ON THE WEIGHTED COMPOSITION OPERATORS ON WEIGHTED BANACH SPACES

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Abstract: In this paper we give conditions under which the numerical range of a weighted composition operator acting on $H^p(\beta)$ is closed.

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1. Introduction

Let $\{\beta(n)\}$ be a sequence of nonzero complex numbers with $\beta(0) = 1$ and $1 \leq p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_{\beta}^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p |\beta(n)|^p < \infty.$$

The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ will be used whether or not the series converges for any value of $z \in \mathbb{D}$. These are called formal power series. Let $H^p(\beta)$ denotes the space of such formal power series. It is called a weighted Hardy space. Let $f_k(z) = z^k$, then $\{f_k\}_k$ forms a basis for the reflexive Banach space $H^p(\beta)$. Note that the dual of $H^p(\beta)$ is $H^q(\beta^{p/q})$, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta^{p/q} = \{\beta(n)^{p/q}\}_{n=0}^{\infty}$. If $\{\beta(n)\}_n$ is a sequence of positive numbers, with

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$\lim \beta(n)^{\frac{1}{n}} = 1$, then $H^p(\beta)$ is a Banach space of analytic functions on $\mathbb{D}$. The Hardy, Bergman and Dirichlet spaces can be viewed as we consider respectively that $\beta(n) \equiv 1$, $\beta(n) = (n+1)^{-1/2}$ and $\beta(n) = (n+1)^{-1/2}$.

Let λ be a complex number, the functional of evaluation at λ , e_λ , is defined by $e_\lambda(p) = p(\lambda)$, for all polynomials p . Also, λ is said to be a bounded point evaluation on $H^p(\beta)$ if the function e_λ extends to be a bounded linear functional on $H^p(\beta)$. In this case we have $e_\lambda(f) = f(\lambda)$, $f \in H^p(\beta)$.

Let φ be an analytic self map of $\mathbb{D}$ and ψ belongs to the set of multipliers of $H^p(\beta)$ that is denoted by $M(H^p(\beta))$. A weighted composition operator $C_{\psi, \varphi}$ maps an analytic function $f \in H^2(\beta)$ into $C_{\psi, \varphi}f(z) = \psi(z)f(\varphi(z))$. The function φ is called the composition map and the function ψ is called the multiplier map. We will use the notations $H(\mathbb{D})$ and $C(\overline{\mathbb{D}})$ to denote the set of analytic functions on $\mathbb{D}$ and the set of continuous functions on $\overline{\mathbb{D}}$, the closure of $\mathbb{D}$.

In this paper we will give a definition for the numerical range of an operator acting on Banach spaces and then we will investigate its closedness for weighted composition operators acting on the weighted Hardy spaces $H^p(\beta)$. For some sources on the topics of this paper one can refer to [1–7].

2. Main Result

In this section we investigate the closedness of the numerical range of composition operators acting on the sequence Banach spaces $H^p(\beta)$. from now on we consider that $H^p(\beta) \subset H(\mathbb{D})$.

Definition 2.1. Let X be a reflexive Banach space and $T \in B(X)$. The numerical range of T is defined by

$$W(T) = co(V(T)),$$

where $co(V(T))$ is the convex hull of $V(T)$ and

$$V(T) = \{x^*(T(x)) : x \in X, x^* \in X^*; \|x\| = \|x^*\| = x^*(x) = 1\}.$$

Theorem 2.2. Let $\frac{1}{p} + \frac{1}{q} = 1$ and $\|e_\lambda\| \rightarrow \infty$ as $\lambda \rightarrow z_0$ for some $z_0 \in \partial\mathbb{D}$. Suppose φ is any holomorphic self-map of the unit disk $\mathbb{D}$ such that $\sum_{n \geq 0} \frac{\varphi(z_0)^n}{z_0^n \beta(n)^q}$ is finite and let $\psi \in C(\overline{\mathbb{D}}) \cap M(H^p(\beta))$. If $C_{\psi, \varphi}$ is bounded on $H^p(\beta)$, then $0 \in \overline{W}(C_{\psi, \varphi})$.

Proof. Let $0 \neq \lambda \in \mathbb{D}$, and α_λ be such that $|\lambda| = \lambda \alpha_\lambda$. Note that

$$e_\lambda(z) = \sum_{n=0}^{\infty} \frac{\bar{\lambda}^n}{\beta(n)^p} z^n$$

and put

$$f_\lambda(z) = \sum_{n=0}^{\infty} \frac{\alpha_\lambda^{nq} \lambda^{\frac{nq}{p}}}{\beta(n)^q} z^n.$$

Then, $e_\lambda \in H^p(\beta)^*$ and $f_\lambda \in H^p(\beta)$. Indeed, $\|e_\lambda\|_q^q = \|f_\lambda\|_p^p$. Hence, $\|f_\lambda\|_p = \|e_\lambda\|_q^{\frac{q}{p}}$ and so

$$\|f_\lambda\|_p \|e_\lambda\|_q = \|e_\lambda\|_q^q. \quad (*)$$

Also,

$$\begin{aligned} \langle f_\lambda, e_\lambda \rangle &= \sum_{n=0}^{\infty} \frac{\alpha_\lambda^{nq} \lambda^{\frac{nq}{p}}}{\beta(n)^q} \lambda^n = \sum_{n=0}^{\infty} \frac{(\alpha_\lambda \lambda)^{nq}}{\beta(n)^q} \\ &= \sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q} = \|e_\lambda\|_q^q. \end{aligned}$$

So by (*), we get

$$\langle \frac{f_\lambda}{\|f_\lambda\|_p}, \frac{e_\lambda}{\|e_\lambda\|_q} \rangle = 1.$$

Put

$$L_\lambda = \langle C_{\psi, \varphi}(\frac{f_\lambda}{\|f_\lambda\|_p}), \frac{e_\lambda}{\|e_\lambda\|_q} \rangle.$$

Then $L_\lambda \in V(C_{\psi, \varphi}) \subseteq W(C_{\psi, \varphi})$. But,

$$\begin{aligned} L_\lambda &= \langle \frac{f_\lambda}{\|f_\lambda\|_p}, C_{\psi, \varphi}^*(\frac{e_\lambda}{\|e_\lambda\|_q}) \rangle \\ &= \langle \frac{f_\lambda}{\|f_\lambda\|_p}, \psi(\lambda) \frac{e_{\varphi(\lambda)}}{\|e_\lambda\|_q} \rangle \\ &= \frac{\psi(\lambda)}{\|e_\lambda\|_q^q} f_\lambda(\varphi(\lambda)) \\ &= \frac{\psi(\lambda)}{\|e_\lambda\|_q^q} \sum_{n=0}^{\infty} \frac{\alpha_\lambda^{nq} \lambda^{\frac{nq}{p}}}{\beta(n)^q} \varphi(\lambda)^n \\ &= \frac{\psi(\lambda)}{\|e_\lambda\|_q^q} \sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\lambda^n \beta(n)^q} \varphi(\lambda)^n. \end{aligned}$$

So if $\lambda \rightarrow z_0 \in \partial\mathbb{D}$, then we have

$$\|e_\lambda\|_q^q L_\lambda \rightarrow \psi(z_0) \sum_{n \geq 0} \frac{\varphi(z_0)^n}{z_0^n \beta(n)^q}.$$

Since $\sum_{n \geq 0} \frac{1}{\beta(n)^q} = \infty$, we get $L_\lambda \rightarrow 0$ as $\lambda \rightarrow z_0$. Hence,

$$0 \in \overline{V}(C_{\psi,\varphi}) \subseteq \overline{W}(C_{\psi,\varphi})$$

and this completes the proof. $\square$

Theorem 2.3. Let $\frac{1}{p} + \frac{1}{q} = 1$ and $\|e_\lambda\| \rightarrow \infty$ as $\lambda \rightarrow z_0$ for some $z_0 \in \partial\mathbb{D}$. Suppose φ is any holomorphic self-map of the unit disk $\mathbb{D}$ such that $\sum_{n \geq 0} \frac{\varphi(z_0)^n}{z_0^n \beta(n)^q}$ is finite and let $\psi \in C(\overline{D}) \cap M(H^p(\beta))$. If $C_{\psi,\varphi}$ is compact, then $0 \in W(C_{\psi,\varphi})$ if and only if $W(C_{\psi,\varphi})$ is closed.

Proof. If $W(C_{\psi,\varphi})$ is closed, then by Theorem 2.2, $0 \in W(C_{\psi,\varphi})$. Let $\alpha \in \overline{W}(C_{\psi,\varphi})$. This implies that there exists a constant $0 \leq c \leq 1$ such that $\alpha \in c W(C_{\psi,\varphi})$. If $\alpha \neq 0$ and $c \neq 0$, then $\frac{\alpha}{c} \in W(C_{\psi,\varphi})$. Since $W(C_{\psi,\varphi})$ is convex and $0 \in W(C_{\psi,\varphi})$, thus $\alpha \in W(C_{\psi,\varphi})$. Therefore,

$$\overline{W}(C_{\psi,\varphi}) \subseteq W(C_{\psi,\varphi}) \subseteq \overline{W}(C_{\psi,\varphi}).$$

This implies that $W(C_{\psi,\varphi})$ is closed and so the proof is complete. $\square$

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