MORE ON WEAK DECOMPOSITION OF CONTINUITY

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Abstract: Using the notion of w -space on a set X and the concept of locally w -semi open set, we introduce, study and characterize the notions of w -s-Kernel of a subset A of X . Also we introduce and study a new forms of weak decomposition of continuity.

AMS Subject Classification: 54C05, 54C08, 54C10

Key Words: μ -st-set, μ -sB-set, (μ, σ) -s-continuous functions, w -s- $Ker(A)$

1. Introduction and Preliminaries

In the last years, different variants of open sets are being studied. Recently, a

Received: June 13, 2017

Revised: July 23, 2017

Published: August 30, 2017

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url: www.acadpubl.eu

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significant contribution to the theory of generalized open sets have been presented by A. Császár [1], [2], [3]. Specifically, in 2002, A. Császár [1], introduced the notions of generalized topology and generalized continuity. It is observed that a large numbers of articles are devoted to the study of generalized open sets and certain type of sets associated to a topological spaces, containing the class of open sets and possessing properties more or less to those open sets. Bishwambhar. et al. [4] studied some type of decomposition of continuity using generalized topologies and in [5], studied some weak forms of continuity. Rosas E. et al. in [10], give a new theory of decomposition of continuous functions using generalized topologies. In 2015, W. K. Min et al. [7], introduced and studied the notions of weak structures on a nonempty set X . In 2016, W. K. Min et al. introduced the notions of w -semiopen sets and w -semi continuity in w -spaces. Later in 2017, W. K. Min in [6], introduced and studied the notions of weakly $w_\tau g$ -closed set and weakly $w_\tau g$ -open set as a generalization of the $w_\tau g$ -closed set and $w_\tau g$ -open set in associated w -spaces. E. Rosas et al in [9], introduce the concepts of locally w -regular closed sets and locally w -semi regular semi closed and a new weak decomposition of some type of weak continuity functions are studied and characterized. In this article, using the notion of w -semi open set, we introduce the concept of locally w -semi open set as a generalization of locally w -closed and give a new theory of weak decomposition of continuity and some weak form of continuity are studied. Throughout this paper $cl(A)$ (respectively $int(A)$) denotes the closure (respectively interior) of A in a topological space X .

2. Preliminaries

Definition 2.1. [7] Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a weak structure on X if it satisfies the following :

1. $\emptyset \in w_X$ and $X \in w_X$.
2. For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$

Then the pair (X, w_X) is called a w -space on X . An element $U \in w_X$ is called w -open set and the complement of a w -open set is a w -closed set

Definition 2.2. [7] Let (X, w_X) be a w -space. For a subset A of X ,

1. The w -closure of A is defined as $wC(A) = \bigcap \{F : A \subseteq F, X \setminus F \in w_X\}$.
2. The w -interior of A is defined as $wI(A) = \bigcup \{U : U \subseteq A, U \in w_X\}$.

Theorem 2.3. [7] Let (X, w_X) be a w -space on X . A, B subsets of X . Then the following hold:

1. If $A \subseteq B$, then $wI(A) \subseteq wI(B)$ and $wC(A) \subseteq wC(B)$.
2. $wI(wI(A)) = wI(A)$ and $wC(wC(A)) = wC(A)$
3. $wC(X \setminus A) = X \setminus wI(A)$ and $wI(X \setminus A) = X \setminus wC(A)$
4. If A is w -closed (resp. w -open), then $wC(A) = A$ (resp. $wI(A) = A$)

Definition 2.4. [8] Let (X, w_X) be a w -space on X . A subset A of X is called w -semi open if $A \subseteq wC(wI(A))$. The complement of a w -semi open set is called w -semi closed.

The collection of all w -semi open sets is denoted by $wSO(X, w_X)$ and the collection of all w -semi closed sets is denoted by $wSC(X, w_X)$

Definition 2.5. [8] Let (X, w_X) be a w -space on X . For a subset A of X

1. The w -semi closure of A is defined as $wsC(A) = \bigcap \{F : A \subseteq F, X \setminus F \in wSO(X, w_X)\}$.
2. The w -semi interior of A is defined as $wsI(A) = \bigcup \{U : U \subseteq A, U \in wSO(X, w_X)\}$.

Theorem 2.6. [8] Let (X, w_X) be a w -space on X . A, B subsets of X . Then the following hold:

1. $wsI(A) \subseteq A$ and $A \subseteq wsC(A)$.
2. If $A \subseteq B$, then $wsI(A) \subseteq wsI(B)$ and $wsC(A) \subseteq wsC(B)$.
2. $wsI(wsI(A)) = wsI(A)$ and $wsC(wsC(A)) = wsC(A)$.
3. $wsC(X \setminus A) = X \setminus wsI(A)$ and $wsI(X \setminus A) = X \setminus wsC(A)$.
4. A is w -semi closed (resp. w -semi open), if and only if $wsC(A) = A$ (resp. $wsI(A) = A$).

Theorem 2.7. [8] Let (X, w_X) be a w -space on X and A a subset of X . Then $wsC(A)$ is an w -semi closed.

Theorem 2.8. Let (X, w_X) be a w -space on X and A a subset of X . Then $A \cup wI(wC(A)) \subseteq wsC(A)$.

The following example shows that the reverse contention in the above theorem is not necessarily true.

Example 2.9. Let $X = N$ be the set of natural numbers. Define $w_X = \{\emptyset, \{1\}, N\} \cup P(\{2n : n \in N\})$. The set of w -closed sets $= \{\emptyset, N, N \setminus \{1\}\} \cup \{A^c : A \in P(\{2n : n \in N\})\}$. The set of w -semiopen sets $= \{\emptyset, \{1\}, N, F_1, F_2\} \cup P(\{2n : n \in N\})$ where $F_1 \cap \{2n : n \in N\} \neq \emptyset$ and $1 \in F_2$. If we take $A = \{3\}$, $wsC(A) = \{3\}$, $wC(A) = \{2n + 1 : n \in N\}$ and $wI(wC(A)) = \{1\}$. Observe that $A \cup wI(wC(A)) = \{1, 3\} \supset \{3\} = wsC(A)$.

Theorem 2.10. Let (X, w_X) be a w -space on X and A, B subsets of X . Then

1. $x \in wsC(A)$ if and only if $A \cap V \neq \emptyset$, for every w -semiopen set V containing x .
2. $wsC(A \cap B) = wsC(A) \cap wsC(B)$.

3. New Types of w -Closed Sets

Throughout this paper (X, w_X, τ) a weak topological space denotes (X, w_X) is a w -space and (X, τ) is a topological space.

Definition 3.1. [4] Let (X, w_X, τ) be a weak topological space. A subset A of X is called locally w -closed if $A = U \cap F$ where $U \in \tau$ and F is w -closed.

Remark 3.2. If (X, w_X, τ) is a weak topological space, then every open set as well as a w -closed set is locally w -closed.

Theorem 3.3. Let (X, w_X, τ) be a weak topological space. If $A \subseteq X$ is locally w -closed then there exists an open set U such that $A = U \cap wC(A)$.

Proof. Let A be a locally w -closed subset of X , then $A = U \cap F$, where $U \in \tau$ and F is w -closed. It follows that $A = A \cap U \subseteq U \cap wC(A) \subseteq U \cap wC(F) = U \cap F = A$. In consequence, $A = U \cap wC(A)$.

Example 3.4. In Example 3.12, $\{a, b\} = \{a, b, c\} \cap wsC(\{a, b\})$, but $\{a, b\}$ is not a locally w -closed set.

□

Definition 3.5. Let (X, w_X, τ) be a weak topological space. A subset A of X is called:

1. w -t-set if $\text{int}(A) = \text{int}(wC(A))$.
2. w -B-set if $A = U \cap V$, $U \in \tau$, V is a w -t-set.
3. w^* -open set if $A \subseteq \text{cl}(wI(A))$.
4. w -open set if $A \subseteq \text{int}(wC(A))$.

Definition 3.6. Let (X, w_X, τ) be a weak topological space. A subset A of X is called generalized w -closed or simple a gw -closed if $wC(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$.

Definition 3.7. [9] Let (X, w_X, τ) be a weak topological space. A subset A of X is called locally w -semi closed or simple locally w -s-closed if $A = U \cap F$ where $U \in \tau$ and F is w -semi closed.

Remark 3.8. If (X, w_X, τ) is a weak topological space, then every open set as well as a w -semi closed set is locally w -semi closed, also every locally w -closed set is locally w -semi closed.

Definition 3.9. Let (X, w_X, τ) be a weak topological space. A subset A of X is called:

1. w -st-set if $\text{int}(A) = \text{int}(wsC(A))$.
2. w -sB-set if $A = U \cap V$, $U \in \tau$, V is a w -st-set.
3. w^* -semi open set (briefly w^* -s-open set) if $A \subseteq \text{cl}(wsI(A))$.
4. w -semi open set (briefly w -s-open set) if $A \subseteq \text{int}(wsC(A))$.

The complement of a w^* -semi open set is called w^* -semi closed set (briefly w^* -s-closed set).

The following theorems characterizes the locally w -semi closed sets.

Theorem 3.10. Let (X, w_X, τ) be a weak topological space. $A \subseteq X$ is locally w -semi closed if and only if $X - A$ is the union of a closed set and a w -semi open set.

Proof. Suppose that A is locally w -semi closed, then $A = U \cap F$ where $U \in \tau$ and F is w -semi closed. It follows that $X - A = X \cap (U \cap F)^c = (U^c \cup F^c) = (X - U) \cup (X - F)$. Conversely, suppose that $X - A = W \cup G$, W closed set and G w -semi open. Then $A = X - (X - A) = X - (W \cup G) = (X - W) \cap (X - G)$. $\square$

Theorem 3.11. *Let (X, w_X, τ) be a weak topological space. $A \subseteq X$ is locally w -semi closed if and only if there exists an open set U such that $A = U \cap wsC(A)$.*

Proof. Let A be a locally w -semi closed subset of X , then $A = U \cap F$, where $U \in \tau$ and F is w -semi closed. It follows that $A = A \cap U \subseteq U \cap wsC(A) \subseteq U \cap wsC(F) = U \cap F = A$. In consequence, $A = U \cap wsC(A)$. Conversely, since $wsC(A)$ is a w -semi closed, it follows that A is locally w -semi closed. $\square$

In the following example, we can see that there exists a locally w -semi closed set that is not open as well as w -semi closed, w -st-set that is not w -s-set, w -sB-set that is not w -B-set.

Example 3.12. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b, c\}, \{c, d\}, \{c\}\}$ and $w_X = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$ be a weak structure on (X, τ) . Then:

1. w -closed sets $= \{\emptyset, X, \{c, d\}, \{a, d\}, \{a, c, d\}\}$.
2. locally w -closed $= \{\emptyset, X, \{a, b, c\}, \{a, c, d\}, \{c, d\}, \{c\}, \{a, d\}, \{a\}, \{a, c\}, \{d\}\}$.
3. $wSO(X, w_X) = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$.
4. $wSC(X, w_X) = \{\emptyset, X, \{a, c, d\}, \{c, d\}, \{a, d\}, \{a, c\}, \{d\}, \{c\}, \{b\}\}$.
5. locally w -s-closed $= \{\emptyset, X, \{a, b, c\}, \{c, d\}, \{c\}, \{a, c, d\}, \{c, d\}, \{a, d\}, \{a, c\}, \{d\}, \{b\}, \{a\}\}$.
6. w -t-set $= \{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$.
7. w -st-set $= \{\emptyset, X, \{a\}, \{b\}, \{d\}, \{a, d\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$.
8. w -B-set $= \{\emptyset, X, \{a, b, c\}, \{c, d\}, \{c\}, \{a\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}, \{a, c\}\}$.
9. w -sB-set $= \{\emptyset, X, \{a, b, c\}, \{c, d\}, \{c\}, \{a\}, \{b\}, \{d\}, \{a, d\}, \{a, c\}, \{a, c, d\}\}$.
10. w^* -open $= \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$.
11. w^* -s-open $= \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.

12. w -open $= \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}\}$.
13. w -s-open $= \{\emptyset, X, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.

The following examples will be useful to better understand the development of this article.

Example 3.13. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $w_X = \{\emptyset, X, \{a\}, \{a, c\}\}$ be a weak structure on (X, τ) . Then:

1. w -closed sets $= \{X, \{b, c\}, \{b\}\}$.
2. locally w -closed $= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$.
3. $wSO(X, w_X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$.
4. $wSC(X, w_X) = \{\emptyset, X, \{b, c\}, \{c\}, \{b\}, \{a, c\}\}$.
5. locally w -s-closed $= \{\emptyset, X, \{b, c\}, \{c\}, \{b\}, \{a\}, \{a, b\}, \{a, c\}\}$.
6. w -t-set $= \{X, \{b\}, \{b, c\}\}$.
7. w -st-set $= \{\emptyset, X, \{b\}, \{c\}, \{a\}, \{b, c\}, \{a, c\}\}$.
8. w -B-set $= \{\emptyset, X, \{b\}, \{a\}, \{b, c\}, \{a, b\}\}$.
9. w -sB-set $= \{\emptyset, X, \{b\}, \{c\}, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.
10. w^* -open $= \{\emptyset, \{a\}, \{b\}, \{a, c\}\}$.
11. w^* -s-open $= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.
12. w -open $= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$.
13. w -s-open $= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.

In a weak topological space (X, w_X, τ) , always $wsC(A) \subseteq wC(A)$ and $wI(A) \subseteq wsI(A)$ for all $A \subseteq X$. Therefore, we obtain the relationship between the sets given in Definitions 3.5 and 3.7.

Theorem 3.14. Let (X, w_X, τ) be a weak topological space and $A \subset X$. Then the following holds:

1. If A is a w -t-set, then A is a w -st-set.

2. If A is a w - B -set, then A is a w - sB -set.
3. If A is a w^* -open, then A is a w^* - s -open.
4. If A is a w - s -open, then A is a w -open.

Example 3.15. Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and weak structure $w_X = \{\emptyset, X, \{b, c\}\}$ and on (X, τ) . Observe that τ is not contained in w . If we take $A = \{a, b\}$, A is locally w -semi closed, because $A = \{a, b\} \cap X$, $wsC(A) = X$, $wsC(A) - A = \{c\}$ is not w -semi closed. $A \cup (X - wsC(A)) = A = \{a, b\}$ is not a w -semi open set, also A is not contained in $wsI(A \cup (X - wsC(A)))$, because $A \cup (X - wsC(A)) = \{a, b\}$ and $wsI(\{a, b\}) = \emptyset$.

In the case that $\tau \subset w_X$, we have the following theorem.

Theorem 3.16. Let (X, w_X, τ) be a weak topological space and $\tau \subset w_X$. If A is locally w -semi closed, then:

1. $wsC(A) - A$ is w -semi closed.
2. $A \cup (X - wsC(A))$ is w -semi open set.
3. A is contained in $wsI(A \cup (X - wsC(A)))$.

Proof. 1.- Suppose that A is a locally w -semi closed subset of X , then there exists an open set U such that $A = U \cap wsC(A)$. It follows that: $wsC(A) - A = wsC(A) - (U \cap wsC(A)) = wsC(A) \cap (X - (U \cap wsC(A))) = wsC(A) \cap ((X - U) \cup (X - wsC(A))) = wsC(A) \cap (X - U) \cup wsC(A) \cap (X - wsC(A)) = wsC(A) \cap (X - U)$. Now, as $wsC(A)$ is w -semi closed, $X - U$ is closed and $\tau \subset \mu$, we obtain that $X - U$ is w -semi closed and then $wsC(A) \cap (X - U)$ is w -semi closed.

2.- Using (1), $wsC(A) - A$ is w -semi closed, then its complement $X - (wsC(A) - A)$ is w -semi open, but $X - (wsC(A) - A) = X - (wsC(A) \cap (X - A)) = A \cup (X - wsC(A))$.

3.- Using (2), $A \subset (A \cup (X - wsC(A))) = wsI(A \cup (X - wsC(A)))$. $\square$

Definition 3.17. Let (X, w_X, τ) be a weak topological space. A subset A of X is called generalized w -closed or simple a gw -closed if $wC(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$.

Definition 3.18. Let (X, w_X, τ) be a weak topological space. A subset A of X is called generalized w -semi closed or simple a gw - s -closed if $wsC(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$.

Remark 3.19. In a weak topological space (X, w_X, τ) , every generalized w -closed set is a generalized w -semi closed set, but the converse is not necessarily true as we can see in the following example.

Example 3.20. In Example 3.12,

1. locally w -closed $= \{\emptyset, X, \{a, b, c\}, \{a, c, d\}, \{c, d\}, \{c\}, \{a, d\}, \{a\}, \{a, c\}, \{d\}\}$
2. gw -closed $= \{\emptyset, X, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}\}$.
3. w -closed sets $= \{\emptyset, X, \{c, d\}, \{a, d\}, \{a, c, d\}\}$.
4. locally w -s-closed $= \{\emptyset, X, \{a, b, c\}, \{c, d\}, \{c\}, \{a, c, d\}, \{a, d\}, \{a, c\}, \{d\}, \{b\}, \{a\}\}$.
5. gw -s-closed $= \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, d\}, \{a, c\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.
6. $wSC(X, w_X) = \{\emptyset, X, \{a, c, d\}, \{c, d\}, \{a, d\}, \{a, c\}, \{d\}, \{c\}, \{b\}\}$.

The following theorems characterize: the w -closed sets in terms of gw -closed sets and locally w -closed sets and the w -semi closed sets in terms of gw -semi closed sets and locally w -semi closed sets.

Theorem 3.21. *Let (X, w_X, τ) be a weak topological space. $A \subset X$ is w -closed if and only if A is gw -closed and locally w -closed.*

Proof. Suppose that A is w -closed in X and $A \subset U$, with $U \in \tau$. Since $A = wC(A)$, we obtain that A is gw -closed and locally w -closed. Conversely, suppose that A is gw -closed and locally w -closed, then $A = U \cap F$, where $U \in \tau$ and F is w -closed, therefore, $A \subset U$ and $A \subset F$, in consequence, $wC(A) \subset U$ and $wC(A) \subset F$ and hence $wC(A) \subset U \cap F = A$. So A is w -closed. $\square$

Theorem 3.22. *Let (X, w_X, τ) be a weak topological space. $A \subset X$ is w -semi closed if and only if A is gw -semi closed and locally w -semi closed.*

Proof. Suppose that A is w -semi closed in X and $A \subset U$, with $U \in \tau$. Since $A = wsC(A)$, we obtain that A is gw -semi closed and locally w -semi closed. Conversely, suppose that A is gw -semi closed and locally w -semi closed, then $A = U \cap F$, where $U \in \tau$ and F is w -semi closed, therefore, $A \subset U$ and $A \subset F$, in

consequence, $wsC(A) \subset U$ and $wsC(A) \subset F$ and hence $wsC(A) \subset U \cap F = A$. So A is w -semi closed. $\square$

Theorem 3.23. *Let (X, w_X, τ) be a weak topological space and A, B subsets of X .*

1. *A is a w -st-set if and only if A is a w^* -s-closed.*
2. *If A is w -semi closed, then A is w -st-set.*
3. *If A and B are w -st-sets, then $A \cap B$ is w -st-set.*
4. *If A is w -st-set, then A is w -sB-set.*
5. *Every locally w -semi closed set is w -sB-set.*

Proof. 1-. Suppose that A is a w -st-set, then $int(A) = int(wsC(A))$ and hence $int(wsC(A)) \subset A$, in consequence, A is a w^* -s-closed. Conversely, if A is a w^* -s-closed, then $int(wsC(A)) \subset A$ and hence $int(wsC(A)) \subset int(A) \subset A \subset int(wsC(A))$. Therefore, $int(wsC(A)) \subset int(A) \subset int(A) \subset int(wsC(A))$. In consequence, $int(wsC(A)) = A$, so A is a w -st-set.

2-. If A is w -semi closed, then $A = wsC(A)$, and hence $int(A) = int(wsC(A))$. Therefore, A is w -st-set.

3-. Suppose that A and B are w -st-sets. Since $A \cap B \subseteq wsC(A \cap B)$, we obtain that $int(A \cap B) \subseteq int(wsC(A \cap B)) \subseteq int(wsC(A) \cap wsC(B)) = int(wsC(A)) \cap int(wsC(B)) = int(A) \cap int(B) = int(A \cap B)$. In consequence, $int(A \cap B) = int(wsC(A \cap B))$.

4-. Since $X \in \tau$ and $A = A \cap X$, then A is a μ -sB-set.

5-. Suppose that A is locally w -semi closed set of X , then $A = U \cap F$, where $U \in \tau$ and F is a w -semi closed. Using (2), F is w -st-set, then by (4), follows that $A = U \cap F$, where $U \in \tau$ and F is a w -st-set and therefore, A is w -sB-set. $\square$

In the following examples show that the converse of the above theorem is not necessarily true.

Example 3.24. In Example 3.12, $\{c\}$ is w -sB-set, but is not w -st-set. Also $\{a\}$ is w -st-set, but is not w -semi closed.

Example 3.25. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$ and $w_X = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then $\{a\}$ is a w -sB-set but is not locally w -semi closed.

Theorem 3.26. *Let (X, w_X, τ) be a weak topological space. $A \subset X$ is open if and only if A is w -s-open and w -sB-set.*

Proof. Let A be an open set, then $A = \text{int}(A) \subseteq \text{int}(wsC(A))$ and hence, A is w -s-open set. Since $A = A \cap X$, where X is w -st-set, then A is w -sB-set. Conversely, since A is a w -sB-set, $A = U \cap V$, where $U \in \tau$ and V is a w -st-set. By hypothesis, $A \subseteq \text{int}(wsC(A)) = \text{int}(wsC(U \cap V)) \subseteq \text{int}(wsC(U) \cap wsC(V)) = \text{int}(wsC(U)) \cap \text{int}(wsC(V)) = \text{int}(wsC(U)) \cap \text{int}(V)$. But $A = U \cap V = (U \cap V) \cap U \subseteq (\text{int}(wsC(U)) \cap \text{int}(V)) \cap U = (\text{int}(wsC(U)) \cap U) \cap \text{int}(V) = U \cap \text{int}(V) \subseteq U \cap V = A$. Therefore, A is an open set. $\square$

Example 3.27. In Example 3.12, $\{b, c, d\}$ is w -s-open but not w -sB-set. In the same form, $\{a\}$ is w -sB-set but is not w -s-open, in consequence, is not open.

4. (w, σ) -s-Continuous Functions

Definition 4.1. Let (X, w_X, τ) be a weak topological space and (Y, σ) be a topological space. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be (w, σ) -continuous if $f^{-1}(V)$ is w -open in X for each open set V of Y .

Definition 4.2. Let (X, w_X, τ) be a weak topological space and (Y, σ) be a topological space. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be (w, σ) -s-continuous if $f^{-1}(V)$ is w -semi open in X for each open set V of Y .

Theorem 4.3. *Every (w, σ) -continuous function is (μ, σ) -s-continuous but not conversely.*

Example 4.4. Let $X = \mathbb{R}$ be the set of real numbers, $w_X = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}$ and $\tau = \sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ where $\mathbb{Q}$ denotes the set of all rational numbers and $\mathbb{R} \setminus \mathbb{Q}$ denotes the set of all irrational numbers. Define $f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \sigma)$ as the identity function. Then f is (w, σ) -s-continuous but not (w, σ) -continuous.

Theorem 4.5. *Let (X, w_X, τ) be a weak topological space, (Y, σ) be a topological space and $f : (X, \tau) \rightarrow (Y, \sigma)$ a function. Then the following are equivalent :*

1. f is (w, σ) -s-continuous.
2. For each $x \in X$ and each open set V of Y with $f(x) \in V$, there exists a w -semi open set U containing x such that $f(U) \subseteq V$.

3. For each $x \in X$ and each open set V of Y with $f(x) \in V$, $f^{-1}(V)$ is a w -semi open neighborhood of x .
4. The inverse image of each closed set in Y is w -semi closed.
5. $wsC(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for every $B \subseteq Y$.
6. $f(wsC(A)) \subseteq cl(f(A))$ for every $A \subseteq X$.
7. $f^{-1}(int(B)) \subseteq wsI(f^{-1}(B))$ for every $B \subseteq Y$.

Proof. 1. $\Rightarrow$ 2. Let $x \in X$ and V any open set in Y such that $f(x) \in V$. Since f is (w, σ) -s-continuous, $f^{-1}(V)$ is w -semi open. By putting $U = f^{-1}(V)$, $x \in U$ and $f(U) \subseteq V$.

2. $\Rightarrow$ 3. Let $x \in X$ and V an open in Y such that $f(x) \in V$. By 2, there exists a w -semi open set U containing x such that $f(U) \subseteq V$. So each $x \in U \subseteq f^{-1}(V)$ and hence $f^{-1}(V)$ is a w -semi open neighborhood of x .

3. $\Rightarrow$ 1. Let $x \in X$ and V an open in Y such that $f(x) \in V$. By 3, $f^{-1}(V)$ is a w -semi open neighborhood of x . Thus for each $x \in f^{-1}(V)$, there exists a w -semi open set U_x containing x such that $x \in U_x \subseteq f^{-1}(V)$. Hence $f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x$ and so $f^{-1}(V) \in wSO(X)$.

1. $\Leftrightarrow$ 4. It is obvious.

1. $\Rightarrow$ 5. Let B be a subset of Y . Since $cl(B)$ is closed and f is (w, σ) -s-continuous, $f^{-1}(cl(B))$ is w -semi closed. Therefore,

$$wsC(f^{-1}(B)) \subseteq wsC(f^{-1}(cl(B))) = f^{-1}(cl(B)).$$

5. $\Rightarrow$ 6. Let A be a subset of X . By 5, we have

$$wsC(f^{-1}(f(A))) \subseteq f^{-1}(cl(f(A))).$$

But $wsC(A) \subseteq wsC(f^{-1}(f(A)))$. Therefore $f(wsC(A)) \subseteq cl(f(A))$.

6. $\Rightarrow$ 7. Let B be a subset of Y . By 6,

$$f(wsC(X \setminus f^{-1}(B))) \subseteq cl(f(X \setminus f^{-1}(B)))$$

and

$$f(X \setminus wsI(f^{-1}(B))) \subseteq cl(Y \setminus B) = Y \setminus int(B).$$

Therefore $X \setminus wsI(f^{-1}(B)) \subseteq f^{-1}(Y \setminus int(B))$ and $f^{-1}(int(B)) \subseteq wsI(f^{-1}(B))$.

7. $\Rightarrow$ 1. Let B be an open in Y and $f^{-1}(int(B)) \subseteq wsI(f^{-1}(B))$. Then

$$f^{-1}(B) \subseteq wsI(f^{-1}(B)).$$

But $wsI(f^{-1}(B)) \subseteq f^{-1}(B)$. Hence $f^{-1}(B) = wsI(f^{-1}(B))$. Therefore $f^{-1}(B)$ is w -semi open. $\square$

As immediate consequence of Theorem 4.5, we have the following result.

Corollary 4.6. *Let (X, w_X, τ) be a weak topological space and (Y, σ) be a topological space and $f : (X, \tau) \rightarrow (Y, \sigma)$ a (w, σ) - s -continuous function, then the following are equivalent:*

1. $wsI(wsC(f^{-1}(B))) \cap wsC(wsI(f^{-1}(B))) \subseteq f^{-1}(cl(B))$ for each B in Y .
2. $f[wsI(wsC(A)) \cap wsC(wsI(A))] \subseteq cl(f(A))$ for each A in X .

Definition 4.7. Let (X, w_X, τ) be a weak topological space and $A \subseteq X$. Then the w - s -kernel of A denoted by w - s - $Ker(A)$ is defined to be the set, w - s - $Ker(A) = \cap \{U : U \in wSO(X), A \subseteq U\}$.

Theorem 4.8. *Let (X, w_X, τ) be a weak topological space and $x \in X$. Then $y \in w$ - s - $Ker(\{x\})$ if and only if $x \in wsC(\{y\})$.*

Proof. Assume that $y \notin w$ - s - $Ker(\{x\})$. Then there exists a w -semi open set U containing x such that $y \notin U$. Therefore, we have $x \notin wsC(\{y\})$. The converse is similarly shown. $\square$

Theorem 4.9. *Let (X, w_X, τ) be a weak topological space and A a subset of X . Then w - s - $Ker(A) = \{x \in X : wsC(\{x\}) \cap A \neq \emptyset\}$.*

Proof. Let $x \in w$ - s - $Ker(A)$ and $wsC(\{x\}) \cap A = \emptyset$. Then, $x \notin X \setminus wsC(\{x\})$ which is a w -semi open set containing A . But this is impossible, since $x \in w$ - s - $Ker(A)$. Consequently, $wsC(\{x\}) \cap A \neq \emptyset$.

Conversely, let $x \in X$ such that $wsC(\{x\}) \cap A \neq \emptyset$. Suppose that $x \notin w$ - s - $Ker(A)$. Then there exists a w -semi open set U containing A and $x \notin U$. Let $y \in wsC(\{x\}) \cap A$. Then $y \in wsC(\{x\})$ and $y \in A$. Thus $x \in w$ - s - $Ker(\{y\})$ and $y \in U \in w$ implies $x \in U \in SO(X)$. By this contradiction, $x \in w$ - s - $Ker(A)$. $\square$

Theorem 4.10. *The following are equivalent for any points x and y in a weak space (X, w_X) :*

1. w - s - $Ker(\{x\}) \neq w$ - s - $Ker(\{y\})$.
2. $wsC(\{x\}) \neq wsC(\{y\})$.

Theorem 4.11. *Let (X, w_X, τ) be a weak topological space and $A \subseteq X$. Then*

1. $x \in w$ - s - $Ker(A)$ if and only if $A \cap F \neq \emptyset$ for any w -semi closed subset F of X with $x \in F$.

2. $A \subseteq w\text{-}s\text{-}Ker(A)$ and $A = w\text{-}s\text{-}Ker(A)$ if A is w -semi open in X .
3. If $A \subseteq B$ then $w\text{-}s\text{-}Ker(A) \subseteq w\text{-}s\text{-}Ker(B)$.

Proof. 1. Let $x \in w\text{-}s\text{-}Ker(A)$. Then by Theorem 4.9, $A \cap wsC(\{x\}) \neq \emptyset$. Conversely, assume that $A \cap F \neq \emptyset$. By taking $F = wsC(\{x\})$, we have $A \cap wsC(\{x\}) \neq \emptyset$ which implies $x \in w\text{-}s\text{-}Ker(A)$.

2. Let A be w -semi open in X . Then always $A \subseteq w\text{-}s\text{-}Ker(A)$. On the other hand, assume that $x \in w\text{-}s\text{-}Ker(A)$. Then $x \in \bigcap \{U : U \in wSO(X), A \subseteq U\}$. Since A is w -semi open implies that $x \in A$. Thus $w\text{-}s\text{-}Ker(A) \subseteq A$. Hence $A = w\text{-}s\text{-}Ker(A)$.

3. It is obvious. □

As immediate consequence of Theorems 4.5 and 4.11, we have the following result.

Corollary 4.12. *Let (X, w_X, τ) be a weak topological space, (Y, σ) be a topological space and $f : (X, \tau) \rightarrow (Y, \sigma)$ a function (w, σ) - s -continuous. Then the following are equivalent :*

1. For every subset A of X , $f(wsI(A)) \subset Ker(f(A))$.
2. For every subset B of Y , $wsI(f^{-1}(B)) \subseteq f^{-1}(Ker(B))$.

Definition 4.13. Let (X, w_X, τ) be a weak topological space and (Y, σ) be a topological space. Then $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be gw - s -continuous (respectively contra locally w - s -continuous) if $f^{-1}(F)$ is a gw - s -closed (respectively locally w - s -closed) for each closed set F of (Y, σ) .

Example 4.14. In Example 3.13, take $f : (X, \tau) \rightarrow (X, \tau)$, defined as: $f(a) = b$, $f(b) = c$ and $f(c) = a$, then f is contra locally w - s -continuous but is not gw - s -continuous. In the same form if in Example 3.25, we define $f : (X, \tau) \rightarrow (X, \tau)$, as: $f(a) = b$, $f(b) = a$ and $f(c) = c$, then f is w - s -continuous but is not contra locally w - s -continuous. Observe that in each case f is not (w, τ) - s -continuous.

The following theorem is a direct consequence of Theorem 4.5 and Theorem 3.22

Theorem 4.15. *Let (X, w_X, τ) be a weak topological space and (Y, σ) be a topological space. Then $f : (X, \tau) \rightarrow (Y, \sigma)$ is (w, σ) - s -continuous if and only if it is gw - s -continuous and contra locally w - s -continuous.*

The following example shows the existence of a function that is contra locally w -s-continuous but not is gw -s-continuous, in consequence is not (w, τ) -s-continuous.

Example 4.16. In Example 3.12, define $f : (X, \tau) \rightarrow (X, \tau)$ as follows: $f(a) = a, f(b) = d, f(c) = b$ and $f(d) = c$. According with 3.20, f is contra locally w -s-continuous but not is gw -s-continuous, in consequence is not (μ, τ) -s-continuous.

Theorem 4.17. Let (X, w_X, τ) be a weak topological space and (Y, σ) be a topological space. Then a contra continuous function $f : (X, \tau) \rightarrow (Y, \sigma)$ is (w, σ) -s-continuous if and only if it is gw -s-continuous

Proof. Suppose that f is contra continuous and (w, σ) -s-continuous. Let F be a closed set in Y , then $f^{-1}(F)$ is open and w -semi closed in X . Since each w -semi closed is gw -s-closed, then f is gw -s-continuous.

Conversely, let F be a closed set in Y , then $f^{-1}(F)$ is open and gw -s-closed in X . Since each open set is locally w -s-closed, then $f^{-1}(F)$ is locally w -s-closed and gw -s-closed, by Theorem 3.22, f is (w, σ) -s-continuous. $\square$

Example 4.18. In Example 4.4, f is (μ, σ) -s-continuous, $f(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R} \setminus \mathbb{Q}$ is w -open but is not w -s-closed in consequence, f is not contra continuous.

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