THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A NONLINEAR FRACTIONAL CAPUTO-LANGEVIN EQUATION WITH MULTI-POINT BOUNDARY CONDITIONS

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Abstract: This article investigates a nonlinear fractional Caputo-Langevin equation subject to the multi-point boundary conditions

$$\begin{cases} {}^c D^\alpha ({}^c D^\beta + k)u(t) = f(t, u(t)), & 0 \leq t \leq 1, \\ u(0) = u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i). \end{cases}$$

where $0 < \alpha, \beta \leq 1$ and $1 < \alpha + \beta < 2$, ${}^c D^\alpha$ denotes the Caputo's fractional derivative. Some new existence and uniqueness results are obtained by applying standard fixed point theorems.

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1. Introduction

Analysis of fractional differential equations have gained considerable importance and continuous attention during the past three decades, due mainly to their varied applications in many fields [1, 2, 3]. The differential equations involving Riemann-Liouville and Caputo operators of fractional orders arise in

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many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, polymer rheology, economic, control theory, signal and image processing, biophysics, etc. [13, 14, 15, 16]. Boundary-value problems for fractional differential equations have been discussed in [4, 5, 6, 8, 10]. In [9], the authors have investigated the existence and uniqueness of solutions for Langevin equation with two fractional orders. In [7], the authors study the existence and uniqueness of solutions for a nonlinear fractional integro-differential Langevin-equation with two fractional orders subject to three-point multi-term fractional integral boundary conditions. Motivated by the previous results, in this paper, we study the existence and uniqueness of solutions for the fractional Caputo-Langevin equation with, multi-point boundary-conditions

$$\begin{cases} {}^c D^\alpha ({}^c D^\beta + k)u(t) = f(t, u(t)), & 0 \leq t \leq 1, \\ u(0) = u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i). \end{cases} \quad (1)$$

where $0 < \alpha, \beta < 1$; $1 < \alpha + \beta < 2$, ${}^c D^\alpha$ denotes the Caputo's fractional derivative, $a_i \geq 0$ for $i = 1, 2, \dots, m-2$; $\xi_i : 0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$ with $\sum_{i=1}^{m-2} a_i < 1$ and $k \in (-1, 1)$; $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

The organization of this paper is as follows. In section 2, we introduce some notations and definitions of fractional calculus [1, 2] and present preliminary results needed in our proof later. In section 3 we discuss the existence and uniqueness of solutions for the BVP (1). Examples are given to illustrate our theoretical results.

2. Preliminaries

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$, Γ is the gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt$.

The gamma function satisfies the following basic properties:

i) For any $z \in \mathbb{R}$

$$\Gamma(z+1) = z\Gamma(z);$$

ii) For any $\alpha \in \mathbb{R}$, $\alpha > 1$, then

$$\frac{\alpha + 1}{\Gamma(\alpha + 1)} = \frac{\alpha + 1}{\alpha \Gamma(\alpha)} < \frac{2}{\Gamma(\alpha)}.$$

Definition 2.2. For an at least n -times continuously differentiable function $f : (0, \infty) \rightarrow \mathbb{R}$, The Caputo derivative of fractional order $\alpha > 0$ is defined as

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{f^{(n)}(s) ds}{(t - s)^{\alpha - n + 1}},$$

$n - 1 < \alpha \leq n$, $n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of the real number α .

Remark 2.3. Under natural condition on the function $f(x)$ if $\alpha = n$, the Caputo's derivative of order α is the usual derivative of order n . We have the following properties:

i) $I^\alpha I^\beta f(t) = I^{\alpha + \beta} f(t)$, ${}^c D^\alpha I^\alpha f(t) = f(t)$, for $\alpha, \beta > 0$, $f \in L^1(0, 1)$ and $I^\alpha : C[0, 1] \rightarrow C[0, 1]$, $\alpha > 0$.

ii) $I^\alpha t^\mu = \frac{\Gamma(\mu + 1)}{\Gamma(\alpha + \mu + 1)} t^{\alpha + \mu}$, $\mu > -1$.

Lemma 2.4. The general solution of the equation

$${}^c D^\alpha u(t) = 0,$$

is given by

$$u(t) = c_0 + c_1 t + \dots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n - 1$ and $n = [\alpha] + 1$.

Lemma 2.5. Suppose that $u \in C[0, 1] \cap L^1[0, 1]$ with a derivative of order n that belong to $C[0, 1] \cap L^1[0, 1]$. Then

$$I^\alpha {}^c D^\alpha u(t) = u(t) + c_0 + c_1 t + \dots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n - 1$ and $n = [\alpha] + 1$.

Lemma 2.6. Let $h(t) \in C[0, 1]$ be a given function, and $0 < \alpha, \beta \leq 1$; $1 < \alpha + \beta < 2$. Then linear BVP

$$\begin{cases} {}^c D^\alpha ({}^c D^\beta + k) u(t) = h(t), & 0 \leq t \leq 1, \\ u(0) = u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i). \end{cases} \quad (2)$$

has a unique solution given by

$$\begin{aligned} u(t) = & I^{\alpha+\beta}h(t) - kI^\beta u(t) + t^\beta \left(kI^\beta u(1) - I^{\alpha+\beta}h(s)(1) \right) \\ & + \frac{\sum_{i=1}^{m-2} a_i \left(I^{\alpha+\beta}h(s)(\xi_i) - kI^\beta u(\xi_i) \right)}{1 - \sum_{i=1}^{m-2} a_i} \\ & + \frac{\sum_{i=1}^{m-2} a_i \xi_i^\beta (kI^\beta u(1) - I^{\alpha+\beta}h(s)(1))}{1 - \sum_{i=1}^{m-2} a_i}. \end{aligned}$$

Proof. The general solution of 2 is given by

$$\begin{aligned} u(t) = & I^{\alpha+\beta}h(t) - kI^\beta u(t) + c_0 \frac{t^\beta}{\Gamma(\beta+1)} + c_1 \\ = & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s) ds - \frac{k}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s) ds \\ & + c_0 \frac{t^\beta}{\Gamma(\beta+1)} + c_1. \end{aligned} \tag{3}$$

using the boundary conditions, we have

$$u(0) = c_1,$$

and

$$u(1) = I^{\alpha+\beta}h(1) - kI^\beta u(1) + \frac{c_0}{\Gamma(\beta+1)} + c_1,$$

and

$$u(\xi_i) = I^{\alpha+\beta}h(\xi_i) - kI^\beta u(\xi_i) + c_0 \frac{\xi_i^\beta}{\Gamma(\beta+1)} + c_1,$$

and

$$\sum_{i=1}^{m-2} a_i u(\xi_i) = \sum_{i=1}^{m-2} a_i I^{\alpha+\beta}h(\xi_i) - k \sum_{i=1}^{m-2} a_i I^\beta u(\xi_i) + c_0 \frac{\sum_{i=1}^{m-2} a_i \xi_i^\beta}{\Gamma(\beta+1)} + c_1 \sum_{i=1}^{m-2} a_i.$$

It follows that

$$c_0 = \Gamma(\beta+1) \left(kI^\beta u(1) - I^{\alpha+\beta}h(1) \right),$$

and

$$c_1 = \frac{\sum_{i=1}^{m-2} a_i \left(I^{\alpha+\beta}h(\xi_i) - kI^\beta u(\xi_i) \right)}{1 - \sum_{i=1}^{m-2} a_i} + \frac{\sum_{i=1}^{m-2} a_i \xi_i^\beta \left(kI^\beta u(1) - I^{\alpha+\beta}h(1) \right)}{1 - \sum_{i=1}^{m-2} a_i}.$$

Substituting c_0 and c_1 into (3) we obtain $u(t)$. □

3. THE MAIN RESULTS

Let $E = C([0, 1], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, 1] \rightarrow \mathbb{R}$ endowed with the norm defined by $\|u\| = \sup_{0 \leq t \leq 1} |u(t)|$.

To define a fixed point problem equivalent to (1), we make use of the lemma 2.6 to define an operator $T : E \rightarrow E$ as

$$\begin{aligned} Tu(t) &= I^{\alpha+\beta} f(s, u(s))(t) - kI^\beta u(t) + t^\beta \left(kI^\beta u(1) - I^{\alpha+\beta} f(s, u(s))(1) \right) \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i (I^{\alpha+\beta} f(s, u(s))(\xi_i) - kI^\beta u(\xi_i))}{1 - \sum_{i=1}^{m-2} a_i} \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i \xi_i^\beta (kI^\beta u(1) - I^{\alpha+\beta} f(s, u(s))(1))}{1 - \sum_{i=1}^{m-2} a_i}. \end{aligned}$$

Observe that the problem (1) has solutions if and only if the operator T has fixed points.

For the forthcoming analysis, we set

$$\delta_1 = \frac{2}{\Gamma(\alpha + \beta + 1)} + \frac{\sum_{i=1}^{m-2} a_i (\xi_i^\alpha + 1) \xi_i^\beta}{\Gamma(\alpha + \beta + 1) \left(1 - \sum_{i=1}^{m-2} a_i \right)}$$

and

$$\delta_2 = |k| \left(\frac{2}{\Gamma(\beta + 1)} + \frac{2 \sum_{i=1}^{m-2} a_i \xi_i^\beta}{\Gamma(\beta + 1) \left(1 - \sum_{i=1}^{m-2} a_i \right)} \right).$$

Theorem 3.1. Assume that $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying the assumption

$$(H_1) \quad |f(t, u) - f(t, v)| \leq L|u - v|, \quad \forall t \in [0, 1], \quad x, y \in \mathbb{R}, \quad L > 0.$$

Then the BVP (1) has a unique solution on $[0, 1]$ if $L\delta_1 + \delta_2 < 1$.

Proof. Let us set $\sup_{0 \leq t \leq 1} |f(t, 0)| = C$ and define $B_r = \{u \in E : \|u\| \leq r\}$, where

$$r \geq \frac{C\delta_1}{1 - (L\delta_1 + \delta_2)}.$$

Set

$$\Delta_1 = \frac{\sum_{i=1}^{m-2} a_i (\xi_i^\alpha + 1) \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i \right) \Gamma(\alpha + \beta + 1)},$$

$$\Delta_2 = \frac{2 \sum_{i=1}^{m-2} a_i \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\beta + 1)}.$$

As a first step, we show that $TB_r \subset B_r$. For $u \in B_r$, we have

$$\begin{aligned} |Tu(t)| &\leq I^{\alpha+\beta}|f(s, u(s))|(t) + |k|I^\beta|u(s)|(t) \\ &\quad + |k|t^\beta I^\beta|u(s)|(1) + t^\beta I^{\alpha+\beta}|f(s, u(s))|(1) \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i (I^{\alpha+\beta}|f(s, u(s))|(\xi_i) + |k|I^\beta|u(s)|(\xi_i))}{1 - \sum_{i=1}^{m-2} a_i} \\ &\quad + \frac{|k| \sum_{i=1}^{m-2} a_i \xi_i^\beta (I^\beta|u(s)|(1) + I^{\alpha+\beta}|f(s, u(s))|(1))}{1 - \sum_{i=1}^{m-2} a_i} \\ &\leq I^{\alpha+\beta}(|f(s, u(s)) - f(s, 0)| + |f(s, 0)|)(t) + |k|I^\beta|u(s)|(t) \\ &\quad + t^\beta \left(I^{\alpha+\beta}(|f(s, u(s)) - f(s, 0)| + |f(s, 0)|)(1) \right) \\ &\quad + |k|t^\beta I^\beta|u(s)|(1) + \frac{|k| \sum_{i=1}^{m-2} a_i I^\beta|u(s)|(\xi_i)}{1 - \sum_{i=1}^{m-2} a_i} \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i (I^{\alpha+\beta}(|f(s, u(s)) - f(s, 0)| + |f(s, 0)|)(\xi_i))}{1 - \sum_{i=1}^{m-2} a_i} \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i \xi_i^\beta (I^{\alpha+\beta}(|f(s, u(s)) - f(s, 0)| + |f(s, 0)|)(1))}{1 - \sum_{i=1}^{m-2} a_i} \\ &\quad + \frac{|k| \sum_{i=1}^{m-2} a_i \xi_i^\beta I^\beta|u(s)|(1)}{1 - \sum_{i=1}^{m-2} a_i}. \end{aligned}$$

Thus we get

$$\begin{aligned} |Tu(t)| &\leq (Lr + C) \left(\frac{2}{\Gamma(\alpha + \beta + 1)} + \frac{\sum_{i=1}^{m-2} a_i (\xi_i^\alpha + 1) \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\alpha + \beta + 1)} \right) \\ &\quad + r|k| \left(\frac{2}{\Gamma(\beta + 1)} + \frac{2 \sum_{i=1}^{m-2} a_i \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\beta + 1)} \right) \\ &= (Lr + C)\delta_1 + r\delta_2 \\ &= (Lr + C)S_1 + rS_2 \leq r. \end{aligned}$$

It follows that

$$\|Tu\| \leq r.$$

Thus, $TB_r \subset B_r$. Now for any $u, v \in E$, we obtain

$$\begin{aligned}
|Tu(t) - Tv(t)| &\leq I^{\alpha+\beta}|f(s, u(s)) - f(s, v(s))|(t) + |k|I^\beta|u(s) - v(s)|(t) \\
&\quad + |k|t^\beta I^\beta|u(s) - v(s)|(1) \\
&\quad + t^\beta I^{\alpha+\beta}|f(s, u(s)) - f(s, v(s))|(1) \\
&\quad + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i I^{\alpha+\beta}|f(s, u(s)) - f(s, v(s))|(\xi_i) \\
&\quad + \frac{|k|}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i I^\beta|u(s) - v(s)|(\xi_i) \\
&\quad + \frac{|k| \sum_{i=1}^{m-2} a_i \xi_i^\beta}{1 - \sum_{i=1}^{m-2} a_i} I^\beta|u(s) - v(s)|(1) \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i \xi_i^\beta}{1 - \sum_{i=1}^{m-2} a_i} I^{\alpha+\beta}|f(s, u(s)) - f(s, v(s))|(1) \\
&\leq \|u - v\| \left\{ \frac{L}{\Gamma(\alpha + \beta + 1)} + \frac{|k|}{\Gamma(\beta + 1)} + \frac{|k|}{\Gamma(\beta + 1)} \right. \\
&\quad + \frac{L}{\Gamma(\alpha + \beta + 1)} + \frac{L \sum_{i=1}^{m-2} a_i \xi_i^{\alpha+\beta}}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\alpha + \beta + 1)} \\
&\quad + \frac{|k| \sum_{i=1}^{m-2} a_i \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\beta + 1)} + \frac{|k| \sum_{i=1}^{m-2} a_i \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\beta + 1)} \\
&\quad \left. + \frac{L \sum_{i=1}^{m-2} a_i \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\alpha + \beta + 1)} \right\} \\
&= \|u - v\|(L\delta_1 + \delta_2).
\end{aligned}$$

Since $L\delta_1 + \delta_2 < 1$, therefore, the operator T is a contraction. Hence, by Banach's contraction principle, the problem 1 has a unique solution on $[0, 1]$. $\square$

Theorem 3.2. *Let M be a closed convex and nonempty subset of a Banach space E . Let A, B the operators such that*

- (i) $Au + Bv \in M$ whenever $u, v \in M$;
- (ii) A is compact and continuous;
- (iii) B is a contraction mapping.

Then there exists $w \in M$ such that $w = Aw + Bw$.

Theorem 3.3. Let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying the assumptions (H_1) and

(H_2) $|f(t, u)| \leq \gamma(t)$, $\forall t \in [0, 1]$, $u \in \mathbb{R}$, where $\gamma \in C([0, 1], \mathbb{R}^+)$.

If $L\Delta_1 + |k|\Delta_2 < 1$, then the problem 1 has at least one solution.

Proof. Let $\|\gamma\| = \sup_{0 \leq t \leq 1} \gamma(t)$, and consider $B_R = \{u \in E : \|u\| \leq R\}$, where $R \geq \frac{\|\gamma\|\delta_1}{1 - \delta_2}$, $\delta_2 < 1$. We define the operators T_1 and T_2 on B_R as

$$\begin{aligned} T_1 u(t) &= I^{\alpha+\beta} f(s, u(s))(t) - k I^\beta u(t) + k t^\beta I^\beta u(1) \\ &\quad - t^\beta I^{\alpha+\beta} f(s, u(s))(1), \end{aligned}$$

$$\begin{aligned} T_2 u(t) &= \frac{\sum_{i=1}^{m-2} a_i I^{\alpha+\beta} f(s, u(s))(\xi_i)}{1 - \sum_{i=1}^{m-2} a_i} - \frac{k \sum_{i=1}^{m-2} a_i I^\beta u(\xi_i)}{1 - \sum_{i=1}^{m-2} a_i} \\ &\quad + \frac{k \sum_{i=1}^{m-2} a_i \xi_i^\beta I^\beta u(1)}{1 - \sum_{i=1}^{m-2} a_i} - \frac{\sum_{i=1}^{m-2} a_i \xi_i^\beta I^{\alpha+\beta} f(s, u(s))(1)}{1 - \sum_{i=1}^{m-2} a_i}. \end{aligned}$$

For $u, v \in B_R$ then

$$\begin{aligned} |T_1 u(t) + T_2 u(t)| &\leq \int_0^t \frac{(t-s)^{\alpha+\beta-1} \gamma(s) ds}{\Gamma(\alpha+\beta)} + |k| \int_0^t \frac{(t-s)^{\beta-1} |u(s)| ds}{\Gamma(\beta)} \\ &\quad + |k| t^\beta \int_0^1 \frac{(1-s)^{\beta-1} |u(s)| ds}{\Gamma(\beta)} \\ &\quad + t^\beta \int_0^1 \frac{(1-s)^{\alpha+\beta-1} \gamma(s) ds}{\Gamma(\alpha+\beta)} \\ &\quad + \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha+\beta-1} \gamma(s) ds}{\Gamma(\alpha+\beta)} \\ &\quad + \frac{|k|}{1 - \sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\beta-1} |u(s)| ds}{\Gamma(\beta)} \\ &\quad + \frac{|k| \sum_{i=1}^{m-2} a_i \xi_i^\beta}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 \frac{(1-s)^{\beta-1} |u(s)| ds}{\Gamma(\beta)} \end{aligned}$$

$$+ \frac{\sum_{i=1}^{m-2} a_i \xi_i^\beta}{1 - \sum_{i=1}^{m-2} a_i} \int_0^1 \frac{(1-s)^{\alpha+\beta-1} \gamma(s) ds}{\Gamma(\alpha+\beta)},$$

from which we deduce that

$$\begin{aligned} &\leq \|\gamma\| \frac{2}{\Gamma(\alpha+\beta+1)} + \frac{\sum_{i=1}^{m-2} a_i (\xi_i^\alpha + 1) \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\alpha+\beta+1)} \\ &\quad + R|k| \frac{2}{\Gamma(\beta+1)} + \frac{2 \sum_{i=1}^{m-2} a_i \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\beta+1)} \\ &= \|\gamma\| \delta_1 + R \delta_2 \leq R. \end{aligned}$$

Then

$$\|T_1 u + T_2 v\| \leq R.$$

Thus $T_1 u + T_2 v \in B_R$. We prove that T_2 is a contraction mapping. Let $u, v \in B_R$, we have

$$\begin{aligned} |T_2 u(t) - T_2 v(t)| &\leq \|u - v\| \left\{ L \left(\frac{\sum_{i=1}^{m-2} a_i (\xi_i^\alpha + 1) \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\alpha+\beta+1)} \right) \right. \\ &\quad \left. + |k| \left(\frac{2 \sum_{i=1}^{m-2} a_i \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\beta+1)} \right) \right\} \\ &= \|u - v\| (L \Delta_1 + |k| \Delta_2) \leq (L \Delta_1 + |k| \Delta_2) \|u - v\|. \end{aligned}$$

Then

$$\|T_2 u - T_2 v\| \leq (L \Delta_1 + |k| \Delta_2) \|u - v\|.$$

Which implies that T_2 is a contraction mapping.

We prove the continuity of T_1 . Let $\{u_n\}_{n=0}^\infty \subset M$ and $u \in M$ such that $u_n \rightarrow u$ as $n \rightarrow \infty$, so continuity of f implies that $\lim_{n \rightarrow \infty} f(t, u_n(t)) = f(t, u(t))$. Then

$$\begin{aligned} |T_1 u_n(t) - T_1 u(t)| &\leq \int_0^t \frac{(t-s)^{\alpha+\beta-1} |f(s, u_n(s)) - f(s, u(s))| ds}{\Gamma(\alpha+\beta)} \\ &\quad + |k| \int_0^t \frac{(t-s)^{\beta-1} |u_n(s) - u(s)| ds}{\Gamma(\beta)} \end{aligned}$$

$$\begin{aligned}
& + |k| t^\beta \int_0^1 \frac{(1-s)^{\beta-1} |u_n(s) - u(s)| ds}{\Gamma(\beta)} \\
& + t^\beta \int_0^1 \frac{(1-s)^{\alpha+\beta-1} |f(s, u_n(s)) - f(s, u(s))| ds}{\Gamma(\alpha + \beta)},
\end{aligned}$$

it follows that $|T_1 u_n(t) - T_1 u(t)| \rightarrow 0$ as $n \rightarrow \infty$, which implies the continuity of T_1 . Now we prove the compactness of the operator T_1 . We define

$$C_0 = \sup_{(t,u) \in [0,1] \times B_R} |f(t, u)|.$$

For $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned}
& |T_1 u(t_2) - T_1 u(t_1)| \\
\leq & \frac{C_0}{\Gamma(\alpha + \beta)} \left(\left| \int_{t_1}^{t_2} (t_2 - s)^{\alpha+\beta-1} ds \right| \right. \\
& + \left| \int_0^{t_1} [(t_2 - s)^{\alpha+\beta-1} - (t_1 - s)^{\alpha+\beta-1}] ds \right| \Big) \\
& + \frac{|k|R}{\Gamma(\beta)} \left(\left| \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} ds \right| \right. \\
& + \left| \int_0^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] ds \right| \Big) \\
& + \frac{|k|R (t_2^\beta - t_1^\beta)}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} ds \\
& + \frac{C_0(t_2^\beta - t_1^\beta)}{\Gamma(\alpha + \beta)} \int_0^1 (1-s)^{\alpha+\beta-1} ds \\
= & \frac{C_0}{\Gamma(\alpha + \beta + 1)} \left((t_2 - t_1)^{\alpha+\beta} - (t_2 - t_1)^{\alpha+\beta} + t_2^{\alpha+\beta} - t_1^{\alpha+\beta} \right) \\
& + \frac{|k|R}{\Gamma(\beta + 1)} \left((t_2 - t_1)^\beta - (t_2 - t_1)^\beta + t_2^\beta - t_1^\beta \right) \\
& + \frac{|k|R}{\Gamma(\beta + 1)} (t_2^\beta - t_1^\beta) + \frac{C_0}{\Gamma(\alpha + \beta + 1)} (t_2^\beta - t_1^\beta),
\end{aligned}$$

which is independent of u and tends to zero as $t_2 \rightarrow t_1$. So T_1 is relatively compact on B_R . Hence, by the Arzela-Ascoli theorem, T_1 is compact on B_R . Thus all the assumptions of theorem 3.2 are satisfied. Therefore the problem (1) has at least one solution on $[0,1]$. $\square$

Our next existence result is based on the following fixed point theorem [17].

Theorem 3.4. *Let E be a Banach space. Assume that $T : E \rightarrow E$ is completely continuous operator and the set $\Omega = \{u \in E : u = \mu Tu, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in E .*

Theorem 3.5. *Assume that there exists $C > 0$ such that $|f(t, u)| \leq C, \forall t \in [0, 1], u \in \mathbb{R}$. Then the problem 1 has at least one solution.*

Proof. We show that the operator T is completely continuous. Let B a bounded set in $E = C[0, 1]$. Then there exists $C > 0$ such that $|f(t, u)| \leq C, \forall t \in [0, 1], u \in B$, we get

$$\begin{aligned}
 |Tu(t)| &\leq I^{\alpha+\beta}|f(s, u(s))|(t) + |k|I^\beta|u(s)|(t) \\
 &\quad + |k|t^\beta I^\beta|u(s)|(1) + t^\beta I^{\alpha+\beta}|f(s, u(s))|(1) \\
 &\quad + \frac{\sum_{i=1}^{m-2} a_i (I^{\alpha+\beta}|f(s, u(s))|(\xi_i) + |k|I^\beta|u(s)|\xi_i)}{1 - \sum_{i=1}^{m-2} a_i} \\
 &\quad + \frac{\sum_{i=1}^{m-2} a_i \xi_i^\beta (|k|I^\beta|u(s)|(1) + I^{\alpha+\beta}|f(s, u(s))|(1))}{1 - \sum_{i=1}^{m-2} a_i} \\
 &\leq C \left(\frac{2}{\Gamma(\alpha + \beta + 1)} + \frac{\sum_{i=1}^{m-2} a_i (\xi_i^\alpha + 1) \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\alpha + \beta + 1)} \right) \\
 &\quad + \|u\| |k| \frac{2}{\Gamma(\beta + 1)} + \frac{\sum_{i=1}^{m-2} a_i \xi_i^\beta}{\left(1 - \sum_{i=1}^{m-2} a_i\right) \Gamma(\beta + 1)} \\
 &= C\delta_1 + \|u\|\delta_2,
 \end{aligned}$$

which implies that

$$\|Tu\| \leq C\delta_1 + \|u\|\delta_2 < \infty.$$

Hence $T(B)$ is uniformly bounded.

For any $t_1, t_2 \in [0, 1], t_1 < t_2; u \in B$, we have

$$\begin{aligned}
 &|Tu(t_2) - Tu(t_1)| \\
 = &\left| \int_0^{t_2} \frac{(t_2 - s)^{\alpha+\beta-1} f(s, u(s))}{\Gamma(\alpha + \beta)} ds - \int_0^{t_1} \frac{(t_1 - s)^{\alpha+\beta-1} f(s, u(s))}{\Gamma(\alpha + \beta)} ds \right. \\
 &\left. + k \int_0^{t_2} \frac{(t_2 - s)^{\beta-1} u(s)}{\Gamma(\beta)} ds - k \int_0^{t_1} \frac{(t_1 - s)^{\beta-1} u(s)}{\Gamma(\beta)} ds \right|
 \end{aligned}$$

$$\begin{aligned}
& +k(t_2 - t_1)^\beta \int_0^1 \frac{(1-s)^{\beta-1}u(s)}{\Gamma(\beta)} ds \\
& + (t_2 - t_1)^\beta \int_0^1 \frac{(1-s)^{\alpha+\beta-1}f(s, u(s))}{\Gamma(\alpha + \beta)} ds \Big|
\end{aligned}$$

then

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| & \leq C \left| \frac{1}{\Gamma(\alpha + \beta)} \int_0^{t_2} ((t_2 - s)^{\alpha+\beta-1} - (t_1 - s)^{\alpha+\beta-1}) ds \right. \\
& \quad \left. - \frac{1}{\Gamma(\alpha + \beta)} \int_{t_1}^{t_2} (t_1 - s)^{\alpha+\beta-1} ds \right| \\
& \quad + \|u\| \|k\| \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} ((t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}) ds \right. \\
& \quad \left. - \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_1 - s)^{\beta-1} ds \right| \\
& \quad + \|u\| \|k\| (t_2 - t_1)^\beta \frac{1}{\Gamma(\beta)} \left(\int_0^1 (1-s)^{\beta-1} ds \right) \\
& \quad + C(t_2 - t_1)^\beta \frac{1}{\Gamma(\alpha + \beta)} \int_0^1 (1-s)^{\alpha+\beta-1} ds \\
& \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1.
\end{aligned}$$

Thus the operator T is equicontinuous. Hence by Arzela-Ascoli theorem, $T : E \rightarrow E$ is completely continuous. Now consider the set $\Omega = \{u \in E : u = \mu Tu, \ 0 < \mu < 1\}$. In order to show that Ω is bounded, let $u \in \Omega$, then $u = \mu Tu$, $0 < \mu < 1$, for $t \in [0, 1]$, we have

$$\begin{aligned}
|u(t)| & = |\mu Tu(t)| \\
& \leq \mu \left\{ I^{\alpha+\beta} |f(s, u(s))|(t) + |k| I^\beta |u(s)|(t) \right. \\
& \quad + |k| t^\beta I^\beta |u(s)|(1) + t^\beta I^{\alpha+\beta} |f(s, u(s))|(1) \\
& \quad + \frac{\sum_{i=1}^{m-2} a_i (I^{\alpha+\beta} |f(s, u(s))|(\xi_i) + |k| I^\beta |u(s)|\xi_i)}{1 - \sum_{i=1}^{m-2} a_i} \\
& \quad \left. + \frac{\sum_{i=1}^{m-2} a_i \xi_i^\beta (|k| I^\beta |u(s)|(1) + I^{\alpha+\beta} |f(s, u(s))|(1))}{1 - \sum_{i=1}^{m-2} a_i} \right\} \\
& \leq C\delta_1 + |u|\delta_2 < \infty.
\end{aligned}$$

It follows that the set Ω is bounded independently of μ . From theorem 3.4, the operator T has at least a fixed point, which implies that the BVP 1 has at least one solution on $[0,1]$. $\square$

Example 3.6. Consider the five-point problem

$$\begin{cases} {}^c D^{\frac{3}{4}} \left({}^c D^{\frac{1}{2}} + \frac{1}{7} \right) u(t) = \frac{|u(t)|}{(t^2 + 2)^3 (2 + |u(t)|)} + \cos^2 t, & t \in [0, 1], \\ u(0) = u(1) = \sum_{i=1}^3 a_i u(\xi_i), \end{cases} \quad (4)$$

where $f(t, u(t)) = \frac{|u(t)|}{(t^2 + 2)^3 (2 + |u(t)|)} + \cos^2 t$, and $\alpha = \frac{3}{4}$, $\beta = \frac{1}{2}$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{3}$, $\xi_3 = \frac{1}{5}$; $a_1 = \frac{1}{6}$, $a_2 = \frac{1}{3}$, $a_3 = \frac{1}{5}$, $k = -\frac{1}{13}$, $\Gamma\left(\frac{9}{4}\right) = 1,133$; $\Gamma\left(\frac{3}{2}\right) = 0,886$.
Let

$$\begin{aligned} \sum_{i=1}^{m-2} a_i (\xi_i^\alpha + 1) \xi_i^\beta &= a_1 \xi_1^{\alpha+\beta} + a_1 \xi_1^\beta + a_2 \xi_2^{\alpha+\beta} + a_2 \xi_2^\beta + a_3 \xi_3^{\alpha+\beta} + a_3 \xi_3^\beta \\ &= \frac{1}{6} \left(\frac{1}{4}\right)^{\frac{5}{4}} + \frac{1}{6} \left(\frac{1}{4}\right)^{\frac{1}{2}} + \frac{1}{3} \left(\frac{1}{3}\right)^{\frac{5}{4}} + \frac{1}{3} \left(\frac{1}{3}\right)^{\frac{1}{2}} \\ &\quad + \frac{1}{5} \left(\frac{1}{5}\right)^{\frac{5}{4}} + \frac{1}{5} \left(\frac{1}{5}\right)^{\frac{1}{2}} \\ &\approx 2,601; \end{aligned}$$

and

$$1 - \sum_{i=1}^{m-2} a_i = 1 - \left(\frac{1}{6} + \frac{1}{3} + \frac{1}{5} \right) = \frac{3}{7} = 0.428.$$

We get

$$\begin{aligned} \delta_1 &= \frac{2}{\Gamma(\alpha + \beta + 1)} + \frac{\sum_{i=1}^{m-2} a_i (\xi_i^\alpha + 1) \xi_i^\beta}{\Gamma(\alpha + \beta + 1) \left(1 - \sum_{i=1}^{m-2} a_i \right)} \\ &= \frac{2}{1,133} + \frac{2,601}{1,133 \times 0.428} \end{aligned}$$

$$= 1,765 + 5,337 = 7,102.$$

We have that $|f(t, u(t)) - f(t, v(t))| \leq \left(\frac{1}{16}\right) |u(t) - v(t)|$, and $L = \frac{1}{16}$. Thus we get

$$\begin{aligned} \delta_2 &= |k| \left(\frac{2}{\Gamma(\beta+1)} + \frac{2 \sum_{i=1}^{m-2} a_i \xi_i^\beta}{\Gamma(\beta+1) \left(1 - \sum_{i=1}^{m-2} a_i\right)} \right) \\ &= \frac{1}{13} \left(\frac{2}{0,886} + \frac{\frac{1}{6} \left(\frac{1}{4}\right)^{\frac{1}{2}} + \frac{1}{3} \left(\frac{1}{3}\right)^{\frac{1}{2}} + \frac{1}{5} \left(\frac{1}{5}\right)^{\frac{1}{2}}}{0,886 \times 0.428} \right) \\ &= 0,394. \end{aligned}$$

Finally we have that $\delta_1 \approx 7,102$ and $\delta_2 \approx 0,394$. Thus $L\delta_1 + \delta_2 = \frac{1}{16} \times 7,102 + 0,394 = 0,838 < 1$. Hence by theorem 3.1 the BVP (4) has a unique solution on $[0,1]$.

Example 3.7. Consider the problem

$$\begin{cases} {}^c D^{\frac{3}{4}} \left({}^c D^{\frac{1}{2}} + \frac{1}{7} \right) u(t) = \frac{2e^{-\sin^2 u} e^{-t}}{\cos u + 4}, & t \in [0, 1], \\ u(0) = u(1) = \sum_{i=1}^3 a_i u(\xi_i), \end{cases} \quad (5)$$

where $f(t, u(t)) = \frac{2e^{-\sin^2 u} e^{-t}}{\cos u + 4}$, and where $\alpha = \frac{3}{4}$, $\beta = \frac{1}{2}$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{3}$, $\xi_3 = \frac{1}{5}$; $a_1 = \frac{1}{6}$, $a_2 = \frac{1}{3}$, $a_3 = \frac{1}{5}$, $k = \frac{1}{13}$; $\Gamma(\frac{9}{4}) = 1,133$; $\Gamma(\frac{3}{2}) = 0,886$.

It follows that $|f(t, u(t))| = \left| \frac{5e^{-\sin^2 u(t)} e^{-t}}{\cos u(t) + 4} \right| \leq \frac{5}{3} = C$. Therefore, by theorem 3.5, the BVP (5) has at least one solution on $[0,1]$.

Example 3.8. Consider the following fractional five-point boundary problem

$$\begin{cases} {}^c D^{\frac{4}{5}} \left({}^c D^{\frac{3}{4}} + \frac{1}{7} \right) u(t) = \frac{te^{-t}(|u(t)| + 1)}{t + 2|u(t)| + 1} e^{2t}, & t \in [0, 1], \\ u(0) = u(1) = \sum_{i=1}^3 a_i u(\xi_i). \end{cases} \quad (6)$$

where $\alpha = \frac{4}{5}$, $\beta = \frac{3}{4}$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{3}$, $\xi_3 = \frac{1}{5}$; $a_1 = \frac{1}{6}$, $a_2 = \frac{1}{3}$, $a_3 = \frac{1}{5}$, $k = \frac{1}{11}$; $\Gamma(1, 75) = 0, 919$; $\Gamma(2, 55) = 1, 337$.

It follows that $f(t, u(t)) = \frac{te^{-t}(|u(t)| + 1)}{t + 2|u(t)| + 1}e^{2t}$, and $|f(t, u(t))| \leq e^{2t} = \gamma(t)$.

We get $\delta_2 = |k| \frac{2}{\Gamma(\beta + 1)} + \frac{2 \sum_{i=1}^{m-2} a_i \xi_i^\beta}{\Gamma(\beta + 1) \left(1 - \sum_{i=1}^{m-2} a_i\right)} \approx 0, 199 < 1$. Therefore,

by theorem 3.3 the BVP (6) has at least one solution on $[0, 1]$.

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