STRONGLY PRIME RADICALS OF A TERNARY SEMIGROUP

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Abstract: In this article we introduce the notion of right strongly prime radical of a ternary-semigroup and study it. We also introduce the notion of a super sp-system of a ternary-semigroup and characterize it.

AMS Subject Classification: 20M12, 20M17
Key Words: right strongly prime ternary semigroup, right strongly prime ideal, super sp-system, right strongly prime radical

1. Introduction

The literature of the theory of ternary operations is vast and scatter over diverse areas of mathematics. In 1932 Lehmer[6] introduced the concept of ternary algebraic systems. The algebraic system called triplexes was investigated by Lehmer which turn out to be commutative ternary groups. Kasner[3] also studied the structures and give the ideal of $n$-ary algebras. Ternary semigroups are universal algebras with one associative ternary operation. The ideal theory in ternary-
semigroup was introduced by Sioson [8]. Shabir and Bashir [9] launched prime ideals in ternary semigroups. The class of strongly prime semigroups is an interesting class of semigroups because it has some interesting properties which are similar to the properties of commutation domains. Some earlier works of ternary semigroup may be found in [1, 2, 4, 5, 7]. Throughout this paper T will always denote a ternary semigroup with zero and \( T^* = T \setminus \{0\} \). In this paper we introduce and study the notion of right strongly prime radical in ternary semigroup.

2. Right Strongly Prime Radicals of a Ternary Semigroup

**Definition 2.1.** A ternary semigroup \( T \) is called a right stronglyprime if for every \( x \in T^* \) there exist finite subsets \( F_1, F_2, F_3 \) of \( T \) depending on \( x \) such that \( xF_1F_2F_3y = \{0\} \Rightarrow y = 0 \) for all \( y \in T \).

**Example 2.2.** Let \( T = \{ ri : r \in \mathbb{R}, i^2 = -1 \} \), where \( \mathbb{R} \) is the set of all real numbers. Then together with usual ternary multiplication, \( T \) forms a ternary semigroup. Let \( ri (\neq 0) \in T \) and \( F = \{ ri \} \). Then \( (ri) FFFy = 0 \) implies that \( y = 0 \) for all \( y \in T \). Hence \( T \) is a right stronglyprime ternary semigroup.

**Definition 2.3.** Let \( A \) be a nonempty subset of a ternary semigroup \( T \). Then the right annihilator of \( A \) w.r.t a nonempty subset \( B \) of \( T \), denoted by \( r_{s}(A, B) \) is defined by

\[
r_{s}(A, B) = \{ x \in T : ABx = \{0\} \}
\]

[left annihilator, lateral annihilator, annihilator].

**Theorem 2.4.** A ternary semigroup \( T \) with identity is a right stronglyprime ternary semigroup if and only if every nonzero ideal of \( T \) contains a finitely generated left ideal whose right annihilator w.r.t some finite subset of \( T \) is zero.

**Proof.** Suppose \( T \) is a right stronglyprime ternary semigroup and \( I \) is a non zero ideal of \( T \). Let \( r \neq 0 \) be an element of \( I \). So \( r \in T^* \). Now since \( T \) is a right stronglyprime ternary semigroup, corresponding to \( r \), there exists a finite subset \( F \) of \( T \), satisfying \( rFFFT = 0 \Rightarrow t = 0 \). Now \( rFF \subseteq I \) and \( rFF \) is finite. Let \( L \) be the left ideal of \( T \) generated by \( rFF \). i.e \( L = TTtrFF \). So \( L \subseteq I \). Now let \( LFt = 0 \). As \( T \) contains identity , \( rFF \subseteq TTtrFF \Rightarrow rFFt \subseteq TTtrFFt = LFt = 0 \), hence \( t = 0 \). Thus \( I \) contains the finitely generated left ideal \( L \) whose right annihilator w.r.t \( F \) is zero.

Conversely, let \( x (\neq 0) \in T \). Then \( \langle x \rangle \) is a nonzero ideal of \( T \). Then by hypothesis, there exists a finite subset \( F' \) such that right annihilator of the left ideal \( L \) generated by \( F' \) w.r.t finite subset \( F \) of \( T \) is zero. In other words
LFy = \{0\} implies that y = 0, for all y ∈ T. If possible let xTT = \{0\}. Then ⟨x⟩TT = \{0\}. Since FFa ⊆ ⟨a⟩ = TTaTT ⊆ ⟨a⟩TT = \{0\} (since T contains identity), we have FFa = \{0\} ⇒ LFx = \{0\}, as L = TTF; but LFx = \{0\} ⇒ x = 0, a contradiction. Thus xTT ≠ 0. So there exist r, a ∈ T such that xra ≠ 0. Let I = ⟨xra⟩, I ≠ \{0\}.

Then by hypothesis, there exist a finite subset G of I and H = TTG satisfying HH'y = \{0\} ⇒ y = 0 for all y ∈ T where H is a finite subset of T.

Let G = \left\{ \bigcup_{p=1}^{t} \xi_p\gamma_pxra\lambda_p\eta_p \right\} \text{ where } t ∈ Z^+ \text{ and } \xi_p, \gamma_p, \lambda_p, \eta_p ∈ T. \text{ Let } F'' = \{a, a\lambda_p\eta_p\}; \text{ } p = 1, 2, ..., t \text{ where } t ∈ Z^+; \text{ then } F'' \text{ is a finite subset of } T. \text{ Let } F_1 = \{r\}; F_2 = F'' \text{ and } F_3 = H'. \text{ Now } xx_1F_2F_3y = \{0\}, HH'y = \{0\} ⇒ y = 0. \text{ So } T \text{ is right stronglyprime.}

**Theorem 2.5.** A ternary semigroup T with identity is right stronglyprime if and only if every nonzero ideal of T contains a finite subset B such that the right annihilator of B w.r.t a finite subset of T is zero.

**Proof.** Suppose T is right stronglyprime ternary semigroup and A is any non zero ideal of T. Let x (≠ 0) ∈ A. Since T is a right stronglyprime ternary semigroup, x has a right insulator F. Let B = xFF, then B is a finite subset of A and right annihilator of B w.r.t F is zero. i.e rs(B,F) = 0.

Conversely, suppose that every non zero ideal of T contains a finite subset whose right annihilator w.r.t a finite subset F of T is zero. Let x (≠ 0) ∈ T. Then ⟨x⟩ is a non zero ideal of T. Then there exists finite subsets F of ⟨a⟩ and a finite subset F of T such that F FY = \{0\} ⇒ y = 0 for all y ∈ T. Now proceeding as in the proof of the theorem 2.4 we can show that T is right stronglyprime ternary semigroup.

**Proposition 2.6.** Every simple ternary semigroup with identity is a right stronglyprime.

**Proof.** Let T be a simple ternary semigroup with identity. Since T admits identity there exist elements \{(e_i, f_i) ∈ T × T (i = 1, 2, ..., n)\} such that \bigcup_{i=1}^{n} e_ifix = \bigcup_{i=1}^{n} e_ixfi = \bigcup_{i=1}^{n} xe_ifi = x \text{ for all } x ∈ T. \text{ Since T is simple, T is the only non zero ideal of T. Now } G = \{e_i\} (i = 1, 2, ..., n), H = \{f_i\} (i = 1, 2, ..., n) \text{ are finite subsets of T and right annihilator of G w.r.t H is } \{0\}, \text{ since } GHy = \{0\} ⇒ \sum_{i=1}^{n} e_if_iy = 0 ⇒ y = 0. \text{ So by theorem 2.5, T is right stronglyprime ternary semigroup.}

**Definition 2.7.** A class ρ of a ternary semigroup is called hereditary A is
Proposition 2.8. The class of all right stronglyprime ternarysemigroups is hereditary.

Proof. Let $T$ be a right stronglyprime ternarysemigroup and $A$ be an ideal of $T$. If $A = (0)$, then $A$ is trivially right stronglyprime ternarysemigroup. So we assume that $A \neq (0)$ and let $x$ be a non zero element of $A$. Then $x \in T^*$. Since $T$ is right stronglyprime ternarysemigroup, there exists a finite subset $F$ of $T$ depending on $x$ such that $xF F F y = 0 \Rightarrow y = 0 \Rightarrow (1)$.

Obviously $F_1 = FFFx$ and $F_2 = \{x\}$ are two finite subsets of $A$. Now suppose that $xF_1 F_2 F_1 y = 0$ where $y \in I$. Then $xF F F x F F F x F F F y = 0$. Now by (1) we have $xF F F x F F F x F F F y = 0$. Continuing this process we get $y = 0$. So $A$ is right stronglyprime ternarysemigroup. Hence the class of all right stronglyprime ternarysemigroups is hereditary.

Definition 2.9. An element $x$ of a ternarysemigroup $T$ is called an idempotent if $x^3 = x$.

Proposition 2.10. If $T$ is a right stronglyprime ternarysemigroup and $i$ is a non zero idempotent element in $T$, then $iT i$ is a right stronglyprime ternarysubsemigroup of $T$.

Proof. Obviously $iT i$ is a ternarysubsemigroup of $T$. Let $iti$ be a non zero element of $iT i$. Then $iti \in T$. Since $T$ is right stronglyprime ternarysemigroup, corresponding to $iti$ there exist a right insulator $F = \{f_1, f_2, ..., f_k\}$ (say) in $T$. Let $F_1 = \{i\}$, $F_2 = \{if_r f_s f_t i / f_r, f_s, f_t \in F where 1 \leq r; s, t \leq k\}$. Consider $(isi) F_1 F_2 F_1 (i) y _i = 0 \Rightarrow (isi) i (if_r f_s f_t i i) y _i = 0 where f_r, f_s, f_t \in F \Rightarrow is (iii) f_r, f_s, f_t (iii) y _i = 0 \Rightarrow (isi) f_r, f_s, f_t (iii) y _i = 0 \Rightarrow (isi) F F F (iii) y _i = 0$ which implies $iy = 0$. This $iT i$ is a right stronglyprime ternarysubsemigroup of $T$.

Definition 2.11. An ideal $A$ of a ternarysemigroup $T$ is called right stronglyprime ideal of $T$ if for $x \notin A$, there exist a finite subset $F$ of $T$ such that $F F y \subseteq A \Rightarrow y \in A$.

Theorem 2.12. Every proper right stronglyprime ideal of a ternarysemigroup $T$ is a prime ideal of $T$.

Proof. Suppose $A$ right stronglyprime ideal of $T$. Let $X, Y, Z$ be three ideals of $T$ such that $XYZ \subseteq A$. Suppose that $X \not\subseteq A$ and $Y \not\subseteq A$. Let $x \in X \setminus A$ and $y \in Y \setminus A$. Since $A$ is a right stronglyprime ideal of $T$ and $X \notin A$, there exists a finite subsets $F$ of $T$ such that $F F a \subseteq A \Rightarrow a \in A. \Rightarrow(1)$
Let $y_i \in Y$, $t \in T$ and $z_i \in Z$. Now we have $F' F y_i t z_i \subseteq X T Y T Z \subseteq A$. So by (1) $y_1 t z_1 \in A$. Thus $Y T Z \subseteq A$. Also $y \notin A$, so there exists a finite subset $F''$ of $\langle y \rangle$ and a finite subset $F_1$ of $T$ such that $F'' F_1 b \subseteq A \to b \in A$. 

Now let $z \in Z$, then $F'' F_1 z \subseteq Y T Z \subseteq A \to z \in A$ by (2). Thus $Z \subseteq A$. Again let $X \not\subseteq A$ and $Z \not\subseteq A$. If possible, let $Y \not\subseteq A$. Then by above argument we have $Z \subseteq A$, a contradiction. So, $Y \subseteq A$. Similarly, if $Z \not\subseteq A$ and $Y \not\subseteq A$, then we have $X \subseteq A$. Thus $X Y Z \subseteq A \Rightarrow X \subseteq A$ or $Y \subseteq A$ or $Z \subseteq A$. So $A$ is a prime ideal of $T$.

**Definition 2.13.** Right stronglyprime radical of a ternarysemigroup $T$, denoted by $T \, P (T)$ is defined by

$$T \, P (T) = \bigcap \{ A : A \text{ is a right stronglyprime ideal of } T \}.$$ 

**Definition 2.14.** A non empty subset $G$ of a ternarysemigroup $T$ is called an sp-system if for any $g \in G$ there is a finite subset $F_1 \langle g \rangle$ and a finite subset $F_2$ of $T$ such that $F_1 F_2 z \cap G \neq \emptyset$ for all $z \in G$.

**Theorem 2.15.** A proper ideal $A$ of a ternarysemigroup $T$ is right stronglyprime if and only if $T \setminus A$ is an sp-system.

**Proof.** Let $A$ be a right strongly prime ideal of $T$. Let $g \in T \setminus A$. Then $g \notin A$. So there exists a finite subsets $F'$ of $\langle g \rangle$ and $F$ of $T$ such that $F' F b \subseteq A$ implies that $b \in A$. This implies that $F' F z \cap T \setminus A \neq \emptyset$ for all $z \in T \setminus A$. Hence $T \setminus A$ is an sp-system.

Conversely, suppose that $T \setminus A$ is an sp-system. Let $a \notin A$. Then $a \in T \setminus A$. So there exists a finite subset $F'$ of $\langle a \rangle$ and $F$ of $T$ such that $F' F z \cap T \setminus A \neq \emptyset$ for all $z \in T \setminus A$. Let $F' F b \subseteq A$. Then $F' F b \cap (T \setminus A) = \emptyset$. If possible, let $b \notin A$. Then $b \in T \setminus A$ which implies that $F' F b \cap (T \setminus A) \neq \emptyset$, a contradiction. So $b \in A$ and hence $A$ is a right strongly prime ideal of $T$.

**Definition 2.16.** A pair of subsets $(G, I)$ where $I$ is an ideal of a ternarysemigroup $T$ and $G$ is a nonempty subset of $T$ is called a super sp-system of $T$. $G \cap I$ contains no non zero element of $T$ and for any $g \in G$, there exist a finite subset $F$ of $\langle g \rangle$ and a finite subset $F'$ of $T$ such that $F F' z \cap G \neq \emptyset$ for all $z \notin I$.

**Proposition 2.17.** An ideal $A$ of a ternarysemigroup $T$ is a right stronglyprime if and only if $(T \setminus A, A)$ is a super sp-system of $T$.

**Proof.** Let $A$ be a right stronglyprime ideal of $T$. So $T \setminus A$ is an sp-system by theorem 2.15. Thus for any $y \in T \setminus A$, there exists a finite subset $F$ of $\langle y \rangle$ and a finite subset $F'$ of $T$ such that $F F' z \cap (T \setminus A) \neq \emptyset$ for all $z \notin A$. Also
(T \setminus A) \cap A contains no non zero element of T. Thus the pair (T \setminus A, A) is a super sp-system of T.

Converse follows from the definition.

**Theorem 2.18.** For any ternary semigroup T, TP(T) = \{ x \in T / \text{whenever} (G, P) \text{ is a super sp-system for some ideal } P \text{ of } T \text{ and } x \in G, \text{ then } 0 \in G \}.

**Proof.** Let x \in TP(T). If possible, let (G, P) be a super sp-system with x \in G and 0 \notin G. Then G \cap P = \phi. By Zorn’s lemma, choose an ideal Q with P \subseteq Q and Q is maximal with respect to G \cap Q = \phi. We now prove that Q is a right strongly prime ideal of T. Let a \notin Q. Then there exist g \in G such that \langle g \rangle \subseteq Q \cup \langle a \rangle. Since (G, P) is a super sp-system, there exists a finite subset F = \{f_1, f_2, ..., f_k\} \subseteq \langle g \rangle and a finite subset F' of T such that F'z \cap G \neq \phi for all z \notin P \rightarrow (1).

Since F \subseteq \langle g \rangle \subseteq Q \cup \langle a \rangle each f_i is of the form f_i = q_i \cup a_i for some q_i \in Q and a_i \in \langle a \rangle. Let F^* = \{a_1, a_2, ..., a_k\} then F \subseteq \langle a \rangle. Let z \in T be such that F^*F'z \subseteq Q. Now if z \notin Q, then z \notin P so from (1) we have FF'z \cap G \neq \phi; but f_iF'z = (q_i \cup a_i)F'z = q_iF'z \cup a_iF'z \subseteq Q \cup F^*F'z \subseteq Q \cup Q = Q for all i \in \{1, 2, ..., k\}. So FF'z \subseteq Q. Hence G \cap Q = \phi, a contradiction. Hence z \in Q. So Q is a right strongly prime ideal of T. Now as TP(T) \subseteq Q. So x \in Q. But by assumption x \in G, a contradiction. Hence 0 \in G.

Conversely, suppose B = \{ x \in T / \text{when ever} (G, P) \text{ is a super sp-system for some ideal } P \text{ of } T \text{ and } x \in G, \text{ then } 0 \in G \}. Let x \in B. If possible let x \notin TP(T). Then there exist a right strongly prime ideal A of T such that x \notin A. Then (T \setminus A, A) is a super sp-system, where x \in T \setminus A but 0 \notin T \setminus A, a contradiction. Hence the converse part.

**References**


