AN UPPER ESTIMATE FOR THE LARGEST SINGULAR VALUE OF A SPECIAL MATRIX, III

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Abstract: Our goal is to obtain an upper bound for the largest singular value of the matrix in the form

$$A = D^{-1} E (E^t D^{-1} E)^{-1} E^t.$$

Here $\cdot^t$ denotes the matrix transpose, D is a non-singular diagonal matrix, and E is thin matrix with maximal rank.

AMS Subject Classification: 15A18

Key Words: singular values, largest singular value

1. Introduction

In many situation, we have to deal with a matrix in the form ($\cdot^t$ is matrix transpose)

$$A = D^{-1} E (E^t D^{-1} E)^{-1} E^t,$$

where:

1. D is a symmetric, positive definite $(m \times m)$ -matrix.
2. E is a $(m \times n)$ -matrix, $n \leq m$ and $\text{rank}(E) = n$.
3. It follows from (2) that $E^t D^{-1} E$ is a non-singular matrix.

Our goal is to establish an upper bound for $\|A\|$.

Received: June 16, 2017

Revised: July 5, 2017

Published: August 30, 2017

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url: www.acadpubl.eu

In many applications $\mathbf{D}$ is a diagonal matrix, see for example the results in [1, 2, 4, 5, 6, 7, 11] for simple constructions of moving least squares approximations.

2. Main Result

First, we will obtain an upper bound for the largest singular value of the symmetric matrix $\mathbf{A}\mathbf{D}^{-1}$.

Lemma 2.1. *Let the following conditions hold true:*

1. $\mathbf{D}$ is a non-singular diagonal $(m \times m)$ -matrix, with positive entries on main diagonal.
2. $\mathbf{E}$ is a $(m \times n)$ -matrix, $m \geq n$, and $\text{rank}(\mathbf{E}) = n$.
3. $\mathbf{A} \neq \mathbf{I}$.
4. All diagonal elements of the matrix $\mathbf{A}$ are non-negative numbers.

Then the largest singular value $\sigma_{\max}(\mathbf{A}\mathbf{D}^{-1})$ of matrix $\mathbf{A}\mathbf{D}^{-1}$ satisfies

$$\|\mathbf{A}\mathbf{D}^{-1}\|_2 = \sigma_{\max}(\mathbf{A}\mathbf{D}^{-1}) \leq (m-1) \text{trace}(\mathbf{D}^{-1}).$$

Proof. We separate the proof into several simple steps.

Step 1. First, we will collect some properties of $\mathbf{A}$:

1. The matrix $\mathbf{A}$ is an idempotent matrix:

$$\mathbf{A}^2 = \mathbf{D}^{-1}\mathbf{E}(\mathbf{E}^t\mathbf{D}^{-1}\mathbf{E})^{-1}\mathbf{E}^t\mathbf{D}^{-1}\mathbf{E}(\mathbf{E}^t\mathbf{D}^{-1}\mathbf{E})^{-1}\mathbf{E}^t = \mathbf{A}.$$

2. Again, by a direct calculation:

$$\mathbf{E}^t\mathbf{A} = \mathbf{A}.$$

3. All eigenvalues λ_i of $\mathbf{A}$ are 0 or 1.

Indeed, let $\mathbf{x} \neq \mathbf{0}$ be m -dimensional eigenvector corresponding to the eigenvalue λ . Then

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

$$\mathbf{A}^2\mathbf{x} = \mathbf{A}\mathbf{A}\mathbf{x} = \mathbf{A}\lambda\mathbf{x} = \lambda\mathbf{A}\mathbf{x} = \lambda^2\mathbf{x},$$

i.e. $\lambda\mathbf{x} = \lambda^2\mathbf{x}$ or $\lambda - \lambda^2 = 0$. Hence $\lambda = 0$ or $\lambda = 1$.

4. If all eigenvalues of $\mathbf{A}$ are equal to 1, then $\mathbf{A} = \mathbf{I}$.

Indeed, in this case $\det(\mathbf{A}) = \prod_{i=1}^m 1 = 1 \neq 0$. Hence $\mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}^{-1}\mathbf{A}$ or $\mathbf{A} = \mathbf{I}$.

5. We have

$$0 \leq \text{trace}(\mathbf{A}) = \sum_{i=1}^m \lambda_i \leq m - 1, \quad (1)$$

because $\mathbf{A} \neq \mathbf{I}$ and then at least one eigenvalue is equal to zero.

Step 2. Let us prove the inequality

$$\text{trace}(\mathbf{A}\mathbf{D}^{-1}) \leq (m - 1) \text{trace}(\mathbf{D}^{-1}). \quad (2)$$

Indeed ($\mathbf{D}$ is a diagonal matrix)

$$(\mathbf{A}\mathbf{D}^{-1})_{ii} = (\mathbf{A})_{ii} (\mathbf{D}^{-1})_{ii} = (\mathbf{A})_{ii} d_i, \quad i = 1, \dots, m.$$

Hence,

$$\text{trace}(\mathbf{A}\mathbf{D}^{-1}) = \sum_{i=1}^m (\mathbf{A})_{ii} (\mathbf{D}^{-1})_{ii}.$$

and using Cauchy-Schwarz inequality, we receive

$$|\text{trace}(\mathbf{A}\mathbf{D}^{-1})| \leq \left(\sum_{i=1}^m (\mathbf{A})_{ii}^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^m (\mathbf{D}^{-1})_{ii}^2 \right)^{\frac{1}{2}}.$$

Therefore (using well-know inequality $\|\cdot\|_2 \leq \|\cdot\|_1$)

$$|\text{trace}(\mathbf{A}\mathbf{D}^{-1})| \leq \text{trace}(|\mathbf{A}|) \text{trace}(|\mathbf{D}^{-1}|).$$

where, we used also the notation $(|\mathbf{A}|) = (|(\mathbf{A})_{ii}|)$ and analogous for $|\mathbf{D}^{-1}|$.

The matrices $\mathbf{A}$ and $\mathbf{D}^{-1}$ have non-negative diagonal entries, then we get the result

$$\text{trace}(\mathbf{A}\mathbf{D}^{-1}) \leq \text{trace}(\mathbf{A}) \text{trace}(\mathbf{D}^{-1}).$$

Now, (1) implies (2).

Step 3. We have

$$\sigma_{\max}^2(\mathbf{A}\mathbf{D}^{-1}) \leq \text{trace}(\mathbf{D}^{-1}\mathbf{A}^t\mathbf{A}\mathbf{D}^{-1}) \leq m\sigma_{\max}^2(\mathbf{A}\mathbf{D}^{-1}). \quad (3)$$

Indeed, let $\mathbf{A}\mathbf{D}^{-1} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^t$ be the singular value decomposition of the symmetric $(m \times m)$ -matrix $\mathbf{A}\mathbf{D}^{-1}$. Then

$$\begin{aligned}
 \text{trace} \left((\mathbf{A}\mathbf{D}^{-1})^t \mathbf{A}\mathbf{D}^{-1} \right) &= \text{trace} \left((\mathbf{U}\mathbf{\Sigma}\mathbf{U}^t)^t (\mathbf{U}\mathbf{\Sigma}\mathbf{U}^t) \right) \\
 &= \text{trace} \left(\mathbf{U}\mathbf{\Sigma}\mathbf{U}^t \mathbf{U}\mathbf{\Sigma}\mathbf{U}^t \right) \\
 &= \text{trace} \left(\mathbf{U}\mathbf{\Sigma}^2 \mathbf{U}^t \right) \\
 &= \text{trace} \left(\mathbf{U}\mathbf{U}^t \mathbf{\Sigma}^2 \right) \\
 &= \text{trace} \left(\mathbf{\Sigma}^2 \right) \\
 &= \sum_{i=1}^m \sigma_i^2 \left(\mathbf{A}\mathbf{D}^{-1} \right). \tag{4}
 \end{aligned}$$

But $\sigma_{\max}^2 \left(\mathbf{A}\mathbf{D}^{-1} \right) \leq \sum_{i=1}^m \sigma_i^2 \left(\mathbf{A}\mathbf{D}^{-1} \right) \leq m \sigma_{\max}^2 \left(\mathbf{A}\mathbf{D}^{-1} \right)$, hence the inequalities (3) hold true.

Step 4. The matrix $\mathbf{A}\mathbf{D}^{-1}$ is symmetric and positive semi-definite.

Indeed, let $\mathbf{x}$ be an arbitrary m -dimensional vector, and let us set $\mathbf{y} = \mathbf{E}^t \mathbf{D}^{-1} \mathbf{x}$, $\mathbf{z} = (\mathbf{E}^t \mathbf{D}^{-1} \mathbf{E})^{-1} \mathbf{y}$. Then

$$\begin{aligned}
 \langle \mathbf{A}\mathbf{D}^{-1} \mathbf{x}, \mathbf{x} \rangle &= (\mathbf{A}\mathbf{D}^{-1} \mathbf{x})^t \mathbf{x} \\
 &= \mathbf{x}^t \mathbf{D}^{-1} \mathbf{E} (\mathbf{E}^t \mathbf{D}^{-1} \mathbf{E})^{-1} \mathbf{E}^t \mathbf{D}^{-1} \mathbf{x} \\
 &= \mathbf{y}^t (\mathbf{E}^t \mathbf{D}^{-1} \mathbf{E})^{-1} \mathbf{y} \\
 &= \mathbf{z}^t \mathbf{E}^t \mathbf{D}^{-1} \mathbf{E} \mathbf{z} \\
 &= (\mathbf{E} \mathbf{z})^t \mathbf{D}^{-1} (\mathbf{E} \mathbf{z}) \\
 &= \langle (\mathbf{E} \mathbf{z}), \mathbf{D}^{-1} (\mathbf{E} \mathbf{z}) \rangle \geq 0.
 \end{aligned}$$

Step 5. In the space of symmetric positive semi-definite matrices, trace is an inner-product. Hence it obeys Cauchy-Schwarz inequality. Therefore

$$\begin{aligned}
 \text{trace}^2 \left((\mathbf{A}\mathbf{D}^{-1})^t (\mathbf{A}\mathbf{D}^{-1}) \right) &\leq \text{trace} \left((\mathbf{A}\mathbf{D}^{-1})^t \right) \text{trace} \left(\mathbf{A}\mathbf{D}^{-1} \right) \\
 &\leq \text{trace}^2 \left(\mathbf{A}\mathbf{D}^{-1} \right). \tag{5}
 \end{aligned}$$

The same result follows from (4), Cauchy-Schwarz inequality, and the fact that $\mathbf{A}\mathbf{D}^{-1}$ is a symmetric non-negative definite matrix.

Step 6. It follows from (3) and (5) that

$$\begin{aligned}\sigma_{\max}^2(\mathbf{A}\mathbf{D}^{-1}) &\leq \text{trace}(\mathbf{D}^{-1}\mathbf{A}^t\mathbf{A}\mathbf{D}^{-1}) \\ &= \text{trace}\left((\mathbf{A}\mathbf{D}^{-1})^t(\mathbf{A}\mathbf{D}^{-1})\right) \\ &\leq \text{trace}^2(\mathbf{A}\mathbf{D}^{-1}).\end{aligned}\tag{6}$$

On the end (see (2)), we receive

$$\sigma_{\max}(\mathbf{A}\mathbf{D}^{-1}) \leq (m-1) \text{trace}(\mathbf{D}^{-1}). \quad \square$$

Example 2.1. Let us consider a typical situation in moving least squares method. Let

$$\mathbf{E} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using a simple **Maple** code, it is not hard to generate discrete graphs of $\|\mathbf{A}\mathbf{D}^{-1}\|_2$ and $(m-1) \text{trace}(\mathbf{D}^{-1})$ when $\alpha = 0.001i$, $i = 1, \dots, 200$, see Figure 1.

Theorem 2.1. *Let the following conditions hold true:*

1. $\mathbf{D} = \text{diag}(d_1, d_2, \dots, d_m)$ is a diagonal $(m \times m)$ -matrix, and let $d_i > 0$, $i = 1, \dots, m$.
2. $\mathbf{E}$ is a $(m \times n)$ -matrix, $m \geq n$, and $\text{rank}(\mathbf{E}) = n$.
3. $\mathbf{A} \neq \mathbf{I}$.
4. All diagonal elements of the matrix $\mathbf{A}$ are non-negative numbers.

Then the largest singular value $\sigma_{\max}(\mathbf{A})$ of matrix $\mathbf{A}$ satisfies

$$\sigma_{\max}(\mathbf{A}) \leq (m-1) \text{trace}(\mathbf{D}^{-1}) \max\{d_1, \dots, d_m\}.$$

Proof. We refer to well-known inequality

$$\sigma_{\max}(\mathbf{A}) \sigma_{\min}(\mathbf{D}^{-1}) \leq \sigma_{\max}(\mathbf{A}\mathbf{D}^{-1})$$

see [3, P15-SVD].

Therefore, by Lemma 2.1, we receive

$$\sigma_{\max}(\mathbf{A}) \sigma_{\min}(\mathbf{D}^{-1}) \leq \sigma_{\max}(\mathbf{A}\mathbf{D}^{-1})$$

Figure 1: Graph of $(m-1) \text{trace}(\mathbf{D}^{-1})$ in blue and graph of $\|\mathbf{A}\mathbf{D}^{-1}\|_2$ in red

$$\leq (m-1) \text{trace}(\mathbf{D}^{-1})$$

or

$$\begin{aligned} \sigma_{\max}(\mathbf{A}) &\leq (m-1) \frac{\text{trace}(\mathbf{D}^{-1})}{\sigma_{\min}(\mathbf{D}^{-1})} \\ &= (m-1) \text{trace}(\mathbf{D}^{-1}) \sigma_{\max}(\mathbf{D}) \\ &= (m-1) \text{trace}(\mathbf{D}^{-1}) \max\{d_1, \dots, d_m\}, \end{aligned}$$

where we used $\sigma_{\max}(\mathbf{D}) = \frac{1}{\sigma_{\min}(\mathbf{D}^{-1})}$, see [3, P3-SVD]. □

Corollary 2.1. *Let the conditions of Theorem 2.1 be fulfilled. Then*

$$\|\mathbf{A}\| \leq \min \left\{ (m-1) \text{trace}(\mathbf{D}^{-1}) \max\{d_1, \dots, d_m\}, \frac{\sigma_{\max}^2(\mathbf{E})}{\sigma_{\min}^2(\mathbf{E})} \frac{d_{\max}}{d_{\min}} \right\}.$$

The proof follows from [8, Lemma 2.1].

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