AN UPPER ESTIMATE FOR THE LARGEST SINGULAR
VALUE OF A SPECIAL MATRIX, III

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Abstract: Our goal is to obtain an upper bound for the largest singular value of the matrix in the form

\[ A = D^{-1} E (E' D^{-1} E)^{-1} E'. \]

Here \( E' \) denotes the matrix transpose, \( D \) is a non-singular diagonal matrix, and \( E \) is thin matrix with maximal rank.

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1. Introduction

In many situation, we have to deal with a matrix in the form (\( E' \) is matrix transpose)

\[ A = D^{-1} E (E' D^{-1} E)^{-1} E', \]

where:

1. \( D \) is a symmetric, positive definite \((m \times m)\)-matrix.
2. \( E \) is a \((m \times n)\)-matrix, \( n \leq m \) and \( \text{rank}(E) = n \).
3. It follows from (2) that \( E' D^{-1} E \) is a non-singular matrix.

Our goal is to establish an upper bound for \( \|A\| \).
In many applications $D$ is a diagonal matrix, see for example the results in [1, 2, 4, 5, 6, 7, 11] for simple constructions of moving least squares approximations.

2. Main Result

First, we will obtain an upper bound for the largest singular value of the symmetric matrix $AD^{-1}$.

Lemma 2.1. Let the following conditions hold true:

1. $D$ is a non-singular diagonal $(m \times m)$-matrix, with positive entries on main diagonal.
2. $E$ is a $(m \times n)$-matrix, $m \geq n$, and $\text{rank}(E) = n$.
3. $A \neq I$.
4. All diagonal elements of the matrix $A$ are non-negative numbers.

Then the largest singular value $\sigma_{\text{max}}(AD^{-1})$ of matrix $AD^{-1}$ satisfies

$$\|AD^{-1}\|_2 = \sigma_{\text{max}}(AD^{-1}) \leq (m - 1) \text{trace}(D^{-1}).$$

Proof. We separate the proof into several simple steps.

Step 1. First, we will collect some properties of $A$:

1. The matrix $A$ is an idempotent matrix:

$$A^2 = D^{-1}E(E^tD^{-1}E)^{-1}E^tD^{-1}E(E^tD^{-1}E)^{-1}E^t = A.$$

2. Again, by a direct calculation:

$$E^tA = A.$$

3. All eigenvalues $\lambda_i$ of $A$ are 0 or 1.

   Indeed, let $x \neq 0$ be $m$-dimensional eigenvector corresponding to the eigenvalue $\lambda$. Then

$$Ax = \lambda x,$$

$$A^2x = AAx = A(\lambda x) = \lambda Ax = \lambda^2 x,$$

i.e. $\lambda x = \lambda^2 x$ or $\lambda - \lambda^2 = 0$. Hence $\lambda = 0$ or $\lambda = 1$. 
4. If all eigenvalues of $A$ are equal to 1, then $A = I$.

Indeed, in this case $\det(A) = \prod_{i=1}^{m} 1 = 1 \neq 0$. Hence $A^{-1}A^2 = A^{-1}A$ or $A = I$.

5. We have

$$0 \leq \text{trace}(A) = \sum_{i=1}^{m} \lambda_i \leq m - 1,$$

(1)

because $A \neq I$ and then at least one eigenvalue is equal to zero.

Step 2. Let us prove the inequality

$$\text{trace}(AD^{-1}) \leq (m - 1) \text{trace}(D^{-1}).$$

(2)

Indeed ($D$ is a diagonal matrix)

$$(AD^{-1})_{ii} = (A)_{ii} (D^{-1})_{ii} = (A)_{ii} d_i, \quad i = 1, \ldots, m.$$

Hence,

$$\text{trace}(AD^{-1}) = \sum_{i=1}^{m} (A)_{ii} (D^{-1})_{ii},$$

and using Cauchy-Schwarz inequality, we receive

$$|\text{trace}(AD^{-1})| \leq \left( \sum_{i=1}^{m} (A)_{ii}^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{m} (D^{-1})_{ii}^2 \right)^{\frac{1}{2}}.$$

Therefore (using well-know inequality $\| \cdot \|_2 \leq \| \cdot \|_1$)

$$|\text{trace}(AD^{-1})| \leq \text{trace}(|A|) \text{trace}(|D^{-1}|).$$

where, we used also the notation $|A| = \|(A)_{ii}\|$ and analogous for $|D^{-1}|$.

The matrices $A$ and $D^{-1}$ have non-negative diagonal entries, then we get the result

$$\text{trace}(AD^{-1}) \leq \text{trace}(A) \text{trace}(D^{-1}).$$

Now, (1) implies (2).

Step 3. We have

$$\sigma_{\max}^2(AD^{-1}) \leq \text{trace}(D^{-1}A^tAD^{-1}) \leq m\sigma_{\max}^2(AD^{-1}).$$

(3)
Indeed, let $AD^{-1} = U\Sigma U^t$ be the singular value decomposition of the symmetric $(m \times m)$-matrix $AD^{-1}$. Then

\[
\text{trace} \left( (AD^{-1})^t AD^{-1} \right) = \text{trace} \left( (U\Sigma U^t)^t (U\Sigma U^t) \right) \\
= \text{trace} (U\Sigma U^t U\Sigma U^t) \\
= \text{trace} (U\Sigma^2 U^t) \\
= \text{trace} (UU^t \Sigma^2) \\
= \text{trace} (\Sigma^2) \\
= \sum_{i=1}^{m} \sigma_i^2 (AD^{-1}) .
\]

But $\sigma_{\text{max}}^2 (AD^{-1}) \leq \sum_{i=1}^{m} \sigma_i^2 (AD^{-1}) \leq m \sigma_{\text{max}}^2 (AD^{-1})$, hence the inequalities (3) hold true.

Step 4. The matrix $AD^{-1}$ is symmetric and positive semi-definite. Indeed, let $x$ be an arbitrary $m$-dimensional vector, and let us set $y = E^t D^{-1} x$, $z = (E^t D^{-1} E)^{-1} y$. Then

\[
\langle AD^{-1} x, x \rangle = (AD^{-1} x)^t x \\
= x^t D^{-1} E (E^t D^{-1} E)^{-1} E^t D^{-1} x \\
= y^t (E^t D^{-1} E)^{-1} y \\
= z^t E^t D^{-1} E z \\
= (Ez)^t D^{-1} (Ez) \\
= \langle (Ez), D^{-1} (Ez) \rangle \geq 0 .
\]

Step 5. In the space of symmetric positive semi-definite matrices, trace is an inner-product. Hence it obeys Cauchy-Schwarz inequality. Therefore

\[
\text{trace}^2 \left( (AD^{-1})^t (AD^{-1}) \right) \leq \text{trace} \left( (AD^{-1})^t \right) \text{trace} (AD^{-1}) \\
\leq \text{trace}^2 (AD^{-1}) .
\]

The same result follows from (4), Cauchy-Schwarz inequality, and the fact that $AD^{-1}$ is a symmetric non-negative definite matrix.
Step 6. It follows from (3) and (5) that
\[
\sigma_{\text{max}}^2 (AD^{-1}) \leq \text{trace} \left( D^{-1} A^t AD^{-1} \right) \\
= \text{trace} \left( (AD^{-1})^t (AD^{-1}) \right) \\
\leq \text{trace}^2 (AD^{-1}).
\]

On the end (see (2)), we receive
\[
\sigma_{\text{max}} (AD^{-1}) \leq (m - 1) \text{trace} (D^{-1}).
\]

**Example 2.1.** Let us consider a typical situation in moving least squares method. Let
\[
E = \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 4 \\
1 & 3 & 9 \\
1 & 4 & 16
\end{pmatrix}, \quad D = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Using a simple Maple code, it is not hard to generate discrete graphs of \(\|AD^{-1}\|_2\) and \((m - 1) \text{trace} (D^{-1})\) when \(\alpha = 0.001i, i = 1, \ldots, 200\), see Figure 1.

**Theorem 2.1.** Let the following conditions hold true:

1. \(D = \text{diag}(d_1, d_2, \ldots, d_m)\) is a diagonal \((m \times m)\)-matrix, and let \(d_i > 0, i = 1, \ldots, m\).
2. \(E\) is a \((m \times n)\)-matrix, \(m \geq n\), and \(\text{rank}(E) = n\).
3. \(A \neq I\).
4. All diagonal elements of the matrix \(A\) are non-negative numbers.

Then the largest singular value \(\sigma_{\text{max}} (A)\) of matrix \(A\) satisfies
\[
\sigma_{\text{max}} (A) \leq (m - 1) \text{trace} (D^{-1}) \max \{d_1, \ldots, d_m\}.
\]

**Proof.** We refer to well-known inequality
\[
\sigma_{\text{max}} (A) \sigma_{\text{min}} (D^{-1}) \leq \sigma_{\text{max}} (AD^{-1})
\]
see [3, P15-SVD].

Therefore, by Lemma 2.1, we receive
\[
\sigma_{\text{max}} (A) \sigma_{\text{min}} (D^{-1}) \leq \sigma_{\text{max}} (AD^{-1})
\]
Figure 1: Graph of \((m-1)\text{trace}(D^{-1})\) in blue and graph of \(\|AD^{-1}\|_2\) in red

\[\leq (m-1)\text{trace}(D^{-1})\]

or

\[
\sigma_{\text{max}}(A) \leq (m-1)\frac{\text{trace}(D^{-1})}{\sigma_{\text{min}}(D^{-1})}
= (m-1)\text{trace}(D^{-1})\sigma_{\text{max}}(D)
= (m-1)\text{trace}(D^{-1}) \max \{d_1, \ldots, d_m\},
\]

where we used \(\sigma_{\text{max}}(D) = \frac{1}{\sigma_{\text{min}}(D^{-1})}\), see [3, P3-SVD].

**Corollary 2.1.** Let the conditions of Theorem 2.1 be fulfilled.

Then

\[
\|A\| \leq \min \left\{ (m-1)\text{trace}(D^{-1}) \max \{d_1, \ldots, d_m\}, \frac{\sigma_{\text{max}}^2(E)}{\sigma_{\text{min}}^2(E)} d_{\text{max}} \right\}
\]

The proof follows from [8, Lemma 2.1].
References


