APPROXIMATE CAUCHY–JENSEN MAPPINGS IN NON-ARCHIMEDEAN QUASI- β -NORMED SPACES

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Abstract: In this article, we investigate the generalized Hyers–Ulam stability of the Cauchy–Jensen functional equation

$$f\left(\frac{x+y}{n} + z\right) + f\left(\frac{y+z}{n} + x\right) + f\left(\frac{z+x}{n} + y\right) = \frac{n+2}{n}[f(x) + f(y) + f(z)],$$

in non-Archimedean quasi- β -normed spaces for any fixed nonzero integer n .

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1. Introduction

The stability problem of functional equations originated from a question of S.M. Ulam [22] concerning the stability of group homomorphisms.

In 1941, D.H. Hyers [11] proved the following stability result for the case of approximate additive mappings between Banach spaces.

The method which was provided by D.H. Hyers, and which produces the additive mapping derived from the approximate additive mapping, so called a direct method, can be applied for studying the stability of various functional

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equations. This method is the most important and most powerful tool for studying the stability of various functional equations. Hyers' theorem was generalized by T. Aoki [1] and D.G. Bourgin [3] for additive mappings by considering an unbounded Cauchy difference. In 1978, Th.M. Rassias [18] also provided a generalization of Hyers' theorem for linear mappings which allows the Cauchy difference to be unbounded. In 1991, Z. Gajda [7], following the same approach as in [18], gave an affirmative solution to this question for $p > 1$. However, the counterexamples of Z. Gajda [7] as well as of Th.M. Rassias and P. Šemrl [19] that one cannot prove the Rassias' type theorem for $p = 1$ have stimulated several mathematicians to invent new approximately additive or approximately linear mappings. A generalized result of Rassias' theorem was obtained by P. Găvruta in [8] and S. Jung in [12].

On the other hand, K. Hensel [10] has introduced special normed spaces which do not have the Archimedean property: for any real $x, y > 0$, there exists a natural number n such that $x < ny$. Thus, we recall that a non-Archimedean valuation in a field $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \rightarrow [0, \infty)$ with

- (i) $|r| = 0$ if and only if $r = 0$;
- (ii) $|rs| = |r||s|$ for all $r, s \in \mathbb{K}$;
- (iii) $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$.

Any field endowed with a non-Archimedean valuation is said to be a non-Archimedean field. In any non-Archimedean field, we remark that $|1| = |-1| = 1$, $|n| \leq 1$ for all nonzero integers $n \in \mathbb{Z}$, and $|na| \leq |ka| \leq |a| \leq |\frac{a}{k}| \leq |\frac{a}{n}|$ for any $a \in \mathbb{K}$ and $n, k \in \mathbb{Z}$ with $k|n$.

Definition 1.1. Let X be a linear space over a non-Archimedean field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:

- (i) $\|x\| = 0$ if and only if $x = 0$;
- (ii) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;
- (iii) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$.

In this case, $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Because of the fact

$$\|x_k - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq k-1\} \quad (k > m),$$

a sequence $\{x_m\}$ is Cauchy in the non-Archimedean normed space if and only if $\{x_{m+1} - x_m\}$ converges to zero with respect to the non-Archimedean norm. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. The most important examples of non-Archimedean normed spaces are p -adic numbers. Let p be a fixed prime number. For any

nonzero rational number x , there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p . Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the p -adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x := \sum_{k \geq n_x} a_k p^k$, where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|x| := \sum_{k \geq n_x} a_k p^k = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field (see [9, 20]). A key property of p -adic numbers is that they do not satisfy the Archimedean axiom. It turned out that non-Archimedean normed spaces have many special nice applications in quantum theory, dynamical systems and biological models [13, 16].

In 2007, M.S. Moslehian and Th.M. Rassias [15] proved the generalized Hyers-Ulam stability of the Cauchy and quadratic functional equations in non-Archimedean normed spaces. Some papers ([4],[5],[6]) on the stability of various functional equations and inequalities in non-Archimedean normed spaces have been published after their stability results.

Now, we fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$ in the following definition. Let X be a linear space over $\mathbb{K}$. A quasi- β -norm is a real-valued function $\|\cdot\|$ on X satisfying the followings:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|^\beta \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(3) There is a constant $M \geq 1$ such that $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$. In this case, the pair $(X, \|\cdot\|)$ is called a quasi- β -normed space if $\|\cdot\|$ is a quasi- β -norm on X . The smallest possible M is called the modulus of concavity of the quasi- β -norm $\|\cdot\|$. A quasi- β -Banach space is a complete quasi- β -normed space. In addition, the quasi- β -norm $\|\cdot\|$ is called a (β, p) -norm if there exists a number p ($0 < p \leq 1$) with

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, the quasi- β -Banach space is called a (β, p) -Banach space [17].

In particular, we can refer to [2, 21] for the concept of quasi-normed spaces and p -Banach spaces for $\beta = 1$ as a special case.

Now, we introduce a new concept non-Archimedean quasi- β -normed spaces, which is a generalization of non-Archimedean normed spaces as in the following. Let X be a linear space over a non-Archimedean field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is said to be a non-Archimedean quasi- β -norm if it satisfies the following conditions:

(1) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|^\beta \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(3) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a non-Archimedean quasi- β -normed space if $\|\cdot\|$ is a non-Archimedean quasi- β -norm on X . A non-Archimedean quasi- β -Banach space is a complete non-Archimedean quasi- β -normed space.

For a real number p with $(0 < p \leq 1)$, the non-Archimedean quasi- β -norm $\|\cdot\|$ on X is called a non-Archimedean (β, p) -norm if, moreover, $\|\cdot\|^p$ satisfies the following strong triangle inequality

$$\|x + y\|^p \leq \max\{\|x\|^p, \|y\|^p\}$$

for all $x, y \in X$. In this case, a non-Archimedean quasi- β -Banach space is called a non-Archimedean (β, p) -Banach space.

We remark that for a given non-Archimedean norm $\|\cdot\|_1$ on a non-Archimedean normed space X , the function $\|x\| := \|x\|_1^\beta, x \in X$ defines a non-Archimedean quasi- β -norm on X .

Now, we consider a modified and generalized Cauchy–Jensen type functional equation

$$f\left(\frac{x+y}{n} + z\right) + f\left(\frac{y+z}{n} + x\right) + f\left(\frac{z+x}{n} + y\right) = \frac{n+2}{n}[f(x) + f(y) + f(z)] \quad (1)$$

for any fixed nonzero integer n . For any vector spaces X and Y , it is easily verified that a mapping $f : X \rightarrow Y$ satisfies (1) if and only if f is additive [14]. In this paper, we investigate the generalized Hyers–Ulam stability of (1) in non-Archimedean (β, p) -Banach spaces.

2. Stability of (1) in Non-Archimedean Quasi- β -Normed Spaces

We investigate the stability of the functional equation (1) in non-Archimedean (β, p) -Banach spaces using direct method. From now on, assume that X is a non-Archimedean quasi- α -normed space and that Y is a non-Archimedean (β, p) -Banach space with non-Archimedean (β, p) -norm $\|\cdot\|$ together with non-Archimedean valuation $|\frac{n+2}{n}| \neq 1, 0 < \alpha, \beta, p \leq 1$ without any specific reference.

For notational convenience, given a mapping $f : X \rightarrow Y$, we define a difference operator $Df : X^3 \rightarrow Y$ of the equation (1) by

$$\begin{aligned} Df(x, y, z) := & f\left(\frac{x+y}{n} + z\right) + f\left(\frac{y+z}{n} + x\right) + f\left(\frac{z+x}{n} + y\right) \\ & - \frac{n+2}{n}[f(x) + f(y) + f(z)] \end{aligned}$$

for all $x, y, z \in X$.

Theorem 2.1. *Suppose that a mapping $f : X \rightarrow Y$ satisfies the functional inequality*

$$\|Df(x, y, z)\| \leq \varphi(x, y, z) \quad (2)$$

and that $\varphi : X^3 \rightarrow [0, \infty)$ is a function such that

$$\lim_{m \rightarrow \infty} \frac{1}{|k|^{m\beta p}} \varphi(k^m x, k^m y, k^m z)^p = 0 \quad (3)$$

for all $x, y, z \in X$ and

$$\tilde{\varphi}(x) = \lim_{m \rightarrow \infty} \max \left\{ \frac{1}{|k|^{i\beta p}} \varphi(k^i x, k^i x, k^i x)^p : 0 \leq i < m \right\} \quad (4)$$

exists for all $x \in X$, where $k := \frac{n+2}{n}$. Then there exists an additive mapping $A : X \rightarrow Y$ defined by $A(x) = \lim_{m \rightarrow \infty} \frac{1}{k^m} f(k^m x)$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{|3k|^\beta} [\tilde{\varphi}(x)]^{\frac{1}{p}} \quad (5)$$

for all $x \in X$. Moreover, if

$$\lim_{l \rightarrow \infty} \lim_{m \rightarrow \infty} \max \left\{ \frac{1}{|k|^{i\beta p}} \varphi(k^i x, k^i x, k^i x)^p : l \leq i < m + l \right\} = 0 \quad (6)$$

for all $x \in X$, then A is a unique additive mapping satisfying (5).

Proof. Letting $y = z := x$ in (2), we have

$$\left\| 3f\left(\left(\frac{n+2}{n}\right)x\right) - 3\left(\frac{n+2}{n}\right)f(x) \right\| \leq \varphi(x, x, x), \quad (7)$$

which yields

$$\|3f(kx) - 3kf(x)\| \leq \varphi(x, x, x) \quad (8)$$

for all $x \in X$, where $k := \frac{n+2}{n}$. Dividing both sides on (8) by $|3k|^\beta$, one has

$$\left\| \frac{f(kx)}{k} - f(x) \right\| \leq \frac{1}{|3k|^\beta} \varphi(x, x, x) \quad (9)$$

for all $x \in X$. Replacing x by $k^{i-1}x$ and dividing by $|k|^{(i-1)\beta}$ in (9), we have

$$\left\| \frac{f(k^i x)}{k^i} - \frac{f(k^{i-1} x)}{k^{i-1}} \right\| \leq \frac{1}{|3k|^\beta} \frac{1}{|k|^{(i-1)\beta}} \varphi(k^{i-1} x, k^{i-1} x, k^{i-1} x) \quad (10)$$

for all $x \in X$ and all natural number i . Then it follows from (3) and (10) that a sequence $\{\frac{f(k^m x)}{k^m}\}_{m=1}^\infty$ is Cauchy for all $x \in X$. Thus, the sequence $\{\frac{f(k^m x)}{k^m}\}$ converges in the non-Archimedean (β, p) -Banach space Y . Therefore, we can define a mapping $A : X \rightarrow Y$ as

$$A(x) = \lim_{m \rightarrow \infty} \frac{f(k^m x)}{k^m}, \quad x \in X.$$

It follows from (9) and (10) that

$$\left\| f(x) - \frac{f(k^m x)}{k^m} \right\|^p \leq \frac{1}{|3k|^{\beta p}} \max \left\{ \frac{1}{|k|^{i\beta p}} \varphi(k^i x, k^i x, k^i x)^p : 0 \leq i < m \right\} \quad (11)$$

for all $m \in \mathbb{N}$ and all $x \in X$. By approaching m to infinity in (11) and using (4), one leads to the approximation (5). By (2) and (3), we obtain

$$\|DA(x, y, z)\|^p = \lim_{m \rightarrow \infty} \frac{\|Df(k^m x, k^m y, k^m z)\|^p}{|k|^{m\beta p}} \leq \lim_{m \rightarrow \infty} \frac{\varphi(k^m x, k^m y, k^m z)^p}{|k|^{m\beta p}} = 0$$

for all $x, y, z \in X$. Thus, the mapping A satisfies the equation (1) and so the mapping A is additive.

To prove the uniqueness, we assume that $A' : X \rightarrow Y$ be an additive mapping satisfying the approximation (5). Then, we figure out that for any $x \in X$ and all natural number l ,

$$\begin{aligned} \|A(x) - A'(x)\|^p &= \frac{1}{|k|^{l\beta p}} \|A(k^l x) - A'(k^l x)\|^p \\ &\leq \frac{1}{|k|^{l\beta p}} \max \{ \|A(k^l x) - f(k^l x)\|^p, \|f(k^l x) - A'(k^l x)\|^p \} \\ &\leq \frac{1}{|3k|^{\beta p}} \lim_{m \rightarrow \infty} \max \left\{ \frac{1}{|k|^{i\beta p}} \varphi(k^i x, k^i x, k^i x)^p : l \leq i < m + l \right\}, \end{aligned}$$

which tends to zero as $l \rightarrow \infty$. Thus $A(x) = A'(x)$ for all $x \in X$. This completes the proof. $\square$

Corollary 2.2. *Let $\rho : [0, \infty) \rightarrow [0, \infty)$ be a function satisfying*

$$(i) \ \rho(|k|^\alpha t) \leq \rho(|k|)^\alpha \rho(t) \quad \text{for all } t \geq 0 \quad \text{and} \quad (ii) \ \rho(|k|)^\alpha < |k|^\beta.$$

Suppose that a mapping $f : X \rightarrow Y$ satisfies the inequality

$$\|Df(x, y, z)\| \leq \theta \{ \rho(\|x\|) + \rho(\|y\|) + \rho(\|z\|) \}$$

for all $x, y, z \in X$ and for some $\theta > 0$. Then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3\theta}{|3k|^\beta} \rho(\|x\|)$$

for all $x \in X$.

Proof. Considering $\varphi(x, y, z) := \theta\{\rho(\|x\|) + \rho(\|y\|) + \rho(\|z\|)\}$, and applying Theorem 2.1, we obtain the desired result. $\square$

Theorem 2.3. Suppose that a mapping $f : X \rightarrow Y$ satisfies the functional inequality

$$\|Df(x, y, z)\| \leq \varphi(x, y, z) \quad (12)$$

and that $\varphi : X^3 \rightarrow [0, \infty)$ is a function such that

$$\lim_{m \rightarrow \infty} |k|^{m\beta p} \varphi\left(\frac{x}{k^m}, \frac{y}{k^m}, \frac{z}{k^m}\right)^p = 0 \quad (13)$$

for all $x, y, z \in X$ and

$$\tilde{\varphi}(x) := \lim_{m \rightarrow \infty} \max \left\{ |k|^{i\beta p} \varphi\left(\frac{x}{k^i}, \frac{x}{k^i}, \frac{x}{k^i}\right)^p : 1 \leq i < m+1 \right\} \quad (14)$$

exists for all $x \in X$, where $k := \frac{n+2}{n}$. Then there exists an additive mapping $A : X \rightarrow Y$ defined by $A(x) = \lim_{m \rightarrow \infty} k^m f\left(\frac{x}{k^m}\right)$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{|3k|^\beta} [\tilde{\varphi}(x)]^{\frac{1}{p}} \quad (15)$$

for all $x \in X$. Moreover, if

$$\lim_{l \rightarrow \infty} \lim_{m \rightarrow \infty} \max \left\{ |k|^{i\beta p} \varphi\left(\frac{x}{k^i}, \frac{x}{k^i}, \frac{x}{k^i}\right)^p : l+1 \leq i < l+m+1 \right\} = 0 \quad (16)$$

for all $x \in X$, then A is a unique additive mapping satisfying (15).

Proof. Letting $x := \frac{x}{k}$ in (8) and dividing it by $|3|^\beta$, we obtain

$$\left\| f(x) - kf\left(\frac{x}{k}\right) \right\| \leq \frac{1}{|3|^\beta} \varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right) \quad (17)$$

for all $x \in X$. Replacing x by $\frac{x}{k^{i-1}}$ and multiplying by $|k|^{(i-1)\beta}$ in (17), we have

$$\left\| k^{i-1} f\left(\frac{x}{k^{i-1}}\right) - k^i f\left(\frac{x}{k^i}\right) \right\| \leq \frac{1}{|3k|^\beta} |k|^{i\beta} \varphi\left(\frac{x}{k^i}, \frac{x}{k^i}, \frac{x}{k^i}\right) \quad (18)$$

for all $x \in X$. Thus, it follows from (13) and (18) that a sequence $\{k^m f(\frac{x}{k^m})\}$ is Cauchy for all $x \in X$. Then the sequence $\{k^m f(\frac{x}{k^m})\}$ converges in the non-Archimedean (β, p) -Banach space Y . Therefore, we can define a mapping $A : X \rightarrow Y$ as

$$A(x) = \lim_{m \rightarrow \infty} k^m f\left(\frac{x}{k^m}\right), \quad x \in X.$$

It follows from (17) and (18),

$$\left\| f(x) - k^m f\left(\frac{x}{k^m}\right) \right\|^p \leq \frac{1}{|3k|^{\beta p}} \max \left\{ |k|^{i\beta p} \varphi\left(\frac{x}{k^i}, \frac{x}{k^i}, \frac{x}{k^i}\right)^p : 1 \leq i < 1+m \right\} \quad (19)$$

for all $m \in \mathbb{N}$ and all $x \in X$. By taking m to approach infinity in (19) and using (14), one leads to the approximation (15). Applying the similar argument to the corresponding proof of Theorem 2.1, we get the desired result. $\square$

Corollary 2.4. *Let $\rho : [0, \infty) \rightarrow [0, \infty)$ be a function satisfying*

$$(i) \quad \rho\left(\frac{t}{|k|^\alpha}\right) = \rho\left(\frac{1}{|k|}\right)^\alpha \rho(t) \quad \text{for all } t \geq 0 \quad \text{and} \quad (ii) \quad \rho\left(\frac{1}{|k|}\right)^\alpha < |k|^{-\beta}.$$

Suppose that a mapping $f : X \rightarrow Y$ satisfies the inequality

$$\|Df(x, y, z)\| \leq \theta \{\rho(\|x\|) + \rho(\|y\|) + \rho(\|z\|)\}$$

for all $x, y, z \in X$ and for some $\theta > 0$. Then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3\theta}{|3|^\beta} \rho\left(\frac{1}{|k|}\right)^\alpha \rho(\|x\|) \left(\leq \frac{3\theta}{|3k|^\beta} \rho(\|x\|) \right)$$

for all $x \in X$.

3. Alternative Stability of (1) in Non-Archimedean Quasi- β -Normed Spaces

From now on, we investigate the generalized Hyers–Ulam stability of the functional inequality (1) using the contractive property of perturbing term on the difference operator Df .

Theorem 3.1. Assume that a mapping $f : X \rightarrow Y$ satisfies the functional inequality

$$\|Df(x, y, z)\| \leq \varphi(x, y, z)$$

for all $x, y, z \in X$ and there exists a constant L with $|k|^\alpha L < |k|^\beta$ for which the perturbing function $\varphi : X^3 \rightarrow [0, \infty)$ satisfies

$$\varphi(kx, ky, kz) \leq |k|^\alpha L \varphi(x, y, z) \quad (20)$$

for all $x, y, z \in X$, where $k := \frac{n+2}{n}$. Then, there exists a unique additive mapping $A : X \rightarrow Y$ given by $A(x) = \lim_{m \rightarrow \infty} \frac{1}{k^m} f(k^m x)$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{|3k|^\beta} \varphi(x, x, x) \quad (21)$$

for all $x \in X$.

Proof. It follows from (9) and (20) that

$$\begin{aligned} \left\| \frac{f(k^{m+1}x)}{k^{m+1}} - \frac{f(k^l x)}{k^l} \right\|^p &\leq \frac{1}{|3k|^{\beta p}} \max_{l \leq i \leq m} \frac{1}{|k|^{\beta p i}} \varphi(k^i x, k^i x, k^i x)^p \\ &\leq \frac{1}{|3k|^{\beta p}} \max_{l \leq i \leq m} \frac{(|k|^\alpha L)^{p i}}{|k|^{\beta p i}} \varphi(x, x, x)^p \\ &= \frac{1}{|3k|^{\beta p}} \frac{(|k|^\alpha L)^{p l}}{|k|^{\beta p l}} \varphi(x, x, x)^p \end{aligned}$$

for all nonnegative integers m and l with $m > l \geq 0$ and all $x \in X$. Thus, it follows that the sequence $\{\frac{f(k^m x)}{k^m}\}$ is Cauchy for all $x \in X$. And so we can define a mapping $A : X \rightarrow Y$ by

$$A(x) = \lim_{m \rightarrow \infty} \frac{f(k^m x)}{k^m}, \quad x \in X,$$

using the contractive property $\frac{|k|^\alpha L}{|k|^\beta} < 1$. The rest of the proof is similarly verified as in Theorem 2.1. $\square$

The following corollary is a generalization of the stability result when X is a non-Archimedean normed space with $\alpha = 1$ and Y is a non-Archimedean Banach space with $\beta = 1 = p$.

Corollary 3.2. Assume that there exists a real constant r satisfying either $|k| < 1, \alpha r > \beta$ or $|k| > 1, \alpha r < \beta$, where $k := \frac{n+2}{n}$. If a mapping $f : X \rightarrow Y$ satisfies the following functional inequality

$$\|Df(x, y, z)\| \leq \theta\{\|x\|^r + \|y\|^r + \|z\|^r\}$$

for all $x, y, z \in X$ and for some $\theta \geq 0$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3\theta}{|3k|^\beta} \|x\|^r$$

for all $x \in X$.

Proof. Letting $\varphi(x, y, z) = \theta\{\|x\|^r + \|y\|^r + \|z\|^r\}$ and applying Theorem 3.1 with $L := \frac{|k|^{\alpha r}}{|k|^\alpha}$, we obtain the desired result. $\square$

Theorem 3.3. Assume that a mapping $f : X \rightarrow Y$ satisfies the functional inequality

$$\|Df(x, y, z)\| \leq \varphi(x, y, z)$$

for all $x, y, z \in X$ and there exists a constant L with $|k|^\beta L < |k|^\alpha$ for which the perturbing function $\varphi : X^3 \rightarrow [0, \infty)$ satisfies

$$\varphi\left(\frac{x}{k}, \frac{y}{k}, \frac{z}{k}\right) \leq \frac{L}{|k|^\alpha} \varphi(x, y, z) \quad (22)$$

for all $x, y, z \in X$, where $k := \frac{n+2}{n}$. Then, there exists a unique additive mapping $A : X \rightarrow Y$ given by $A(x) = \lim_{m \rightarrow \infty} k^m f(\frac{x}{k^m})$ such that

$$\|f(x) - A(x)\| \leq \frac{L}{|3|^\beta |k|^\alpha} \varphi(x, x, x) \left(< \frac{1}{|3k|^\beta} \varphi(x, x, x) \right) \quad (23)$$

for all $x \in X$.

Proof. The proof is similar to that of Theorem 3.1. $\square$

Corollary 3.4. Assume that there exists a real constant r satisfying either $|k| < 1, \alpha r < \beta$ or $|k| > 1, \alpha r > \beta$, where $k := \frac{n+2}{n}$. If a mapping $f : X \rightarrow Y$ satisfies the following functional inequality

$$\|Df(x, y, z)\| \leq \theta\{\|x\|^r + \|y\|^r + \|z\|^r\}$$

for all $x, y, z \in X$ and for some $\theta \geq 0$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{3\theta}{|3|^\beta |k|^{\alpha r}} \|x\|^r$$

for all $x \in X$.

Proof. Letting $\varphi(x, y, z) = \theta\{\|x\|^r + \|x\|^r + \|x\|^r\}$ and applying Theorem 3.3 with $L := \frac{|k|^\alpha}{|k|^{\alpha r}}$, one has the desired result. $\square$

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