = 0.1 \quad \text{and} \quad u'(0) = 0. \quad (4.2)$$

To solve above Helmholtz equation by DJM, we integrate eq. (4.1) and use the initial condition $u(0) = 0.1$, to get:

$$u(x) = 0.1 - \int_0^x \int_0^t 2u(x) + u^2(x) \, dt \, dx, \quad (4.3)$$

By using algorithm (2.10) we have:

$$\begin{aligned} u_0 &= 0.1, \\ u_1 &= -0.1050t^2, \\ u_2 &= 0.01924999976 t^4 - 0.0003675 t^6, \end{aligned}$$

$$\vdots$$

Thus

$$\begin{aligned} \sum_{i=0}^3 u_i = & 0.1 - 0.105 t^2 + 0.01925 t^4 - 0.0017791667 t^6 + 7.4206729 \times 10^{-12} t^7 \\ & + 8.6625017 \times 10^{-5} t^8 - 2.9682692 \times 10^{-12} t^9 - 4.9748608 \times 10^{-6} t^{10} \\ & - 7.4206729 \times 10^{-14} t^{11} + 1.071875065 \times 10^{-7} t^{12} \\ & + 7.4206729 \times 10^{-10} t^{14}. \end{aligned} \quad (4.4)$$

In Table 1 below, we show the comparison between the results obtained from VIM, HPM and the Runge-Kutta method, it's very clear that the DJM results are more accuracy than VIM and HPM.

Table 1: Numerical comparisons between RK_4 , u_4 of DJM, VIM [8] and HPM [7]

t	RK_4	DJM	VIM	HPM
1.0	0.012602051	0.012602029	0.012603229	0.012612913
1.5	-0.05599438	-0.05599646	-0.055935391	-0.055421301
2.0	-0.09857378	-0.09862916	-0.097678779	-0.089493632
2.5	-0.09605599	-0.09674898	-0.088817643	-0.021863100
3.0	-0.04954859	-0.05515216	-0.009530719	0.349441660

Example 4.2. Here, we shall study the following non-linear equation:

$$u''(x) + u(x) + 0.1u^2(x)u'(x) = 0, \quad (4.5)$$

The initial conditions for the above equation are:

$$u(0) = 1, \quad u'(0) = 0. \quad (4.6)$$

To solve equation (4.5) by DJM, we integrate it and use (4.6), to get:

$$u(x) = 1 - \int_0^t \int_0^t u(x) + 0.1u^2(x)u'(x) \, dt \, dt, \quad (4.7)$$

By applying the algorithm (2.10), we get:

$$u_0 = 1,$$

$$\begin{aligned}
u_1 &= -0.5 t^2, \\
u_2 &= 0.01667 t^3 + 0.04167 t^4 - 0.00499 t^5 - 2.38095 \times 10^{-11} t^6 + 0.000595 t^7, \\
&\vdots
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{i=0}^3 u_i &= 1 - 3.1954220 \times 10^{-13} t^{22} + 1.9172532 \times 10^{-19} t^{21} \\
&+ 8.8577111 \times 10^{-12} t^{20} - 7.7699284 \times 10^{-11} t^{19} - 0.1547722 \times 10^{-10} t^{18} \\
&+ 2.5009914 \times 10^{-9} t^{17} - 5.5782335 \times 10^{-9} t^{16} - 3.4659493 \times 10^{-8} t^{15} \\
&+ 2.3500174 \times 10^{-7} t^{14} + 8.6189862 \times 10^{-8} t^{13} - 3.6191078 \times 10^{-8} t^{12} \\
&+ 6.6210071 \times 10^{-6} t^{11} + 2.9547939 \times 10^{-5} t^{10} - 0.00013990274 t^9 \\
&- 0.00013938525 t^8 + 0.0013055558 t^7 - 0.0010277779 t^6 - 0.0066666669 t^5 \\
&+ 0.04125 t^4 + 0.016666667 t^3 - 1/2 t^2. \tag{4.8}
\end{aligned}$$

The Table 2 shows the comparison between the results obtained from VIM, HPM and the Runge-Kutta method, it's very clear that the DJM results are more accuracy than HPM and with almost same accuracy with VIM.

Table 2: Numerical comparisons between RK_4 , u_4 of DJM, VIM [8] and HPM [7]

t	RK_4	DJM	VIM	HPM
1.0	0.55136372	0.55136570	0.5513655	0.55164132
1.5	0.09408927	0.0941310	0.0941310	0.10167574
2.0	-0.3851536	-0.3846683	-0.3846678	-0.30760496
2.5	-0.7666063	-0.7624307	-0.7624317	-0.30862750
3.0	-0.9528593	-0.9256864	-0.9256931	0.9485222

Example 4.3. In this example, we will consider the following initial-value problem:

$$u'(x) + u(x) + 0.45u^2(x) = u(x)u'(x), \tag{4.9}$$

With initial conditions:

$$u(0) = 0.1, \quad u'(0) = 0. \tag{4.10}$$

In order to solve Eq. (4.9) by DJM, we integrate it and apply the initial conditions (4.10), to get:

$$u(x) = 0.1 - \int_0^t \int_0^t u(x) + 0.45u^2(x) - u(x)u(x) \, dt dt, \quad (4.11)$$

Applying algorithm (2.10) to get:

$$\begin{aligned} u_0 &= 0.1, \\ u_1 &= -0.05225 t^2, \\ u_2 &= -4.0950937 t^6 \times 10^{-5} + 2.7300622 t^5 \times 10^{-4} \\ &\quad + 0.0047460416 t^4 - 0.0017416667 t^3, \\ &\quad \vdots \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{i=0}^4 u_i &= -8.8926433 \times 10^{-27} t^{30} + 7.3645487 \times 10^{-13} t^{18} - 3.3623733 \times 10^{-12} t^{17} \\ &\quad - 3.9532206 \times 10^{-11} t^{16} + 1.7701918 \times 10^{-11} t^{15} + 8.8926431 \times 10^{-25} t^{29} \\ &\quad - 2.8319707 \times 10^{-23} t^{28} + 1.2832457 \times 10^{-22} t^{27} + 9.3304993 \times 10^{-21} t^{26} \\ &\quad - 1.1767400 \times 10^{-19} t^{25} - 1.2832948 \times 10^{-18} t^{24} + 2.2562560 \times 10^{-17} t^{23} \\ &\quad + 1.1645679 \times 10^{-16} t^{22} - 2.1565357 \times 10^{-15} t^{21} - 9.7555481 \times 10^{-15} t^{20} \\ &\quad + 1.1653930 \times 10^{-13} t^{19} + 1.3370221 \times 10^{-9} t^{14} + 2.7386885 \times 10^{-9} t^{13} \\ &\quad - 2.7434468 \times 10^{-8} t^{12} - 1.3187115 \times 10^{-7} t^{11} + 2.7558057 \times 10^{-7} t^{10} \\ &\quad + 3.3682567 \times 10^{-6} t^9 + 3.0870611 \times 10^{-6} t^8 - 5.1612944 \times 10^{-5} t^7 \\ &\quad - 0.00018892727 t^6 + 0.00046197705 t^5 + 0.0047024998 t^4 \\ &\quad - 0.0017416667 t^3 - 0.05225 t^2 + 0.1. \end{aligned} \quad (4.12)$$

The Table 3 shows the comparison between the results obtained from VIM, HPM and the Runge-Kutta method, we can see that the DJM results are more accuracy than HPM and with almost same accuracy with VIM.

5. Conclusions

In this paper, the Daftardar-Gejji and Jafari method (DJM) is applied to the solution of strongly nonlinear oscillatory systems. It may be concluded that

Table 3: Numerical comparisons between RK_4 , u_4 of DJM, VIM [8] and HPM [7]

t	RK_4	DJM	VIM	HPM
1.0	0.05093886	0.050938846	0.050938846	0.050898844
1.5	0.00105022	0.001050497	0.0010504896	0.000578654
2.0	-0.04916929	-0.04915116	-0.049151160	-0.052121403
2.5	-0.08663561	-0.08636361	-0.086363628	-0.099818707
3.0	-0.10261582	-0.10058473	-0.10058475	-0.14902186

DJM is very powerful and efficient tool for the solutions of nonlinear oscillators. All examples we showed in this paper proved the results obtained from DJM are in excellent agreement with the results obtained from Runge-Kuta method and with better accuracy than VIM and HPM.

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