MORE ON WEAK DECOMPOSITION OF CONTINUITY

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\textbf{Abstract:} Using the notion of $w$-space on a set $X$ and the concept of locally $w$-semi open set, we introduce, study and characterize the notions of $w$-$s$-Kernel of a subset $A$ of $X$. Also we introduce and study a new forms of weak decomposition of continuity.

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\section{1. Introduction and Preliminaries}

In the last years, different variants of open sets are being studied. Recently, a
significant contribution to the theory of generalized open sets have been presented by A. Császár [1], [2], [3]. Specifically, in 2002, A. Császár [1], introduced the notions of generalized topology and generalized continuity. It is observed that a large numbers of articles are devoted to the study of generalized open sets and certain type of sets associated to a topological spaces, containing the class of open sets and possessing properties more or less to those open sets. Bishwambhar. et al. [4] studied some type of decomposition of continuity using generalized topologies and in [5], studied some weak forms of continuity. Rosas E. et al. in [10], give a new theory of decomposition of continuous functions using generalized topologies. In 2015, W. K. Min et al. [7], introduced and studied the notions of weak structures on a nonempty set $X$. In 2016, W. K. Min et al. introduced the notions of $w$-semiopen sets and $w$-semi continuity in $w$-spaces. Later in 2017, W. K. Min in [6], introduced and studied the notions of weakly $w_\tau g$-closed set and weakly $w_\tau g$-open set as a generalization of the $w_\tau g$-closed set and $w_\tau g$-open set in associated $w$-spaces. E. Rosas et al in [9], introduce the concepts of locally $w$-regular closed sets and locally $w$-semi regular semi closed and a new weak decomposition of some type of weak continuity functions are studied and characterized. In this article, using the notion of $w$-semi open set, we introduce the concept of locally $w$-semi open set as a generalization of locally $w$-closed and give a new theory of weak decomposition of continuity and some weak form of continuity are studied. Throughout this paper $cl(A)$ (respectively $int(A)$) denotes the closure (respectively interior) of $A$ in a topological space $X$.

2. Preliminaries

**Definition 2.1.** [7] Let $X$ be a nonempty set. A subfamily $w_X$ of the power set $P(X)$ is called a weak structure on $X$ if it satisfies the following:

1. $\emptyset \in w_X$ and $X \in w_X$.
2. For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$

Then the pair $(X, w_X)$ is called a $w$-space on $X$. An element $U \in w_X$ is called $w$-open set and the complement of a $w$-open set is a $w$-closed set.

**Definition 2.2.** [7] Let $(X, w_X)$ be a $w$-space. For a subset $A$ of $X$,

1. The $w$-closure of $A$ is defined as $wC(A) = \bigcap\{F : A \subseteq F, X \setminus F \in w_X\}$.
2. The $w$-interior of $A$ is defined as $wI(A) = \bigcup\{U : U \subseteq A, U \in w_X\}$.
Theorem 2.3. [7] Let \((X, w_X)\) be a w-space on \(X\). \(A, B\) subsets of \(X\). Then the following hold:

1. If \(A \subseteq B\), then \(wI(A) \subseteq wI(B)\) and \(wC(A) \subseteq wC(B)\).
2. \(wI(wI(A)) = wI(A)\) and \(wC(wC(A)) = wC(A)\)
3. \(wC(X \setminus A) = X \setminus wI(A)\) and \(wI(X \setminus A) = X \setminus wC(A)\)
4. If \(A\) is w-closed (resp. w-open), then \(wC(A) = A\) (resp. \(wI(A) = A\))

Definition 2.4. [8] Let \((X, w_X)\) be a \(\omega\)-space on \(X\). A subset \(A\) of \(X\) is called w-semi open if \(A \subseteq wC(wI(A))\). The complement of a w-semi open set is called w-semi closed.

The collection of all w-semi open sets is denoted by \(wSO(X, w_X)\) and the collection of all w-semi closed sets is denoted by \(wSC(X, w_X)\)

Definition 2.5. [8] Let \((X, w_X)\) be a w-space on \(X\). For a subset \(A\) of \(X\)

1. The w-semi closure of \(A\) is defined as \(wsC(A) = \bigcap\{F : A \subseteq F, X \setminus F \in wSO(X, w_X)\}\).
2. The w-semi interior of \(A\) is defined as \(wsI(A) = \bigcup\{U : U \subseteq A, U \in wSO(X, w_X)\}\).

Theorem 2.6. [8] Let \((X, w_X)\) be a w-space on \(X\). \(A, B\) subsets of \(X\). Then the following hold:

1. \(wsI(A) \subseteq A\) and \(A \subseteq wsC(A)\).
2. If \(A \subseteq B\), then \(wsI(A) \subseteq wsI(B)\) and \(wsC(A) \subseteq wsC(B)\).
3. \(wsI(wsI(A)) = wsI(A)\) and \(wsC(wsC(A)) = wsC(A)\).
4. \(A\) is w-semi closed (resp. w-semi open), if and only if \(wsC(A) = A\) (resp. \(wsI(A) = A\)).

Theorem 2.7. [8] Let \((X, w_X)\) be a w-space on \(X\) and \(A\) a subset of \(X\). Then \(wsC(A)\) is an w-semi closed.

Theorem 2.8. Let \((X, w_X)\) be a w-space on \(X\) and \(A\) a subset of \(X\). Then \(A \cup wI(wC(A)) \subseteq wsC(A)\).
The following example shows that the reverse contention in the above theorem is not necessarily true.

**Example 2.9.** Let \( X = \mathbb{N} \) be the set of natural numbers. Define \( w_X = \{\emptyset, \{1\}, \mathbb{N}\} \cup P(\{2n : n \in \mathbb{N}\}) \). The set of \( w \)-closed sets is \( \emptyset,\{1\},\mathbb{N}\setminus\{1\}\cup\{A^c : A \in P(\{2n : n \in \mathbb{N}\})\} \). The set of \( w \)-semiopen sets is \( \emptyset,\{1\},\mathbb{N},F_1,F_2\cup P(\{2n : n \in \mathbb{N}\})\) where \( F_1 \cap \{2n : n \in \mathbb{N}\} \neq \emptyset \) and \( 1 \in F_2 \). If we take \( A = \{3\} \), \( wsC(A) = \{3\} \), \( wC(A) = \{2n + 1 : n \in \mathbb{N}\} \) and \( wI(wC(A)) = \{1\} \). Observe that \( A \cup wI(wC(A)) = \{1,3\} \supset \{3\} = wsC(A) \).

**Theorem 2.10.** Let \((X,w_X)\) be a \( w \)-space on \( X \) and \( A,B \) subsets of \( X \). Then

1. \( x \in wsC(A) \) if and only if \( A \cap V \neq \emptyset \), for every \( w \)-semiopen set \( V \) containing \( x \).
2. \( wsC(A \cap B) = wsC(A) \cap wsC(B) \).

### 3. New Types of \( w \)-Closed Sets

Throughout this paper \((X,w_X,\tau)\) a weak topological space denotes \((X,w_X)\) is a \( w \)-space and \((X,\tau)\) is a topological space.

**Definition 3.1.** [4] Let \((X,w_X,\tau)\) be a weak topological space. A subset \( A \) of \( X \) is called locally \( w \)-closed if \( A = U \cap F \) where \( U \in \tau \) and \( F \) is \( w \)-closed.

**Remark 3.2.** If \((X,w_X,\tau)\) is a weak topological space, then every open set as well as a \( w \)-closed set is locally \( w \)-closed.

**Theorem 3.3.** Let \((X,w_X,\tau)\) be a weak topological space. If \( A \subseteq X \) is locally \( w \)-closed then there exists an open set \( U \) such that \( A = U \cap wC(A) \).

**Proof.** Let \( A \) be a locally \( w \)-closed subset of \( X \), then \( A = U \cap F \), where \( U \in \tau \) and \( F \) is \( w \)-closed. It follows that \( A = A \cap U \subseteq U \cap wC(A) \subseteq U \cap wC(F) = U \cap F = A \). In consequence, \( A = U \cap wC(A) \).

**Example 3.4.** In Example 3.12, \( \{a,b\} = \{a,b,c\} \cap wsC(\{a,b\}) \), but \( \{a,b\} \) is not a locally \( w \)-closed set.

**Definition 3.5.** Let \((X,w_X,\tau)\) be a weak topological space. A subset \( A \) of \( X \) is called:
1. \( w\)-t-set if \( \text{int}(A) = \text{int}(wC(A)) \).

2. \( w\)-B-set if \( A = U \cap V, \ U \in \tau, \ V \) is a \( w\)-t-set.

3. \( w^\ast\)-open set if \( A \subseteq \text{cl}(wI(A)) \).

4. \( w^\ast\)-open set if \( A \subseteq \text{int}(wC(A)) \).

**Definition 3.6.** Let \((X, w_X, \tau)\) be a weak topological space. A subset \( A \) of \( X \) is called generalized \( w \)-closed or simple a \( gw \)-closed if \( wC(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \in \tau \).

**Definition 3.7.** [9] Let \((X, w_X, \tau)\) be a weak topological space. A subset \( A \) of \( X \) is called locally \( w \)-semi closed or simple locally \( w \)-s-closed if \( A = U \cap V \), \( U \in \tau \), \( V \) is a \( w \)-st-set.

**Remark 3.8.** If \((X, w_X, \tau)\) is a weak topological space, then every open set as well as a \( w \)-semi closed set is locally \( w \)-semi closed, also every locally \( w \)-closed set is locally \( w \)-semi closed.

**Definition 3.9.** Let \((X, w_X, \tau)\) be a weak topological space. A subset \( A \) of \( X \) is called:

1. \( w\)-st-set if \( \text{int}(A) = \text{int}(wsC(A)) \).

2. \( w\)-sB-set if \( A = U \cap V, \ U \in \tau, \ V \) is a \( w \)-st-set.

3. \( w^\ast\)-semi open set (briefly \( w^\ast\)-s-open set) if \( A \subseteq \text{cl}(wsI(A)) \).

4. \( w^\ast\)-semi open set (briefly \( w^\ast\)-s-open set) if \( A \subseteq \text{int}(wsC(A)) \).

The complement of a \( w^\ast\)-semi open set is called \( w^\ast\)-semi closed set (briefly \( w^\ast\)-s-closed set).

The following theorems characterizes the locally \( w \)-semi closed sets.

**Theorem 3.10.** Let \((X, w_X, \tau)\) be a weak topological space. \( A \subseteq X \) is locally \( w \)-semi closed if and only if \( X - A \) is the union of a closed set and a \( w \)-semi open set.

**Proof.** Suppose that \( A \) is locally \( w \)-semi closed, then \( A = U \cap F \) where \( U \in \tau \) and \( F \) is \( w \)-semi closed. It follows that \( X - A = X \cap (U \cap F)^c = (U^c \cup F^c) = (X - U) \cup (X - F) \). Conversely, suppose that \( X - A = W \cup G \), \( W \) closed set and \( G \) \( w \)-semi open. Then \( A = X - (X - A) = X - (W \cup G) = (X - W) \cap (X - G) \). \( \square \)
**Theorem 3.11.** Let \((X, w_X, \tau)\) be a weak topological space. \(A \subseteq X\) is locally \(w\)-semi closed if and only if there exists an open set \(U\) such that \(A = U \cap wsC(A)\).

Proof. Let \(A\) be a locally \(w\)-semi closed subset of \(X\), then \(A = U \cap F\), where \(U \in \tau\) and \(F\) is \(w\)-semi closed. It follows that \(A = A \cap U \subseteq U \cap wsC(A) \subseteq U \cap wsC(F) = U \cap F = A\). In consequence, \(A = U \cap wsC(A)\). Conversely, since \(wsC(A)\) is a \(w\)-semi closed, it follows that \(A\) is locally \(w\)-semi closed. \(\Box\)

In the following example, we can see that there exists a locally \(w\)-semi closed set that is not open as well as \(w\)-semi closed, \(w\)-st-set that is not \(w\)-s-set, \(w\)-sB-set that is not \(w\)-B-set.

**Example 3.12.** Let \(X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{a, b, c\}, \{c, d\}, \{c\}\}\) and \(w_X = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}\) be a weak structure on \((X, \tau)\). Then:

1. \(w\)-closed sets \(= \{\emptyset, X, \{c, d\}, \{a, d\}, \{a, c, d\}\}\).
2. locally \(w\)-closed \(= \{\emptyset, X, \{a, b, c\}, \{a, c, d\}, \{c, d\}, \{c\}, \{a, d\}, \{a\}, \{a, c\}, \{d\}\}\).
3. \(wSO(X, w_X) = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}\).
4. \(wSC(X, w_X) = \{\emptyset, X, \{a, c, d\}, \{c, d\}, \{a, d\}, \{a, c\}, \{d\}, \{c\}, \{b\}\}\).
5. locally \(w\)-s-closed \(= \{\emptyset, X, \{a, b, c\}, \{c, d\}, \{c\}, \{a, c, d\}, \{c, d\}, \{a, d\}, \{a, c\}, \{d\}, \{b\}, \{a\}\}\).
6. \(w\)-t-set \(= \{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}\).
7. \(w\)-st-set \(= \{\emptyset, X, \{a\}, \{b\}, \{d\}, \{a, d\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}\).
8. \(w\)-B-set \(= \{\emptyset, X, \{a, b, c\}, \{c, d\}, \{c\}, \{a\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}, \{a, c\}\}\).
9. \(w\)-sB-set \(= \{\emptyset, X, \{a, b, c\}, \{c, d\}, \{c\}, \{a\}, \{b\}, \{d\}, \{a, d\}, \{a, c\}, \{a, c, d\}\}\).
10. \(w^*\)-open \(= \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}\).
11. \(w^*\)-s-open \(= \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}\).
12. $w$-open $= \{ \emptyset, X, \{ b \}, \{ c \}, \{ a, b \}, \{ b, c \}, \{ b, d \}, \{ c, d \}, \{ a, b, c \}, \{ b, c, d \} \}$.

13. $w$-$s$-open $= \{ \emptyset, X, \{ a, b \}, \{ b, c \}, \{ b, d \}, \{ c, d \}, \{ a, b, d \}, \{ a, c, d \}, \{ b, c, d \} \}$.

The following examples will be useful to better understand the development of this article.

**Example 3.13.** Let $X = \{ a, b, c \}$, $\tau = \{ \emptyset, X, \{ a \}, \{ b \} \}$ and $w_X = \{ \emptyset, X, \{ a \}, \{ a, c \} \}$ be a weak structure on $(X, \tau)$. Then:

1. $w$-closed sets $= \{ X, \{ b, c \}, \{ b \} \}$.

2. locally $w$-closed $= \{ \emptyset, X, \{ a \}, \{ b \}, \{ a, b \} \}$.

3. $wSO(X, w_X) = \{ \emptyset, X, \{ a \}, \{ b \}, \{ a, b \}, \{ a, c \} \}$.

4. $wSC(X, w_X) = \{ \emptyset, X, \{ b, c \}, \{ c \}, \{ b \}, \{ a \} \}$.

5. locally $w$-$s$-closed $= \{ \emptyset, X, \{ b, c \}, \{ c \}, \{ b \}, \{ a \}, \{ a, b \}, \{ a, c \} \}$.

6. $w$-t-set $= \{ X, \{ b \}, \{ b, c \} \}$.

7. $w$-st-set $= \{ \emptyset, X, \{ b \}, \{ c \}, \{ a \}, \{ b, c \}, \{ a, c \} \}$.

8. $w$-B-set $= \{ \emptyset, X, \{ b \}, \{ a \}, \{ b, c \}, \{ a, b \} \}$.

9. $w$sB-set $= \{ \emptyset, X, \{ b \}, \{ a \}, \{ a, b \}, \{ a, c \}, \{ b \} \}$.

10. $w^*$-open $= \{ \emptyset, \{ a \}, \{ b \}, \{ a, c \} \}$.

11. $w^*$-$s$-open $= \{ \emptyset, X, \{ a \}, \{ b \}, \{ a, c \}, \{ b, c \} \}$.

12. $w$-open $= \{ \emptyset, X, \{ a \}, \{ b \}, \{ a, b \}, \{ a, c \} \}$.

13. $w$-$s$-open $= \{ \emptyset, X, \{ a \}, \{ b \}, \{ a, b \} \}$.

In a weak topological space $(X, w_X, \tau)$, always $wSC(A) \subseteq wC(A)$ and $wI(A) \subseteq wsI(A)$ for all $A \subseteq X$. Therefore, we obtain the relationship between the sets given in Definitions 3.5 and 3.7.

**Theorem 3.14.** Let $(X, w_X, \tau)$ be a weak topological space and $A \subset X$. Then the following holds:

1. If $A$ is a $w$-t-set, then $A$ is a $w$-st-set.
2. If $A$ is a $w$-$B$-set, then $A$ is a $w$-$sB$-set.

3. If $A$ is a $w^*$-open, then $A$ is a $w^*$-$s$-open.

4. If $A$ is a $w$-$s$-open, then $A$ is a $w$-open.

**Example 3.15.** Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and weak structure $w_X = \{\emptyset, X, \{b, c\}\}$ and on $(X, \tau)$. Observe that $\tau$ is not contained in $w$. If we take $A = \{a, b\}$, $A$ is locally $w$-semi closed, because $A = \{a, b\} \cap X$, $wsC(A) = X$, $wsC(A) - A = \{c\}$ is not $w$-semi closed. $A \cup (X - wsC(A)) = A = \{a, b\}$ is not a $w$-semi open set, also $A$ is not contained in $wsI(A \cup (X - wsC(A)))$, because $A \cup (X - wsC(A)) = \{a, b\}$ and $wsI(\{a, b\}) = \emptyset$.

In the case that $\tau \subset w_X$, we have the following theorem.

**Theorem 3.16.** Let $(X, w_X, \tau)$ be a weak topological space and $\tau \subset w_X$. If $A$ is locally $w$-semi closed, then:

1. $wsC(A) - A$ is $w$-semi closed.

2. $A \cup (X - wsC(A))$ is $w$-semi open set.

3. $A$ is contained in $wsI(A \cup (X - wsC(A)))$.

**Proof.** 1.- Suppose that $A$ is a locally $w$-semi closed subset of $X$, then there exists an open set $U$ such that $A = U \cap wsC(A)$. It follows that: $wsC(A) - A = wsC(A) - (U \cap wsC(A)) = wsC(A) \cap (X - (U \cap wsC(A))) = wsC(A) \cap ((X - U) \cup (X - wsC(A))) = wsC(A) \cap (X - U) \cup wsC(A) \cap (X - wsC(A)) = wsC(A) \cap (X - U)$. Now, as $wsC(A)$ is $w$-semi closed, $X - U$ is closed and $\tau \subset \mu$, we obtain that $X - U$ is $w$-semi closed and then $wsC(A) \cap (X - U)$ is $w$-semi closed.

2.- Using (1), $wsC(A) - A$ is $w$-semi closed, then its complement $X - (wsC(A) - A)$ is $w$-semi open, but $X - (wsC(A) - A) = X - (wsC(A) \cap (X - A)) = A \cup (X - wsC(A))$.

3.- Using (2), $A \subset (A \cup (X - wsC(A))) = wsI(A \cup (X - wsC(A)))$. $\square$

**Definition 3.17.** Let $(X, w_X, \tau)$ be a weak topological space. A subset $A$ of $X$ is called generalized $w$-closed or simple a $gw$-closed if $wC(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$.

**Definition 3.18.** Let $(X, w_X, \tau)$ be a weak topological space. A subset $A$ of $X$ is called generalized $w$-semi closed or simple a $gw$-s-closed if $wsC(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$. 
Remark 3.19. In a weak topological space \((X, w_X, \tau)\), every generalized \(w\) closed set is a generalized \(w\)-semi closed set, but the converse is not necessarily true as we can see in the following example.

Example 3.20. In Example 3.12,

1. locally \(w\)-closed = \(\emptyset, X, \{a, b, c\}, \{a, c, d\}, \{c, \}, \{a, d\},\{a,\}, \{a, c\}, \{d\}\)

2. \(gw\)-closed = \(\emptyset, X, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, c, d\}, \{a, b, d\},\{b, c, d\}\).

3. \(w\)-closed sets = \(\emptyset, X, \{a, d\}, \{a, c, d\}\).

4. locally \(w\)-s-closed = \(\emptyset, X, \{a, b, c\}, \{c, d\}, \{a, c, d\}, \{a, d\},\{a, c\}, \{d\}, \{b\}, \{a\}\).

5. \(gw\)-s-closed = \(\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, d\}, \{a, c\}, \{c, d\},\{a, b, d\}, \{a, c, d\}, \{b, c, d\}\).

6. \(wSC(X, w_X) = \emptyset, X, \{a, c, d\}, \{c, d\}, \{a, d\}, \{a, c\}, \{d\}, \{c\}, \{b\}\).

The following theorems characterize: the \(w\)-closed sets in terms of \(gw\)-closed sets and locally \(w\)-closed sets and the \(w\)-semi closed sets in terms of \(gw\)-semi closed sets and locally \(w\)-semi closed sets.

Theorem 3.21. Let \((X, w_X, \tau)\) be a weak topological space. \(A \subset X\) is \(w\)-closed if and only if \(A\) is \(gw\)-closed and locally \(w\)-closed.

Proof. Suppose that \(A\) is \(w\)-closed in \(X\) and \(A \subset U\), with \(U \in \tau\). Since \(A = wC(A)\), we obtain that \(A\) is \(gw\)-closed and locally \(w\)-closed. Conversely, suppose that \(A\) is \(gw\)-closed and locally \(w\)-closed, then \(A = U \cap F\), where \(U \in \tau\) and \(F\) is \(w\)-closed, therefore, \(A \subset U\) and \(A \subset F\), in consequence, \(wC(A) \subset U\) and \(wC(A) \subset F\) and hence \(wC(A) \subset U \cap F = A\). So \(A\) is \(w\)-closed.

Theorem 3.22. Let \((X, w_X, \tau)\) be a weak topological space. \(A \subset X\) is \(w\)-semi closed if and only if \(A\) is \(gw\)-semi closed and locally \(w\)-semi closed.

Proof. Suppose that \(A\) is \(w\)-semi closed in \(X\) and \(A \subset U\), with \(U \in \tau\). Since \(A = wsC(A)\), we obtain that \(A\) is \(gw\)-semi closed and locally \(w\)-semi closed. Conversely, suppose that \(A\) is \(gw\)-semi closed and locally \(w\)-semi closed, then \(A = U \cap F\), where \(U \in \tau\) and \(F\) is \(w\)-semi closed, therefore, \(A \subset U\) and \(A \subset F\), in
consequence, \( wsC(A) \subset U \) and \( wsC(A) \subset F \) and hence \( wsC(A) \subset U \cap F = A \). So \( A \) is \( w \)-semi closed.

**Theorem 3.23.** Let \( (X, w_X, \tau) \) be a weak topological space and \( A, B \) subsets of \( X \).

1. \( A \) is a \( w \)-st-set if and only if \( A \) is a \( w^* \)-s-closed.
2. If \( A \) is \( w \)-semi closed, then \( A \) is \( w \)-st-set.
3. If \( A \) and \( B \) are \( w \)-st-sets, then \( A \cap B \) is \( w \)-st-set.
4. If \( A \) is \( w \)-st-set, then \( A \) is \( w \)-sB-set.
5. Every locally \( w \)-semi closed set is \( w \)-sB-set.

**Proof.** 1-. Suppose that \( A \) is a \( w \)-st-set, then \( int(A) = int(wsC(A)) \) and hence \( int(wsC(A)) \subset A \), in consequence, \( A \) is a \( w^* \)-s-closed. Conversely, if \( A \) is a \( w^* \)-s-closed, then \( int(wsC(A)) \subset A \) and hence \( int(wsC(A)) \subset int(A) \subset A \subset int(wsC(A)) \). Therefore, \( int(wsC(A)) \subset int(A) \subset int(A) \subset int(wsC(A)) \). In consequence, \( int(wsC(A)) = A \), so \( A \) is a \( w \)-st-set.

2-. If \( A \) is \( w \)-semi closed, then \( A = wsC(A) \), and hence \( int(A) = int(wsC(A)) \).

3-. Suppose that \( A \) and \( B \) are \( w \)-st-sets. Since \( A \cap B \subset wsC(A \cap B) \), we obtain that \( int(A \cap B) \subset int(wsC(A \cap B)) \subset int(wsC(A) \cap wsC(B)) = int(wsC(A)) \cap int(wsC(B)) = int(A) \cap int(B) = int(A \cap B) \). In consequence, \( int(A \cap B) = int(wsC(A \cap B)) \).

4-. Since \( X \in \tau \) and \( A = A \cap X \), then \( A \) is a \( \mu \)-sB-set.

5-. Suppose that \( A \) is locally \( w \)-semi closed set of \( X \), then \( A = U \cap F \), where \( U \in \tau \) and \( F \) is a \( w \)-semi closed. Using (2), \( F \) is \( w \)-st-set, then by (4), follows that \( A = U \cap F \), where \( U \in \tau \) and \( F \) is a \( w \)-st-set and therefore, \( A \) is \( w \)-sB-set.

In the following examples show that the converse of the above theorem is not necessarily true.

**Example 3.24.** In Example 3.12, \( \{c\} \) is \( w \)-sB-set, but is not \( w \)-st-set. Also \( \{a\} \) is \( w \)-st-set, but is not \( w \)-semi closed.

**Example 3.25.** Let \( X = \{a, b, c\} \), \( \tau = \emptyset, X, \{c\} \) and \( w_X = \emptyset, X, \{a\}, \{a, b\} \). Then \( \{a\} \) is a \( w \)-sB-set but is not locally \( w \)-semi closed.
Theorem 3.26. Let \((X, w_X, \tau)\) be a weak topological space. \(A \subset X\) is open if and only if \(A\) is \(w\)-s-open and \(w\)-sB-set.

Proof. Let \(A\) be an open set, then \(A = \text{int}(A) \subseteq \text{int}(wsC(A))\) and hence, \(A\) is \(w\)-s-open set. Since \(A = A \cap X\), where \(X\) is \(w\)-st-set, then \(A\) is \(w\)-sB-set. Conversely, since \(A\) is a \(w\)-sB-set, \(A = U \cap V\), where \(U \in \tau\) and \(V\) is a \(w\)-st-set. By hypothesis, \(A \subseteq \text{int}(wsC(A)) = \text{int}(wsC(U \cap V)) \subseteq \text{int}(wsC(U) \cap wsC(V)) = \text{int}(wsC(U)) \cap \text{int}(wsC(V)) = \text{int}(wsC(U)) \cap \text{int}(V)\). But \(A = U \cap V = (U \cap V) \cap U \subseteq (\text{int}(wsC(U)) \cap \text{int}(V)) \cap U = (\text{int}(wsC(U)) \cap U \cap \text{int}(V)) = U \cap \text{int}(V) \subseteq U \cap V = A\). Therefore, \(A\) is an open set.

Example 3.27. In Example 3.12, \(\{b, c, d\}\) is \(w\)-s-open but not \(w\)-sB-set. In the same form, \(\{a\}\) is \(w\)-sB-set but is not \(w\)-s-open, in consequence, is not open.

4. \((w, \sigma)\)-s-Continuous Functions

Definition 4.1. Let \((X, w_X, \tau)\) be a weak topological space and \((Y, \sigma)\) be a topological space. A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be \((w, \sigma)\)-continuous if \(f^{-1}(V)\) is \(w\)-open in \(X\) for each open set \(V\) of \(Y\).

Definition 4.2. Let \((X, w_X, \tau)\) be a weak topological space and \((Y, \sigma)\) be a topological space. A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be \((w, \sigma)\)-s-continuous if \(f^{-1}(V)\) is \(w\)-semi open in \(X\) for each open set \(V\) of \(Y\).

Theorem 4.3. Every \((w, \sigma)\)-continuous function is \((\mu, \sigma)\)-s-continuous but not conversely.

Example 4.4. Let \(X = \mathbb{R}\) be the set of real numbers, \(w_X = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}\) and \(\tau = \sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}\}\) where \(\mathbb{Q}\) denotes the set of all rational numbers and \(\mathbb{R} \setminus \mathbb{Q}\) denotes the set of all irrational numbers. Define \(f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \sigma)\) as the identity function. Then \(f\) is \((w, \sigma)\)-s-continuous but not \((w, \sigma)\)-continuous.

Theorem 4.5. Let \((X, w_X, \tau)\) be a weak topological space, \((Y, \sigma)\) be a topological space and \(f : (X, \tau) \rightarrow (Y, \sigma)\) a function. Then the following are equivalent:

1. \(f\) is \((w, \sigma)\)-s-continuous.

2. For each \(x \in X\) and each open set \(V\) of \(Y\) with \(f(x) \in V\), there exists a \(w\)-semi open set \(U\) containing \(x\) such that \(f(U) \subseteq V\).
3. For each \( x \in X \) and each open set \( V \) of \( Y \) with \( f(x) \in V \), \( f^{-1}(V) \) is a \( w \)-semi open neighborhood of \( x \).

4. The inverse image of each closed set in \( Y \) is \( w \)-semi closed.

5. \( wsC(f^{-1}(B)) \subseteq f^{-1}(cl(B)) \) for every \( B \subseteq Y \).

6. \( f(wsC(A)) \subseteq cl(f(A)) \) for every \( A \subseteq X \).

7. \( f^{-1}(int(B)) \subseteq wsI(f^{-1}(B)) \) for every \( B \subseteq Y \).

Proof. 1. \( \Rightarrow \) 2. Let \( x \in X \) and \( V \) any open set in \( Y \) such that \( f(x) \in V \). Since \( f \) is \((w, \sigma)\)-continuous, \( f^{-1}(V) \) is \( w \)-semi open. By putting \( U = f^{-1}(V) \), \( x \in U \) and \( f(U) \subseteq V \).

2. \( \Rightarrow \) 3. Let \( x \in X \) and \( V \) an open in \( Y \) such that \( f(x) \in V \). By 2, there exists a \( w \)-semi open set \( U \) containing \( x \) such that \( f(U) \subseteq V \). So each \( x \in U \subseteq f^{-1}(V) \) and hence \( f^{-1}(V) \) is a \( w \)-semi open neighborhood of \( x \).

3. \( \Rightarrow \) 1. Let \( x \in X \) and \( V \) an open in \( Y \) such that \( f(x) \in V \). By 3, \( f^{-1}(V) \) is a \( w \)-semi open neighborhood of \( x \). Thus for each \( x \in f^{-1}(V) \), there exists a \( w \)-semi open set \( U_x \) containing \( x \) such that \( x \in U_x \subseteq f^{-1}(V) \). Hence \( f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x \) and so \( f^{-1}(V) \in wSO(X) \).

1. \( \Leftrightarrow \) 4. It is obvious.

1. \( \Rightarrow \) 5. Let \( B \) be a subset of \( Y \). Since \( cl(B) \) is closed and \( f \) is \((w, \sigma)\)-continuous, \( f^{-1}(cl(B)) \) is \( w \)-semi closed. Therefore,
\[
wsC(f^{-1}(B)) \subseteq wsC(f^{-1}(cl(B))) = f^{-1}(cl(B)).
\]

5. \( \Rightarrow \) 6. Let \( A \) be a subset of \( X \). By 5, we have
\[
wsC(f^{-1}(f(A))) \subseteq f^{-1}(cl(f(A))).
\]

But \( wsC(A) \subseteq wsC(f^{-1}(f(A))) \). Therefore \( f(wsC(A)) \subseteq cl(f(A)) \).

6. \( \Rightarrow \) 7. Let \( B \) be a subset of \( Y \). By 6,
\[
f(wsC(X \setminus f^{-1}(B))) \subseteq cl(f(X \setminus f^{-1}(B)))
\]

and
\[
f(X \setminus wsI(f^{-1}(B))) \subseteq cl(Y \setminus B) = Y \setminus int(B).
\]

Therefore \( X \setminus wsI(f^{-1}(B)) \subseteq f^{-1}(Y \setminus int(B)) \) and \( f^{-1}(int(B)) \subseteq wsI(f^{-1}(B)) \).

7. \( \Rightarrow \) 1. Let \( B \) be an open in \( Y \) and \( f^{-1}(int(B)) \subseteq wsI(f^{-1}(B)) \). Then
\[
f^{-1}(B) \subseteq wsI(f^{-1}(B)).
\]

But \( wsI(f^{-1}(B)) \subseteq f^{-1}(B) \). Hence \( f^{-1}(B) = wsI(f^{-1}(B)) \). Therefore \( f^{-1}(B) \) is \( w \)-semi open.
\[\square\]
As immediate consequence of Theorem 4.5, we have the following result.

**Corollary 4.6.** Let \((X, w_X, \tau)\) be a weak topological space and \((Y, \sigma)\) be a topological space and \(f : (X, \tau) \to (Y, \sigma)\) a \((w, \sigma)\)-s-continuous function, then the following are equivalent:

1. \(wsI(wsC(f^{-1}(B))) \cap wsC(wsI(f^{-1}(B))) \subseteq f^{-1}(cl(B))\) for each \(B\) in \(Y\).
2. \(f[wsI(wsC(A)) \cap wsC(wsI(A))] \subseteq cl(f(A))\) for each \(A\) in \(X\).

**Definition 4.7.** Let \((X, w_X, \tau)\) be a weak topological space and \(A \subseteq X\). Then the \(w\)-s-kernel of \(A\) denoted by \(w-s-Ker(A)\) is defined to be the set, \(w-s-Ker(A) = \cap \{U : U \in wSO(X), A \subseteq U\}\).

**Theorem 4.8.** Let \((X, w_X, \tau)\) be a weak topological space and \(x \in X\). Then \(y \in w-s-Ker(\{x\})\) if and only if \(x \in wsC(\{y\})\). The converse is similarly shown.

**Proof.** Assume that \(y \notin w-s-Ker(\{x\})\). Then there exists a \(w\)-semi open set \(U\) containing \(x\) such that \(y \notin U\). Therefore, we have \(x \notin wsC(\{y\})\). The converse is similarly shown.

**Theorem 4.9.** Let \((X, w_X, \tau)\) be a weak topological space and \(A\) a subset of \(X\). Then \(w-s-Ker(A) = \{x : wsC(\{x\}) \cap A \neq \emptyset\}\).

**Proof.** Let \(x \in w-s-Ker(A)\) and \(wsC(\{x\}) \cap A = \emptyset\). Then, \(x \notin X \setminus wsC(\{x\})\) which is a \(w\)-semi open set containing \(A\). But this is impossible, since \(x \in w-s-Ker(A)\). Consequently, \(wsC(\{x\}) \cap A \neq \emptyset\).

Conversely, let \(x \in X\) such that \(wsC(\{x\}) \cap A \neq \emptyset\). Suppose that \(x \notin w-s-Ker(A)\). Then there exists a \(w\)-semi open set \(U\) containing \(A\) and \(x \notin U\). Let \(y \in wsC(\{x\}) \cap A\). Then \(y \in wsC(\{x\})\) and \(y \in A\). Thus \(x \in w-s-Ker(\{y\})\) and \(y \in U \in w\) implies \(x \in U \in SO(X)\). By this contradiction, \(x \in w-s-Ker(A)\).

**Theorem 4.10.** The following are equivalent for any points \(x\) and \(y\) in a weak space \((X, w_X)\):

1. \(w-s-Ker(\{x\}) \neq w-s-Ker(\{y\})\).
2. \(wsC(\{x\}) \neq wsC(\{y\})\).

**Theorem 4.11.** Let \((X, w_X, \tau)\) be a weak topological space and \(A \subseteq X\). Then

1. \(x \in w-s-Ker(A)\) if and only if \(A \cap F = \emptyset\) for any \(w\)-semi closed subset \(F\) of \(X\) with \(x \in F\).
2. $A \subseteq w-s-Ker(A)$ and $A = w-s-Ker(A)$ if $A$ is $w$-semi open in $X$.

3. If $A \subseteq B$ then $w-s-Ker(A) \subseteq w-s-Ker(B)$.

Proof. 1. Let $x \in w-s-Ker(A)$. Then by Theorem 4.9, $A \cap wsC(\{x\}) \neq \emptyset$. Conversely, assume that $A \cap F \neq \emptyset$. By taking $F = wsC(\{x\})$, we have $A \cap wsC(\{x\}) \neq \emptyset$ which implies $x \in w-s-Ker(A)$.

2. Let $A$ be $w$-semi open in $X$. Then always $A \subseteq w-s-Ker(A)$. On the other hand, assume that $x \in w-s-Ker(A)$. Then $x \in \bigcap \{U : U \in wSO(X), A \subseteq U\}$. Since $A$ is $w$-semi open implies that $x \in A$. Thus $w-s-Ker(A) \subseteq A$. Hence $A = w-s-Ker(A)$.

3. It is obvious.

As immediate consequence of Theorems 4.5 and 4.11, we have the following result.

Corollary 4.12. Let $(X, w_X, \tau)$ be a weak topological space, $(Y, \sigma)$ be a topological space and $f : (X, \tau) \to (Y, \sigma)$ a function $(w, \sigma)$-s-continuous. Then the following are equivalent:

1. For every subset $A$ of $X$, $f(wsI(A)) \subset Ker(f(A))$.

2. For every subset $B$ of $Y$, $wsI(f^{-1}(B)) \subseteq f^{-1}(Ker(B))$.

Definition 4.13. Let $(X, w_X, \tau)$ be a weak topological space and $(Y, \sigma)$ be a topological space. Then $f : (X, \tau) \to (Y, \sigma)$ is said to be $gw$-s-continuous (respectively contra locally $w$-s-continuous) if $f^{-1}(F)$ is a $gw$-s-closed (respectively locally $w$-s-closed) for each closed set $F$ of $(Y, \sigma)$.

Example 4.14. In Example 3.13, take $f : (X, \tau) \to (X, \tau)$, defined as: $f(a) = b$, $f(b) = c$ and $f(c) = a$, then $f$ is contra locally $w$-s-continuous but is not $gw$-s-continuous. In the same form if in Example 3.25, we define $f : (X, \tau) \to (X, \tau)$, as: $f(a) = b$, $f(b) = a$ and $f(c) = c$, then $f$ is $w$-s-continuous but is not contra locally $w$-s-continuous. Observe that in each case $f$ is not $(w, \tau)$-s-continuous.

The following theorem is a direct consequence of Theorem 4.5 and Theorem 3.22

Theorem 4.15. Let $(X, w_X, \tau)$ be a weak topological space and $(Y, \sigma)$ be a topological space. Then $f : (X, \tau) \to (Y, \sigma)$ is $(w, \sigma)$-s-continuous if and only if it is $gw$-s-continuous and contra locally $w$-s-continuous.
The following example shows the existence of a function that is contra locally $w$-s-continuous but not is $g_w$-s-continuous, in consequence is not $(w, \tau)$-s-continuous.

**Example 4.16.** In Example 3.12, define $f : (X, \tau) \to (X, \tau)$ as follows: $f(a) = a, f(b) = d, f(c) = b$ and $f(d) = c$. According with 3.20, $f$ is contra locally $w$-s-continuous but not is $g_w$-s-continuous, in consequence is not $(\mu, \tau)$-s-continuous.

**Theorem 4.17.** Let $(X, w_X, \tau)$ be a weak topological space and $(Y, \sigma)$ be a topological space. Then a contra continuous function $f : (X, \tau) \to (Y, \sigma)$ is $(w, \sigma)$-s-continuous if and only if it is $g_w$-s-continuous.

**Proof.** Suppose that $f$ is contra continuous and $(w, \sigma)$-s-continuous. Let $F$ be a closed set in $Y$, then $f^{-1}(F)$ is open and $w$-semi closed in $X$. Since each $w$-semi closed is $g_w$-closed, then $f$ is $g_w$-s-continuous.

Conversely, let $F$ be a closed set in $Y$, then $f^{-1}(F)$ is open and $g_w$-s-closed in $X$. Since each open set is locally $w$-s-closed, then $f^{-1}(F)$ is locally $w$-s-closed and $g_w$-s-closed, by Theorem 3.22, $f$ is $(w, \sigma)$-s-continuous. \hfill \Box

**Example 4.18.** In Example 4.4, $f$ is $(\mu, \sigma)$-s-continuous, $f(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R} \setminus \mathbb{Q}$ is $w$-open but is not $w$-s-closed in consequence, $f$ is not contra continuous.

References


