T-SEPARATING SETS FOR COHERENT SEQUENCES

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Abstract: In a previous paper the author used methods of Witzany to give a lower bound for the smallest repeat point of a coherent sequence. Here the notion of a T-separating set is introduced, and the lower bound is improved.

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1. Introduction

In [4] some methods are introduced for constructing separating stationary sets for coherent sequences of normal ultrafilters. Some further results are given in [3]. Here these methods are improved on. Superschemes were introduced in [1], and an improved discussion is given in [2]. The methods here permit the use of superschemes in constructing separating stationary sets.

As noted in [3], by results of Mitchell there is a model $L[\mathcal{U}]$ such that in $L[\mathcal{U}]$, $\mathcal{U}$ is a coherent sequence of normal ultrafilters comprising all the normal ultrafilters. It is well-known that GCH holds in $L[\mathcal{U}]$. Notation for coherent sequences will be as in [3]. Hereafter it will be assumed that GCH holds and $\mathcal{U}$ is maximal, so that for a measurable cardinal κ , $\text{Dom}(\mathcal{U}(\kappa)) = o(\kappa) \leq \kappa^{++}$.

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2. Separating Sets

Say that S is a separating set for $\mathcal{U}(\kappa)$ at α if $S \in \mathcal{U}(\kappa)(\alpha)$ but $S \notin \mathcal{U}(\kappa)(\beta)$ for $\beta < \alpha$; such exists iff α is not a repeat point. Say that S is T-separating if in addition $S \in \mathcal{U}(\kappa)(\beta)$ for $\alpha \leq \beta < \text{Dom}(\mathcal{U}(\kappa))$. These may readily be seen to exist for α up to a bound given in theorem 22.g of [3].

Given a measurable cardinal κ and a function $f : \kappa \mapsto \text{Ord}$ let $D_f^> = \{\lambda \in \text{Card} \cap \kappa : o(\lambda) \geq f(\lambda)\}$. For $\beta < \text{Dom}(\kappa)$, U_β will be used as an abbreviation for $\mathcal{U}(\kappa)(\beta)$.

Theorem 1. *Suppose $\alpha < \text{Dom}(\mathcal{U}(\kappa))$ and f represents α on $[0, \text{Dom}(\mathcal{U}(\kappa)))$. Then $D_f^>$ is T-separating at α .*

Proof. $D_f^> \in U_\beta$ iff $[o]_{U_\beta} \geq [f]_{U_\beta}$ iff $\beta \geq \alpha$. □

It may be easier to construct T-separating sets than representing functions on $[0, \text{Dom}(\mathcal{U}(\kappa)))$, and studying them in their own right is of interest.

Theorem 2. *Suppose S is a separating set at α and $\alpha + 1 < \text{Dom}(\mathcal{U}(\kappa))$. Then there is a T-separating set S' at $\alpha + 1$.*

Proof. Let $S' = \{\lambda \in \text{Card} \cap \kappa : \exists \eta < \text{Dom}(\mathcal{U}(\lambda))(S \cap \lambda \in \mathcal{U}(\lambda)(\eta))\}$. Using coherence and the fact that S is separating, $S' \in U_\beta$ iff $\models_{\text{Ult}_{U_\beta}(V)} \exists \eta < \text{Dom}(\mathcal{U}(\kappa))(S \in \mathcal{U}(\kappa)(\eta))$ iff $\beta > \alpha$. □

Theorem 3. *Suppose $\eta < \kappa$, for $\xi < \eta$ S_ξ is a T-separating set at α_ξ , $\alpha = \sup_{\xi < \eta} \alpha_\xi$, and $\alpha < \text{Dom}(\mathcal{U}(\kappa))$. Let $S = \bigcap_{\xi < \eta} S_\xi$; then S is a T-separating set at α .*

Proof. For $\beta \geq \alpha$, since $S_\xi \in U_\beta$ for $\xi < \eta$ and U_β is κ -complete, $S \in U_\beta$. If $\beta < \alpha$ then $\beta < \alpha_\xi$ for some ξ , so $S_\xi \notin U_\beta$, so $S \notin U_\beta$ since $S \subseteq S_\xi$. □

Theorem 4. *Suppose for $\xi < \kappa$ S_ξ is a T-separating set at α_ξ , $\alpha = \sup_{\xi < \kappa} \alpha_\xi$, and $\alpha < \text{Dom}(\mathcal{U}(\kappa))$. Let $S = \bigtriangleup_{\xi < \kappa} S_\xi$; then S is a T-separating set at α .*

Proof. For $\beta \geq \alpha$, since $S_\xi \in U_\beta$ for $\xi < \kappa$ and U_β is normal, $S \in U_\beta$. If $\beta < \alpha$ then $\beta < \alpha_\xi$ for some ξ , so $S_\xi \notin U_\beta$, so $S \notin U_\beta$ since $S \subseteq_{\mathcal{I}} S_\xi$ where $\mathcal{I}$ is the thin ideal. □

Recall from [2] that for $\kappa \in \text{Card}$ a scheme is a pair $\Sigma = \langle \sigma, \phi \rangle$ where $\sigma < \kappa^+$ and ϕ is a function whose domain is the set of limit ordinals $\alpha \leq \sigma$. For $\alpha \in \text{Dom}(\phi)$, $\phi(\alpha)$ is an increasing function with domain an ordinal $\eta \leq \kappa$,

and whose range is an unbounded subset of α . If $\text{cf}(\alpha) < \kappa$ then $\eta < \kappa$, and if $\text{cf}(\alpha) = \kappa$ then $\eta = \kappa$.

A scheme is a recipe for an iteration. Given a scheme Σ with $\sigma < \text{Dom}(\mathcal{U}(\kappa))$ the subset S_α^Σ may be defined inductively for $\alpha \leq \sigma$ as follows.

0. $S_0^\Sigma = \text{Card} \cap \kappa$.
1. $S_{\beta+1}^\Sigma = (S_\beta^\Sigma)'$, as in theorem 2.
2. $(\alpha \in \text{Lim}, \text{cf}(\alpha) < \kappa) \cap_{\xi < \text{Dom}(\phi(\alpha))} S_{\phi(\alpha)(\xi)}^\Sigma$.
3. $(\alpha \in \text{Lim}, \text{cf}(\alpha) = \kappa) \triangle_{\xi < \kappa} S_{\phi(\alpha)(\xi)}^\Sigma$.

By theorems 2-4 Σ_α^Σ is T-separating for all $\alpha \leq \sigma$.

Suppose there is an α such that there is no T-separating set at α , and let θ be the least such α . For another corollary of theorems 2-4, $\text{cf}(\theta) = \kappa^+$.

3. Superschemes

Recall from [2] that for $\kappa \in \text{Card}$ a superscheme is a pair $\Sigma = \langle \sigma, \phi \rangle$ where $\sigma < \kappa^{++}$ and ϕ is as for a scheme. To use a superscheme for an iteration a method must be specified for obtaining S_α when $\text{cf}(\alpha) = \kappa^+$. Some discussion of this problem will be given here.

Suppose $\alpha = \kappa^+$, U is a normal ultrafilter on κ , and $M = \text{Ult}_U(V)$. Since the well-orders of κ , coded as subsets of κ , are the same in V and M , and the order type function is Δ_1 , $(\kappa^+)^M = \kappa^+$. Thus, the function $\lambda \mapsto \lambda^+$ represents κ^+ mod any normal ultrafilter on κ , and a T-separating set may be constructed using theorem 1.

Alternatively, referring to [3] let $\preceq$ be a Δ_∞^1 WPS with $\Omega(\prec) = \kappa^+$ (in fact there is a $\Sigma_1^1 \preceq$). According to theorem 10 of [3], $\lambda \mapsto \Omega(\prec_\lambda)$ (defined on $D_{\preceq}$) represents κ^+ in any $\text{Ult}_U(V)$.

Another construction may be given by adapting a method of proposition 3.9 of [4]. As in [3] let C denote the map $\alpha \mapsto j_{\mathcal{U}(\kappa)(\alpha)}(\kappa)$. $C_{\mathcal{U}(\kappa)}$ may be written to indicate $\mathcal{U}$ and κ .

Theorem 5. *Suppose S is a separating set at α . For any $\nu \in (\alpha, \min(\text{Dom}(\mathcal{U}(\kappa)), C(\alpha)))$ there is a T-separating set at ν .*

Proof. As in the proof of theorem 2 let $S_1 = \{\lambda \in \text{Card} \cap \kappa : \exists \eta < \text{Dom}(\mathcal{U}(\lambda))(S \cap \lambda \in \mathcal{U}(\lambda)(\eta))\}$, so that $S_1 \in U_\beta$ iff $\beta > \alpha$. For $\lambda \in S_1$ let $f_\alpha(\lambda)$ be the least η ; then for $\beta > \alpha$ the function f_α represents α in $\text{Ult}_{\mathcal{U}(\kappa)(\beta)}(V)$. Choose $g : \kappa \mapsto \kappa$ such that $[g]_{\mathcal{U}(\kappa)(\alpha)} = \nu$. For $\lambda \in S_1$ let $\tilde{g}(\lambda) = [g \upharpoonright \lambda]_{\mathcal{U}(\lambda)(f_\alpha(\lambda))}$; otherwise let $\tilde{g}(\lambda) = 0$. Suppose $\beta > \alpha$ and let M

denote $\mathcal{U}(\kappa)(\beta)$; then $g \in M$, and one verifies (see [4]) that $[\tilde{g}]_{\mathcal{U}(\kappa)}(\beta) = \nu$. The set $S_1 \cap D_{\tilde{g}}^{\geq}$ is T-separating at ν . $\square$

Let $C^{(1)}$ denote the fixed point enumerator of C . If there is an α such that there is no T-separating set at α let θ denote the smallest such. As in theorem 22 of [3], $C(\theta) = \theta$ and $C^{(1)}(\kappa^+) \leq \theta$.

4. Iterated Fixed Points

Let $C^{(\sigma)}$ denote the result of applying the fixed point operator to C σ times. If $\text{Dom}(\mathcal{U}(\kappa)) = \kappa^{++}$ then for $\sigma < \kappa^{++}$ $\text{Ran}(C^{(\sigma)})$ is a club set, and the intersection of all these is empty. If $\text{Dom}(\mathcal{U}(\kappa)) < \kappa^{++}$ then $\text{Dom}(C^{(\sigma)})$ decreases as σ increases, becoming $\emptyset$ for some $\sigma \leq \text{Dom}(\mathcal{U}(\kappa))$.

Theorem 6. *Suppose $\alpha < \text{Dom}(\mathcal{U}(\kappa))$, f_α represents α on $[0, \text{Dom}(\mathcal{U}(\kappa))]$, and $\alpha \in \text{Dom}(C^{(1)})$. Then there is a T-separating set at $C^{(1)}(\alpha)$.*

Proof. Let $S = \{\lambda : o(\lambda) \geq C_{\mathcal{U}(\lambda)}^{(1)}(f_\alpha(\lambda))\}$. Then $S \in U_\beta$ iff $\beta \geq C_{\mathcal{U}(\kappa)}^{(1)}(\alpha)$. The condition on λ holds iff either $\exists \eta < \text{Dom}(\mathcal{U}(\lambda))(\eta = C_{\mathcal{U}(\lambda)}^{(1)}(f_\alpha(\lambda)))$, or $\text{Dom}(\mathcal{U}(\lambda))$ is a limit ordinal and $\forall \zeta < f_\alpha(\lambda) \exists \eta < \text{Dom}(\mathcal{U}(\lambda))(\eta = C_{\mathcal{U}(\lambda)}^{(1)}(\zeta))$ and $\forall \zeta < \text{Dom}(\mathcal{U}(\lambda))(C_{\mathcal{U}(\lambda)}(\zeta) < \text{Dom}(\mathcal{U}(\lambda)))$. $\square$

Theorem 7. *Suppose $\alpha < \text{Dom}(\mathcal{U}(\kappa))$, there is a T-separating set at α , and $\alpha \in \text{Dom}(C^{(1)})$. Then there is a T-separating set at $C^{(1)}(\alpha)$.*

Proof. The proof of theorem 6 may be modified. Let S_1 be a T-separating set at α . Let f_α^E be an E-representing function for α , as defined in [3]. Let S_2 be the set S , as in the proof of theorem 6, with f_α^E used in place of f_α . Then $S_1 \cap S_2$ is T-separating at $C^{(1)}(\alpha)$. $\square$

Theorem 8. *Suppose $\alpha < \text{Dom}(\mathcal{U}(\kappa))$, f_α represents α on $[0, \text{Dom}(\mathcal{U}(\kappa))]$, f_σ represents σ on $[0, \text{Dom}(\mathcal{U}(\kappa))]$, and $\alpha \in \text{Dom}(C^{(\sigma)})$. Then there is a T-separating set at $C^{(\sigma)}(\alpha)$.*

Proof. Let $S = \{\lambda : o(\lambda) \geq C_{\mathcal{U}(\lambda)}^{(f_\sigma(\lambda))}(f_\alpha(\lambda))\}$. Then $S \in U_\beta$ iff $\beta \geq C_{\mathcal{U}(\kappa)}^{(\sigma)}(\alpha)$. The condition on λ holds iff either $\exists \eta < \text{Dom}(\mathcal{U}(\lambda))(\eta = C_{\mathcal{U}(\lambda)}^{(\sigma)}(f_\alpha(\lambda)))$, or $\text{Dom}(\mathcal{U}(\lambda))$ is a limit ordinal and $\forall \zeta < f_\alpha(\lambda) \exists \eta < \text{Dom}(\mathcal{U}(\lambda))(\eta = C_{\mathcal{U}(\lambda)}^{(f_\sigma(\lambda))}(\zeta))$ and $\forall \zeta < \text{Dom}(\mathcal{U}(\lambda)) \forall \tau < f_\sigma(\lambda) (C_{\mathcal{U}(\lambda)}^{(\tau)}(\zeta) < \text{Dom}(\mathcal{U}(\lambda)))$. $\square$

Theorem 9. *Suppose $\alpha < \text{Dom}(\mathcal{U}(\kappa))$, there is a T-separating set at α , there is a T-separating set at σ , and $\alpha \in \text{Dom}(C^{(\sigma)})$. Then there is a T-separating set at $C^{(\sigma)}(\alpha)$.*

Proof. The proof of theorem 8 may be modified. Let $S_{1\alpha}$ be a T-separating set at α . Let f_α^E be an E-representing function for α . Let $S_{1\sigma}$ be a T-separating set at σ . Let f_σ^E be an E-representing function for σ . Let S_2 be the set S , as in the proof of theorem 8, with f_α^E used in place of f_α and f_σ^E used in place of f_σ . Then $S_{1\alpha} \cap S_{1\sigma} \cap S_2$ is T-separating at $C^{(\sigma)}(\alpha)$. $\square$

To show that repeat points do not exist it would suffice to show that for any σ , if $\theta = C^\sigma(\alpha)$ then $\alpha = \theta$; for then $\theta \in \text{Ran}(C^\sigma)$ for all σ , a contradiction. Theorem 9 imposes restrictions on a counterexample.

These methods can be pursued further. Whether there are methods which may be pursued in $L[\mathcal{U}]$ is a question of considerable interest.

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