SOME CYCLIC CODES OF LENGTH $8p^n$

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Abstract: Explicit expressions for primitive idempotents in the semi-simple group algebra FG of cyclic group G of order $8p^n$ over finite field F of order q , where q is primitive root modulo p^n , are obtained. Generating polynomials, minimum distances and dimensions of the corresponding minimal cyclic codes are also obtained.

AMS Subject Classification: 11T71, 11T55, 22D20

Key Words: group algebra, cyclotomic cosets, primitive idempotents, generating polynomials

1. Introduction

Let F be a finite field of prime power order q and G be a cyclic group of order m such that $\gcd(m, q) = 1$. Then FG , the group algebra of the cyclic group G over F , is semi-simple and has only a finite number of primitive idempotents which equals the number of cyclotomic cosets modulo m . Let t be the multiplicative order of q modulo m , then $1 \leq t \leq \phi(m)$ [4]. If $t = \phi(m)$ and $m = 2, 4, p^n, 2p^n$, the complete sets of primitive idempotents were calculated by Pruthi and Arora [1, 6]. The minimal quadratic residue cyclic codes of length p^n

Received: November 3, 2016

Revised: May 2, 2017

Published: September 13, 2017

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were obtained by Batra and Arora [3]. The minimal cyclic codes of length $p^n q$ were discussed by Bakshi and Raka [2]. The minimal cyclic codes of length $8p^n$ over field of prime power order q , where q is of the form $8k + 5$, were obtained by Singh and Arora [7].

It is not easy to find the various parameters of the codes using the form of idempotent generators of the codes available in the literature. Therefore, by taking special cases, we find the explicit expressions of primitive idempotents of the minimal cyclic codes, which help us in finding the various parameters of these codes. It is well known that each cyclotomic coset is associated with an irreducible polynomial in the semi-simple ring $R_{8p^n} = \frac{F[x]}{\langle x^{8p^n} - 1 \rangle} \cong FG$; in turn each of these irreducible polynomial $p(x)$ is associated with a minimal ideal in R_{8p^n} generated by $\frac{x^{8p^n} - 1}{p(x)}$. Thus, the number of minimal ideals, which is also the number of primitive idempotents, is the number of q -cyclotomic cosets modulo m .

In this paper we consider the finite field F of prime power order q , where q is of the form $8k + 7$. If q is primitive root modulo p^n , then the semi-simple group algebra FG of the cyclic group G of order $8p^n$ over F , where p is an odd prime and $n \geq 1$, has $8k + 5$ primitive idempotents. The q -cyclotomic cosets modulo $8p^n$ are obtained in Section 2 and primitive idempotents in Section 3 (Theorem 3.16). In Section 4, we discuss generating polynomials and dimensions for the corresponding minimal cyclic codes of length $8p^n$. The minimum distances or the bounds for these codes are obtained in Section 5. An example to illustrate the parameters of minimal cyclic codes of length 40 for $q = 7$ is discussed in Section 6.

2. Cyclotomic Cosets

Let $S = \{1, 2, \dots, 8p^n\}$. For $a, b \in S$, say that $a \sim b$ iff $a \equiv bq^i \pmod{8p^n}$ for some integer $i \geq 0$. This is an equivalence relation on set S . The equivalence classes due to this relation are called q -cyclotomic cosets modulo $8p^n$. The q -cyclotomic coset containing $s \in S$ is $\Omega_s = \{s, sq, sq^2, \dots, sq^{t_s-1}\}$, where t_s is the smallest positive integer such that $sq^{t_s} \equiv s \pmod{8p^n}$.

Throughout in this paper, $|\Omega_s|$ denotes the order of the q -cyclotomic coset Ω_s containing s .

Lemma 2.1. *If $\phi(p^n)$ is the order of q modulo p^n then order of q modulo p^{n-i} is $\phi(p^{n-i})$ for all i , $0 \leq i \leq n - 1$.*

Proof. Proof is trivial. □

Lemma 2.2. *If q is an odd prime of the form $8k + 7$ and $\phi(p^n)$ is the order of q modulo p^n , then for $0 \leq i \leq n-1$, the order of q modulo $2p^{n-i}$, $4p^{n-i}$, $8p^{n-i}$ is $\phi(p^{n-i})$.*

Proof. Since $\phi(p^n)$ is the order of q modulo p^n therefore by lemma (2.1) order of q modulo p^{n-i} is $\phi(p^{n-i})$, $1 \leq i \leq n-1$. Hence

$$q^{\phi(p^{n-i})} \equiv 1 \pmod{p^{n-i}} \quad (2.1)$$

Since $\phi(p^{n-i})$ is even, therefore, $\phi(p^{n-i}) = 2t$ for some integer t and $q = 8k + 7$, for some integer k , therefore $q \equiv 1 \pmod{2}$. Hence $q^{\phi(p^{n-i})} \equiv 1 \pmod{2}$.

Also $\text{g.c.d.}(2, p^{n-i}) = 1$, therefore $q^{\phi(p^{n-i})} \equiv 1 \pmod{2p^{n-i}}$.

As order of q modulo p^{n-i} is $\phi(p^{n-i})$, therefore $\phi(p^{n-i})$ is the smallest integer for which equation (2.1) holds.

Hence the order of q modulo $2p^{n-i}$ is $\phi(p^{n-i})$. Similarly, the result holds for $4p^{n-i}$ and $8p^{n-i}$. $\square$

Lemma 2.3. *For $0 \leq i \leq n-1$, and $0 \leq k \leq \phi(p^{n-i}) - 1$:*

- (i) $1 + 2p^n \not\equiv q^k \pmod{8p^{n-i}}$. Equivalently, $p^i(1 + 2p^n) \not\equiv p^i q^k \pmod{8p^n}$.
- (ii) $2(1 + 2p^n) \not\equiv q^k \pmod{8p^{n-i}}$. Equivalently, $2p^i(1 + 2p^n) \not\equiv 2p^i q^k \pmod{8p^n}$.
- (iii) $1 + 4p^n \not\equiv q^k \pmod{8p^{n-i}}$. Equivalently, $p^i(1 + 4p^n) \not\equiv p^i q^k \pmod{8p^n}$.
- (iv) $1 + 6p^n \not\equiv q^k \pmod{8p^{n-i}}$. Equivalently, $p^i(1 + 6p^n) \not\equiv p^i q^k \pmod{8p^n}$.

Proof. Proof can be obtained using Lemmas 2.1-2.2. $\square$

Theorem 2.4. *The q -cyclotomic cosets modulo $8p^n$ are $\Omega_{ap^n} = \{ap^n\}$, for $a = 0, 4$;*

$\Omega_{bp^n} = \{bp^n, bp^n q\}$, for $b = 1, 2, \nu$; and

$$\Omega_{tp^j} = \{tp^j, tp^j q, tp^j q^2, \dots, tp^j q^{\phi(p^{n-j})-1}\}.$$

For $0 \leq j \leq n-1$, $t = 1, 2, 4, 8$, $\lambda = 1 + 2p^n$, $\mu = 2(1 + 2p^n)$, $\nu = 1 + 4p^n$, $\chi = 1 + 6p^n$.

Proof. $\Omega_0 = \{0\}$ is trivial.

Since q is odd prime of the form $8k + 7$, so $q^2 \equiv 1 \pmod{8}$, therefore, $p^n q^2 \equiv p^n \pmod{8p^n}$.

Hence $\Omega_{p^n} = \{p^n, p^n q\}$, $\Omega_{2p^n} = \{2p^n, 2p^n q\}$, $\Omega_{\nu p^n} = \{\nu p^n, \nu p^n q\}$.

Since $q \equiv 1 \pmod{2}$, therefore $4p^n q \equiv 4p^n \pmod{8p^n}$ and hence $\Omega_{4p^n} = \{4p^n\}$.

By Lemma 2.2, $q^{\phi(p^{n-i})} \equiv 1 \pmod{8p^{n-i}}$, equivalently

$$p^i q^{\phi(p^{n-i})} \equiv p^i \pmod{8p^n}.$$

Therefore

$$\Omega_{p^i} = \{p^i, p^i q, p^i q^2, \dots, p^i q^{\phi(p^{n-i})-1}\}.$$

Similarly

$$\Omega_{tp^i} = \{tp^i, tp^i q, tp^i q^2, \dots, tp^i q^{\phi(p^{n-i})-1}\},$$

for $t = 2, 4, 8, \lambda, \mu, \nu, \chi$.

Obviously, $|\Omega_0| = 1$, $|\Omega_{p^n}| = |\Omega_{2p^n}| = |\Omega_{\nu p^n}| = 2$, $|\Omega_{4p^n}| = 1$, $|\Omega_{p^i}| = |\Omega_{2p^i}| = |\Omega_{4p^i}| = |\Omega_{8p^i}| = |\Omega_{\lambda p^i}| = |\Omega_{\mu p^i}| = |\Omega_{\nu p^i}| = |\Omega_{\chi p^i}| = \phi(p^{n-i})$.

Therefore

$$\sum_{i=0}^{n-1} |\Omega_{p^i}| = \sum_{i=0}^{n-1} \phi(p^{n-i}) = \phi(p^n) + \phi(p^{n-1}) + \phi(p^{n-2}) + \dots + \phi(p) = p^n - 1.$$

Hence

$$|\Omega_0| + |\Omega_{p^n}| + |\Omega_{2p^n}| + |\Omega_{4p^n}| + |\Omega_{\nu p^n}| + \sum_{i=0}^{n-1} \left\{ \sum_{t=1,2,4,8,\lambda,\mu,\nu,\chi} |\Omega_{tp^i}| \right\} = 8p^n. \quad \square$$

3. Primitive Idempotents

Throughout this paper, we assume that α is a $8p^n$ th root of unity in some extension field of F . Let M_s be the minimal ideal in $R_{8p^n} = \frac{F[x]}{\langle x^{8p^n} - 1 \rangle} \cong FG$;

generated by $\frac{(x^{8p^n} - 1)}{m_s x}$, where $m_s x$ denotes the minimal polynomial for α^s . We denote $\theta_s(x)$, the primitive idempotent in R_{8p^n} , corresponding to the minimal

ideal M_s , given by $\theta_s(x) = \frac{1}{8p^n} \sum_{i=0}^{8p^n-1} \varepsilon_i^s x^i$ where $\varepsilon_i^s = \sum_{j \in \Omega_s} \alpha^{-ij}$ [2]. Also, denote

$$\overline{C}_s = \sum_{s \in \Omega_s} x^s.$$

Lemma 3.1. *For any odd prime p and a positive integer k , if δ is a primitive p^k th root of unity and ζ is a primitive $2p^k$ th root of unity in some extension field of F . Let q be a primitive root modulo p^k , then*

$$\sum_{s=0}^{\phi(p^k)-1} \delta^{q^s} = \begin{cases} -1, & k = 1, \\ 0, & k \geq 2, \end{cases}$$

and

$$\sum_{s=0}^{\phi(2p^k)-1} \zeta^{q^s} = \begin{cases} 1, & k = 1, \\ 0, & k \geq 2, \end{cases}$$

see [7].

Lemma 3.2. *If $p \equiv 1 \pmod{4}$ then $(1 + 6p^n)q^{\frac{\phi(p^n)}{2}} \equiv -1 \pmod{8p^n}$ and if $p \equiv 3 \pmod{4}$ then $q^{\frac{\phi(p^n)}{2}} \equiv -1 \pmod{8p^n}$.*

Proof. Since p is an odd prime, so $\frac{\phi(p^n)}{2}$ is odd iff $p \equiv 3 \pmod{4}$. Also, $q^{\frac{\phi(p^n)}{2}} \equiv -1 \pmod{p^n}$.

Case I. $p \equiv 1 \pmod{4}$.

Since $(1 + 6p^n) \equiv 1 \pmod{p^n}$. Therefore, $(1 + 6p^n)q^{\frac{\phi(p^n)}{2}} \equiv -1 \pmod{p^n}$. Further, $p \equiv 1 \pmod{4}$, so $\frac{\phi(p^n)}{2}$ is even. Also, q is of the form $8k + 7$, therefore, $q^{\frac{\phi(p^n)}{2}}$ is of the form $8k_1 + 1$, so $(1 + 6p^n)q^{\frac{\phi(p^n)}{2}} \equiv -1 \pmod{8}$.

But $\gcd(8, p^n) = 1$, thus $(1 + 6p^n)q^{\frac{\phi(p^n)}{2}} \equiv -1 \pmod{8p^n}$.

Case II. $p \equiv 3 \pmod{4}$.

In this case, $\frac{\phi(p^n)}{2}$ is odd and so $q^{\frac{\phi(p^n)}{2}}$ is of the form $8k_1 + 7$. Thus, $q^{\frac{\phi(p^n)}{2}} \equiv -1 \pmod{8}$. But $\gcd(8, p^n) = 1$, so we have $q^{\frac{\phi(p^n)}{2}} \equiv -1 \pmod{8p^n}$. $\square$

Remark. By Lemma 3.3, we obtain the following, If $p \equiv 1 \pmod{4}$ then $-\Omega_1 = \Omega_\chi$ and if $p \equiv 3 \pmod{4}$ then $-\Omega_1 = \Omega_1$.

Lemma 3.3. *For cyclotomic cosets Ω_{p^i} , $0 \leq i \leq n - 1$.*

$$(i) \quad \lambda^2 \Omega_{p^i} = \Omega_{p^i} = \lambda \Omega_{\lambda p^i}.$$

$$(ii) \quad \nu^2 \Omega_{p^i} = \Omega_{p^i} = \nu \Omega_{\nu p^i}.$$

$$(iii) \quad \chi^2 \Omega_{p^i} = \Omega_{p^i} = \chi \Omega_{\chi p^i}.$$

$$(iv) \quad \mu^2 \Omega_{p^i} = 4 \Omega_{p^i} = \mu \Omega_{\mu p^i} = 2 \Omega_{2p^i} = \Omega_{4p^i}.$$

Proof. Proof of these can be obtained by using the fact that $\lambda^2 \equiv \mu^2 \equiv \nu^2 \equiv \chi^2 \equiv 1 \pmod{8p^n}$. $\square$

Lemma 3.4. *For cyclotomic cosets $\Omega_{p^n}, \Omega_{2p^n}, \Omega_{4p^n}$ and $\Omega_{\nu p^n}$:*

$$(i) \quad -\Omega_{p^n} = \Omega_{p^n}$$

$$(ii) \quad -\Omega_{2p^n} = \Omega_{2p^n}$$

$$(iii) \quad -\Omega_{4p^n} = \Omega_{4p^n}$$

$$(iv) -\Omega_{\nu p^n} = \Omega_{\nu p^n}$$

Proof. (i) We know that $\Omega_{p^n} = \{p^n, p^n q\}$. Since $q \equiv -1 \pmod{8}$. Therefore $p^n q \equiv -p^n \pmod{8p^n}$, or $-\Omega_{p^n} = \Omega_{p^n}$.

The proofs of (ii) – (iv) are similar. $\square$

Lemma 3.5. *If $r = 1, 2, 4, 8, \lambda, \mu, \nu$ or χ then for $0 \leq j \leq n-1$*

$$r\Omega_{8p^j} = \Omega_{8p^j} = 8\Omega_{\lambda p^j} = 8\Omega_{\mu p^j} = 4\Omega_{\nu p^j} = 8\Omega_{\chi p^j} = 8\Omega_{\nu p^j}.$$

Proof. Since $8rp^j \not\equiv lp^jq^k \pmod{8p^n}$ for $l = 1, 2, 4, 8, \lambda, \mu, \nu, \chi$. So, $8rp^j \equiv 8p^jq^k \pmod{8p^n}$.

Hence the required result holds. $\square$

Lemma 3.6. *For cyclotomic cosets $\Omega_{2p^j}, \Omega_{4p^j}, \Omega_{8p^j}, \Omega_{\lambda p^j}, \Omega_{\mu p^j}, \Omega_{\nu p^j}$ and $\Omega_{\chi p^j}$:*

(i) *If $p \equiv 1 \pmod{4}$, then $-\Omega_{2p^j} = \Omega_{\mu p^j}$.*

(ii) *$-\Omega_{4p^j} = \Omega_{4p^j}$ and $-\Omega_{8p^j} = \Omega_{8p^j}$.*

(iii) *If $p \equiv 1 \pmod{4}$, then $-\Omega_{\lambda p^j} = \Omega_{\nu p^j}$.*

(iv) *If $p \equiv 3 \pmod{4}$, then $-\Omega_{2p^j} = \Omega_{2p^j}$, $-\Omega_{\lambda p^j} = \Omega_{\lambda p^j}$, $-\Omega_{\mu p^j} = \Omega_{\mu p^j}$, $-\Omega_{\nu p^j} = \Omega_{\nu p^j}$, $-\Omega_{\chi p^j} = \Omega_{\chi p^j}$.*

Proof. Proof of these can be obtained by using lemma 3.3. $\square$

Notation. Let α be a fixed primitive $8p^n$ th root of unity in some extension field of F . For $0 \leq j \leq n-1$, we define:

$$\begin{aligned} T_j &= p^j \sum_{s \in \Omega_{p^j}} \alpha^s, & S_j &= p^j \sum_{s \in \Omega_{\nu p^j}} \alpha^s, \\ Q_j &= p^j \sum_{s \in \Omega_{\lambda p^j}} \alpha^s, & P_j &= p^j \sum_{s \in \Omega_{\chi p^j}} \alpha^s, \\ R_j &= p^j \sum_{s \in \Omega_{2p^j}} \alpha^s. \end{aligned}$$

Since $(T_j)^q = T_j$. Thus $T_j \in F$. Similarly, S_j, Q_j, P_j and $R_j \in F$.

Lemma 3.7. *For $0 \leq j \leq n-1$, $T_j + S_j = 0$ and $P_j + Q_j = 0$.*

Proof. Since

$$S_j = p^j \sum_{s \in \Omega_{\nu p^j}} \alpha^s = p^j (\alpha^{\nu p^j} + \alpha^{\nu p^j q} + \dots + \alpha^{\nu p^j q^{\phi(p^{n-j})-1}})$$

and

$$\alpha^{\nu p^j} = \alpha^{(1+4p^n)p^j} = \alpha^{p^j} \alpha^{4p^n p^j} = -\alpha^{p^j}$$

then, we have

$$S_j = -p^j (\alpha^{p^j} + \alpha^{p^j q} + \dots + \alpha^{p^j q^{\phi(p^{n-j})-1}}) = -p^j \sum_{s \in \Omega_{p^j}} \alpha^s = -T_j,$$

or, $T_j + S_j = 0$. Similarly, $P_j + Q_j = 0$. □

Lemma 3.8. For $0 \leq i, j \leq n-1$, we have

$$\begin{aligned} \sum_{s \in \Omega_{p^j}} \alpha^{p^i s} &= \sum_{s \in \Omega_{\nu p^j}} \alpha^{\nu p^i s} = \sum_{s \in \Omega_{\lambda p^j}} \alpha^{\lambda p^i s} \\ &= \sum_{s \in \Omega_{\chi p^j}} \alpha^{\chi p^i s} = - \sum_{s \in \Omega_{\lambda p^j}} \alpha^{\chi p^i s} = - \sum_{s \in \Omega_{p^j}} \alpha^{\nu p^i s} \\ &= - \sum_{s \in \Omega_{\nu p^j}} \alpha^{p^i s} = \begin{cases} \frac{\phi(p^{n-j})}{2} (\alpha^{p^{i+j}} + \alpha^{7p^{i+j}}), & \text{if } i+j \geq n, \\ \frac{1}{p^j} T_{i+j}, & \text{if } i+j \leq n-1. \end{cases} \end{aligned}$$

Proof. Let $\delta = \alpha^{p^{i+j}}$. Then

$$\sum_{s \in \Omega_{p^j}} \alpha^{p^i s} = \sum_{t=0}^{\phi(p^{n-j})-1} \alpha^{p^{i+j} q^t} = \sum_{t=0}^{\phi(p^{n-j})-1} \delta^{q^t}.$$

Consider the following cases:

Case I. If $i+j \geq n$, then $\delta = \alpha^{p^n}$ is 8th root of unity. and $\delta^{q^l} \equiv \delta^{q^r}$ iff $q^l \equiv q^r \pmod{8}$ iff $l \equiv r \pmod{2}$.

$$\text{Hence } \sum_{s \in \Omega_{p^j}} \alpha^{p^i s} = \sum_{t=0}^{\phi(p^{n-j})-1} \delta^{q^t} = \frac{\phi(p^{n-j})}{2} (\alpha^{p^{i+j}} + \alpha^{7p^{i+j}}).$$

Case II. If $i+j \leq n-1$, then δ is $8p^{n-i-j}$ th root of unity and $\delta^{q^l} \equiv \delta^{q^r}$ iff $q^l \equiv q^r \pmod{8p^{n-i-j}}$ iff $l \equiv r \pmod{\phi(p^{n-i-j})}$.

Therefore,

$$\begin{aligned} \sum_{s \in \Omega_{pj}} \alpha^{p^i s} &= \frac{\phi(p^{n-j})}{\phi(p^{n-i-j})} \sum_{t=0}^{\phi(p^{n-i-j})-1} \delta^{q^t} = \frac{1}{p^j} p^{i+j} \sum_{t=0}^{\phi(p^{n-i-j})-1} \alpha^{p^{i+j} q^t} \\ &= \frac{1}{p^j} p^{i+j} \sum_{s \in \Omega_{p^{i+j}}} \alpha^s. \end{aligned}$$

Since $p^{i+j} \sum_{s \in \Omega_{p^{i+j}}} \alpha^s = T_{i+j}$, then $\sum_{s \in \Omega_{pj}} \alpha^s = \frac{1}{p^j} T_{i+j}$. $\square$

Lemma 3.9. For $0 \leq i, j \leq n$,

$$\begin{aligned} \sum_{s \in \Omega_{\lambda pj}} \alpha^{p^i s} &= \sum_{s \in \Omega_{pj}} \alpha^{\lambda p^i s} = - \sum_{s \in \Omega_{\lambda pj}} \alpha^{\nu p^i s} \\ &= \sum_{s \in \Omega_{\nu pj}} \alpha^{\chi p^i s} = - \sum_{s \in \Omega_{\nu pj}} \alpha^{\lambda p^i s} = \sum_{s \in \Omega_{\chi pj}} \alpha^{\nu p^i s} = - \sum_{s \in \Omega_{pj}} \alpha^{\chi p^i s} \\ &= \begin{cases} \frac{\phi(p^{n-j})}{2} (\alpha^{p^{i+j}} - \alpha^{7p^{i+j}}) \beta^{p^{i+j}}, & \text{if } i+j \geq n, \\ -\frac{1}{p^j} P_{i+j}, & \text{if } i+j \leq n-1. \end{cases} \end{aligned}$$

Proof. Proof is similar to that of lemma 3.11. $\square$

Lemma 3.10. For $0 \leq i \leq n; 0 \leq j \leq n-1$

$$\begin{aligned} \sum_{s \in \Omega_{pj}} \alpha^{4p^i s} &= \sum_{s \in \Omega_{2pj}} \alpha^{2p^i s} = \sum_{s \in \Omega_{\lambda pj}} \alpha^{4p^i s} = \sum_{s \in \Omega_{4pj}} \alpha^{p^i s} \\ &= \sum_{s \in \Omega_{4pj}} \alpha^{\lambda p^i s} = \sum_{s \in \Omega_{\nu pj}} \alpha^{4p^i s} = \sum_{s \in \Omega_{4pj}} \alpha^{\nu p^i s} \\ &= \sum_{s \in \Omega_{\chi pj}} \alpha^{4p^i s} = \sum_{s \in \Omega_{4pj}} \alpha^{\chi p^i s} \\ &= \sum_{s \in \Omega_{\mu pj}} \alpha^{2p^i s} = \sum_{s \in \Omega_{2pj}} \alpha^{\mu p^i s} = \sum_{s \in \Omega_{\mu pj}} \alpha^{\mu p^i s} \\ &= \begin{cases} -\phi(p^{n-j}), & \text{if } i+j \geq n, \\ p^{n-j-1}, & \text{if } i+j = n-1, \\ 0, & \text{if } i+j < n-1. \end{cases} \end{aligned}$$

Proof. Proof can be obtained by using lemma 3.2 and similar procedure as of lemma 3.11. $\square$

Lemma 3.11. For $0 \leq i \leq n; 0 \leq j \leq n-1$

$$\begin{aligned}
\sum_{s \in \Omega_{p^j}} \alpha^{8p^i s} &= \sum_{s \in \Omega_{4p^j}} \alpha^{2p^i s} = \sum_{s \in \Omega_{2p^j}} \alpha^{4p^i s} = \sum_{s \in \Omega_{\lambda p^j}} \alpha^{8p^i s} = \sum_{s \in \Omega_{8p^j}} \alpha^{\lambda p^i s} = \sum_{s \in \Omega_{\nu p^j}} \alpha^{8p^i s} \\
&= \sum_{s \in \Omega_{8p^j}} \alpha^{\nu p^i s} = \sum_{s \in \Omega_{\chi p^j}} \alpha^{8p^i s} = \sum_{s \in \Omega_{8p^j}} \alpha^{\chi p^i s} = \sum_{s \in \Omega_{\mu p^j}} \alpha^{4p^i s} \\
&= \sum_{s \in \Omega_{4p^j}} \alpha^{\mu p^i s} = \sum_{s \in \Omega_{\mu p^j}} \alpha^{8p^i s} = \sum_{s \in \Omega_{8p^j}} \alpha^{\mu p^i s} \\
&= \sum_{s \in \Omega_{p^j}} \alpha^{16p^i s} = \sum_{s \in \Omega_{\mu p^j}} \alpha^{16p^i s} = \sum_{s \in \Omega_{p^j}} \alpha^{32p^i s} = \sum_{s \in \Omega_{p^j}} \alpha^{64p^i s} \\
&= \begin{cases} \phi(p^{n-j}), & \text{if } i+j \geq n, j=n, \\ -p^{n-j-1}, & \text{if } i+j = n-1, \\ 0, & \text{if } i+j < n-1. \end{cases}
\end{aligned}$$

Proof. Proof can be obtained by using lemma 3.2 and similar procedure as of lemma 3.11. $\square$

Lemma 3.12. For $0 \leq i, j \leq n-1$,

$$\begin{aligned}
\sum_{s \in \Omega_{p^j}} \alpha^{2p^i s} &= \sum_{s \in \Omega_{2p^j}, j \neq n} \alpha^{p^i s} = - \sum_{s \in \Omega_{\mu p^j}, j \neq n} \alpha^{p^i s} = - \sum_{s \in \Omega_{\lambda p^j}, j \neq n} \alpha^{2p^i s} \\
&= \sum_{s \in \Omega_{\nu p^j}, j \neq n} \alpha^{2p^i s} = - \sum_{s \in \Omega_{\nu p^j}, j \neq n} \alpha^{\mu p^i s} = \sum_{s \in \Omega_{\chi p^j}, j \neq n} \alpha^{\mu p^i s} \\
&= - \sum_{s \in \Omega_{\chi p^j}, j \neq n} \alpha^{2p^i s} = \sum_{s \in \Omega_{\lambda p^j}, j \neq n} \alpha^{\mu p^i s} = \sum_{s \in \Omega_{\mu p^j}, j \neq n} \alpha^{\lambda p^i s} \\
&= \begin{cases} 0, & \text{if } i+j \geq n, \\ \frac{1}{p^j} R_{i+j}, & \text{if } i+j \leq n-1. \end{cases}
\end{aligned}$$

Proof. Proof can be obtained by same procedure as of lemma 3.11. and using the fact that if β is $4th$ root of unity in some extension field of F , then $\beta^q + \beta = 0$. $\square$

Theorem 3.13. The explicit expressions for the $8n+5$ primitive idempotents in R_{8p^n} are given by

$$\theta_0(x) = \frac{1}{8p^n} [\overline{C}_0 + \overline{C}_{p^n} + \overline{C}_{2p^n} + \overline{C}_{4p^n} + \overline{C}_{\nu p^n} + \sum_{i=0}^{n-1} \{\overline{C}_{p^i}$$

$$+ \overline{C}_{2p^i} + \overline{C}_{4p^i} + \overline{C}_{8p^i} + \overline{C}_{\lambda p^i} + \overline{C}_{\mu p^i} + \overline{C}_{\nu p^i} + \overline{C}_{\chi p^i} \}],$$

$$\begin{aligned} \theta_{p^n}(x) = & \frac{1}{8p^n} [2\overline{C}_0 + (\alpha^{p^{2n}} + \alpha^{7p^{2n}})\overline{C}_{p^n} - 2\overline{C}_{4p^n} - (\alpha^{p^{2n}} + \alpha^{7p^{2n}})\overline{C}_{\nu p^n} \\ & + \sum_{i=0}^{n-1} \{(\alpha^{p^{n+i}} + \alpha^{7p^{n+i}})\overline{C}_{p^i} - 2\overline{C}_{4p^i} + 2\overline{C}_{8p^i} + (\alpha^{p^{n+i}} - \alpha^{7p^{n+i}})\beta^{p^{n+i}}\overline{C}_{\lambda p^i} \\ & - (\alpha^{p^{n+i}} + \alpha^{7p^{n+i}})\overline{C}_{\nu p^i} - (\alpha^{p^{n+i}} - \alpha^{7p^{n+i}})\beta^{p^{n+i}}\overline{C}_{\chi p^i}\}], \end{aligned}$$

$$\theta_{2p^n}(x) = \frac{1}{4p^n} [\overline{C}_0 - \overline{C}_{2p^n} + \overline{C}_{4p^n} - \sum_{i=0}^{n-1} \{2\overline{C}_{2p^i} - \overline{C}_{4p^i} - \overline{C}_{8p^i} + \overline{C}_{\mu p^i}\}],$$

$$\begin{aligned} \theta_{4p^n}(x) = & \frac{1}{8p^n} [\overline{C}_0 - \overline{C}_{p^n} + \overline{C}_{2p^n} + \overline{C}_{4p^n} - \overline{C}_{\nu p^n} - \sum_{i=0}^{n-1} \{\overline{C}_{p^i} \\ & - \overline{C}_{2p^i} - \overline{C}_{4p^i} - \overline{C}_{8p^i} + \overline{C}_{\lambda p^i} - \overline{C}_{\mu p^i} + \overline{C}_{\nu p^i} + \overline{C}_{\chi p^i}\}], \end{aligned}$$

$$\begin{aligned} \theta_{\nu p^n}(x) = & \frac{1}{8p^n} [2\overline{C}_0 - (\alpha^{p^{2n}} + \alpha^{7p^{2n}})\overline{C}_{p^n} - 2\overline{C}_{4p^n} + (\alpha^{p^{2n}} + \alpha^{7p^{2n}})\overline{C}_{\nu p^n} \\ & - \sum_{i=0}^{n-1} \{(\alpha^{p^{n+i}} + \alpha^{7p^{n+i}})\overline{C}_{p^i} + 2\overline{C}_{4p^i} - 2\overline{C}_{8p^i} + (\alpha^{p^{n+i}} \\ & - \alpha^{7p^{n+i}})\beta^{p^{n+i}}\overline{C}_{\lambda p^i} - (\alpha^{p^{n+i}} + \alpha^{7p^{n+i}})\overline{C}_{\nu p^i} - (\alpha^{p^{n+i}} - \alpha^{7p^{n+i}})\beta^{p^{n+i}}\overline{C}_{\chi p^i}\}]. \end{aligned}$$

For $0 \leq j \leq n-1$:

$$\begin{aligned} \theta_{4p^j}(x) = & \frac{1}{8p^n} [\phi(p^{n-j})\{\overline{C}_0 - \overline{C}_{p^n} + \overline{C}_{2p^n} + \overline{C}_{4p^n} - \overline{C}_{\nu p^n}\} + p^{n-j-1} \\ & \{\overline{C}_{p^{n-j-1}} - \overline{C}_{2p^{n-j-1}} - \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} + \overline{C}_{\lambda p^{n-j-1}} - \overline{C}_{\mu p^{n-j-1}} + \overline{C}_{\nu p^{n-j-1}} \\ & + \overline{C}_{\chi p^{n-j-1}}\} - \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \{\overline{C}_{p^i} - \overline{C}_{2p^i} - \overline{C}_{4p^i} - \overline{C}_{8p^i} + \overline{C}_{\lambda p^i} - \overline{C}_{\mu p^i} + \overline{C}_{\nu p^i} + \overline{C}_{\chi p^i}\}], \end{aligned}$$

$$\begin{aligned} \theta_{8p^j}(x) = & \frac{1}{8p^n} [\phi(p^{n-j})\{\overline{C}_0 + \overline{C}_{p^n} + \overline{C}_{2p^n} + \overline{C}_{4p^n} + \overline{C}_{\nu p^n}\} \\ & - p^{n-j-1}\{\overline{C}_{p^{n-j-1}} + \overline{C}_{2p^{n-j-1}} - \overline{C}_{4p^{n-j-1}} + \overline{C}_{8p^{n-j-1}} + \overline{C}_{\lambda p^{n-j-1}} \end{aligned}$$

$$\begin{aligned}
& + \overline{C}_{\mu p^{n-j-1}} + \overline{C}_{\nu p^{n-j-1}} + \overline{C}_{\chi p^{n-j-1}} \} \\
& + \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \{ \overline{C}_{p^i} + \overline{C}_{2p^i} + \overline{C}_{4p^i} + \overline{C}_{8p^i} + \overline{C}_{\lambda p^i} + \overline{C}_{\mu p^i} + \overline{C}_{\nu p^i} + \overline{C}_{\chi p^i} \}.
\end{aligned}$$

If $p \equiv 3 \pmod{4}$

$$\begin{aligned}
\theta_{p^j}(x) &= \frac{1}{8p^n} [\phi(p^{n-j}) \{ \overline{C}_0 + \frac{(\alpha^{p^{n+j}} + \alpha^{7p^{n+j}})}{2} \overline{C}_{p^n} - \overline{C}_{4p^n} - \frac{(\alpha^{p^{n+j}} + \alpha^{7p^{n+j}})}{2} \overline{C}_{\nu p^n} \} \\
& + p^{n-j-1} \{ \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} \} + \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{ T_{i+j} \overline{C}_{p^i} + R_{i+j} \overline{C}_{2p^i} - P_{i+j} \overline{C}_{\lambda p^i} \\
& \quad - R_{i+j} \overline{C}_{\mu p^i} - T_{i+j} \overline{C}_{\nu p^i} + P_{i+j} \overline{C}_{\chi p^i} \} \\
& + \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \{ \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{p^i} - \overline{C}_{4p^i} + \overline{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}}) \beta^{p^{i+j}}}{2} \overline{C}_{\lambda p^i} \\
& \quad - \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{\nu p^i} - \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}}) \beta^{p^{i+j}}}{2} \overline{C}_{\chi p^i} \},
\end{aligned}$$

$$\begin{aligned}
\theta_{2p^j}(x) &= \frac{1}{8p^n} [\phi(p^{n-j}) \{ \overline{C}_0 - \overline{C}_{2p^n} + \overline{C}_{4p^n} \} \\
& + p^{n-j-1} \{ \overline{C}_{2p^{n-j-1}} - \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} + \overline{C}_{\mu p^{n-j-1}} \} \\
& + \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{ R_{i+j} \overline{C}_{p^i} - R_{i+j} \overline{C}_{\lambda p^i} + R_{i+j} \overline{C}_{\nu p^i} - R_{i+j} \overline{C}_{\chi p^i} \} \\
& - \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \{ \overline{C}_{2p^i} - \overline{C}_{4p^i} - \overline{C}_{8p^i} + \overline{C}_{\mu p^i} \}],
\end{aligned}$$

$$\begin{aligned}
\theta_{\lambda p^j}(x) &= \frac{1}{8p^n} [\phi(p^{n-j}) \{ \overline{C}_0 + \frac{(\alpha^{p^{n+j}} - \alpha^{7p^{n+j}}) \beta^{p^{n+j}}}{2} \overline{C}_{p^n} - \overline{C}_{4p^n} \\
& \quad - \frac{(\alpha^{p^{n+j}} - \alpha^{7p^{n+j}}) \beta^{p^{n+j}}}{2} \overline{C}_{\nu p^n} \} \\
& + p^{n-j-1} \{ \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} \} \\
& + \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{ P_{i+j} \overline{C}_{p^i} + R_{i+j} \overline{C}_{2p^i} - T_{i+j} \overline{C}_{\lambda p^i} - R_{i+j} \overline{C}_{\mu p^i} - P_{i+j} \overline{C}_{\nu p^i} + T_{i+j} \overline{C}_{\chi p^i} \}
\end{aligned}$$

$$+ \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \left\{ \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})}{2} \overline{C}_{p^i} - \overline{C}_{4p^i} + \overline{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{\lambda p^i} \right. \\ \left. - \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})}{2} \overline{C}_{\nu p^i} - \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{\chi p^i} \right\},$$

$$\theta_{\mu p^j}(x) = \frac{1}{8p^n} [\phi(p^{n-j}) \{ \overline{C}_0 - \overline{C}_{2p^n} + \overline{C}_{4p^n} \} \\ + p^{n-j-1} \{ \overline{C}_{2p^{n-j-1}} - \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} + \overline{C}_{\mu p^{n-j-1}} \} \\ - \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{ R_{i+j} \overline{C}_{p^i} - R_{i+j} \overline{C}_{\lambda p^i} + R_{i+j} \overline{C}_{\nu p^i} - R_{i+j} \overline{C}_{\chi p^i} \} \\ - \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \{ \overline{C}_{2p^i} - \overline{C}_{4p^i} - \overline{C}_{8p^i} + \overline{C}_{\mu p^i} \}],$$

$$\theta_{\nu p^j}(x) = \frac{1}{8p^n} [\phi(p^{n-j}) \{ \overline{C}_0 - \frac{(\alpha^{p^{n+j}} + \alpha^{7p^{n+j}})}{2} \overline{C}_{p^n} - \overline{C}_{4p^n} \\ + \frac{(\alpha^{p^{n+j}} + \alpha^{7p^{n+j}})}{2} \overline{C}_{\nu p^n} \} \\ + p^{n-j-1} \{ \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} \} - \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{ T_{i+j} \overline{C}_{p^i} - R_{i+j} \overline{C}_{2p^i} - P_{i+j} \overline{C}_{\lambda p^i} \\ + R_{i+j} \overline{C}_{\mu p^i} - T_{i+j} \overline{C}_{\nu p^i} + P_{i+j} \overline{C}_{\chi p^i} \} \\ - \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \left\{ \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{p^i} + \overline{C}_{4p^i} - \overline{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})}{2} \beta^{p^{i+j}} \overline{C}_{\lambda p^i} \right. \\ \left. - \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{\nu p^i} - \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})}{2} \beta^{p^{i+j}} \overline{C}_{\chi p^i} \right\},$$

$$\theta_{\chi p^j}(x) = \frac{1}{8p^n} [\phi(p^{n-j}) \{ \overline{C}_0 - \frac{(\alpha^{p^{n+j}} - \alpha^{7p^{n+j}})}{2} \beta^{p^{n+j}} \overline{C}_{p^n} - \overline{C}_{4p^n} \\ + \frac{(\alpha^{p^{n+j}} - \alpha^{7p^{n+j}})}{2} \beta^{p^{n+j}} \overline{C}_{\nu p^n} \} \\ + p^{n-j-1} \{ \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} \} + \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{ P_{i+j} \overline{C}_{p^i} - R_{i+j} \overline{C}_{2p^i} \\ - \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})}{2} \beta^{p^{i+j}} \overline{C}_{\lambda p^i} + \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})}{2} \beta^{p^{i+j}} \overline{C}_{\chi p^i} \}],$$

$$\begin{aligned}
& -T_{i+j}\overline{C}_{\lambda p^i} + R_{i+j}\overline{C}_{\mu p^i} - P_{i+j}\overline{C}_{\nu p^i} + T_{i+j}\overline{C}_{\chi p^i} \} \\
& -\phi(p^{n-j}) \sum_{i=n-j}^{n-1} \left\{ \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2} \overline{C}_{p^i} + \overline{C}_{4p^i} - \overline{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{\lambda p^i} \right. \\
& \quad \left. - \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2} \overline{C}_{\nu p^i} - \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{\chi p^i} \right\}.
\end{aligned}$$

If $p \equiv 1 \pmod{4}$

$$\begin{aligned}
\theta_{p^j}(x) &= \frac{1}{8p^n} [\phi(p^{n-j}) \{ \overline{C}_0 - \frac{(\alpha^{p^{n+j}} - \alpha^{7p^{n+j}})\beta^{p^{n+j}}}{2} \overline{C}_{p^n} - \overline{C}_{4p^n} \\
& \quad + \frac{(\alpha^{p^{n+j}} - \alpha^{7p^{n+j}})\beta^{p^{n+j}}}{2} \overline{C}_{\nu p^n} \} + p^{n-j-1} \{ \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} \} \\
& + \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{ P_{i+j}\overline{C}_{p^i} - R_{i+j}\overline{C}_{2p^i} - T_{i+j}\overline{C}_{\lambda p^i} + R_{i+j}\overline{C}_{\mu p^i} - P_{i+j}\overline{C}_{\nu p^i} + T_{i+j}\overline{C}_{\chi p^i} \} \\
& - \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \left\{ \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2} \overline{C}_{p^i} + \overline{C}_{4p^i} - \overline{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{\lambda p^i} \right. \\
& \quad \left. - \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2} \overline{C}_{\nu p^i} - \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{\chi p^i} \right\}]
\end{aligned}$$

$$\begin{aligned}
\theta_{2p^j}(x) &= \frac{1}{8p^n} [\phi(p^{n-j}) \{ \overline{C}_0 - \overline{C}_{2p^n} + \overline{C}_{4p^n} \} \\
& \quad + p^{n-j-1} \{ \overline{C}_{2p^{n-j-1}} - \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} + \overline{C}_{\mu p^{n-j-1}} \} \\
& \quad - \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{ R_{i+j}\overline{C}_{p^i} - R_{i+j}\overline{C}_{\lambda p^i} + R_{i+j}\overline{C}_{\nu p^i} - R_{i+j}\overline{C}_{\chi p^i} \} \\
& \quad - \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \{ \overline{C}_{2p^i} - \overline{C}_{4p^i} - \overline{C}_{8p^i} + \overline{C}_{\mu p^i} \}],
\end{aligned}$$

$$\begin{aligned}
\theta_{\lambda p^j}(x) &= \\
& \frac{1}{8p^n} [\phi(p^{n-j}) \{ \overline{C}_0 - \frac{(\alpha^{p^{n+j}} + \alpha^{7p^{n+j}})}{2} \overline{C}_{p^n} - \overline{C}_{4p^n} + \frac{(\alpha^{p^{n+j}} + \alpha^{7p^{n+j}})}{2} \overline{C}_{\nu p^n} \} \\
& \quad + p^{n-j-1} \{ \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} \} \\
& - \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{ T_{i+j}\overline{C}_{p^i} - R_{i+j}\overline{C}_{2p^i} - P_{i+j}\overline{C}_{\lambda p^i} + R_{i+j}\overline{C}_{\mu p^i} - T_{i+j}\overline{C}_{\nu p^i} + P_{i+j}\overline{C}_{\chi p^i} \}
\end{aligned}$$

$$\begin{aligned}
& -\phi(p^{n-j}) \sum_{i=n-j}^{n-1} \left\{ \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{p^i} + \overline{C}_{4p^i} - \overline{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2} \overline{C}_{\lambda p^i} \right. \\
& \quad \left. - \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{\nu p^i} - \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2} \overline{C}_{\chi p^i} \right\},
\end{aligned}$$

$$\begin{aligned}
\theta_{\mu p^j}(x) &= \frac{1}{8p^n} [\phi(p^{n-j}) \{ \overline{C}_0 - \overline{C}_{2p^n} + \overline{C}_{4p^n} \} \\
& \quad + p^{n-j-1} \{ \overline{C}_{2p^{n-j-1}} - \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} + \overline{C}_{\mu p^{n-j-1}} \} \\
& \quad + \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{ R_{i+j} \overline{C}_{p^i} - R_{i+j} \overline{C}_{\lambda p^i} + R_{i+j} \overline{C}_{\nu p^i} - R_{i+j} \overline{C}_{\chi p^i} \} \\
& \quad - \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \{ \overline{C}_{2p^i} - \overline{C}_{4p^i} - \overline{C}_{8p^i} + \overline{C}_{\mu p^i} \}],
\end{aligned}$$

$$\begin{aligned}
\theta_{\nu p^j}(x) &= \frac{1}{8p^n} [\phi(p^{n-j}) \{ \overline{C}_0 + \frac{(\alpha^{p^{n+j}} - \alpha^{7p^{n+j}})\beta^{p^{i+j}}}{2} \overline{C}_{p^n} - \overline{C}_{4p^n} \\
& \quad - \frac{(\alpha^{p^{n+j}} - \alpha^{7p^{n+j}})\beta^{p^{i+j}}}{2} \overline{C}_{\nu p^n} \} + p^{n-j-1} \{ \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} \} \\
& \quad - \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{ P_{i+j} \overline{C}_{p^i} + R_{i+j} \overline{C}_{2p^i} - T_{i+j} \overline{C}_{\lambda p^i} - R_{i+j} \overline{C}_{\mu p^i} - P_{i+j} \overline{C}_{\nu p^i} + T_{i+j} \overline{C}_{\chi p^i} \} \\
& \quad + \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \left\{ \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2} \overline{C}_{p^i} - \overline{C}_{4p^i} + \overline{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{\lambda p^i} \right. \\
& \quad \left. - \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2} \overline{C}_{\nu p^i} - \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{\chi p^i} \right\},
\end{aligned}$$

$$\begin{aligned}
\theta_{\chi p^j}(x) &= \\
& \frac{1}{8p^n} [\phi(p^{n-j}) \{ \overline{C}_0 + \frac{(\alpha^{p^{n+j}} + \alpha^{7p^{n+j}})}{2} \overline{C}_{p^n} - \overline{C}_{4p^n} - \frac{(\alpha^{p^{n+j}} + \alpha^{7p^{n+j}})}{2} \overline{C}_{\nu p^n} \} \\
& \quad + p^{n-j-1} \{ \overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}} \} \\
& \quad + \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{ T_{i+j} \overline{C}_{p^i} + R_{i+j} \overline{C}_{2p^i} - P_{i+j} \overline{C}_{\lambda p^i} - R_{i+j} \overline{C}_{\mu p^i} - T_{i+j} \overline{C}_{\nu p^i} + P_{i+j} \overline{C}_{\chi p^i} \}
\end{aligned}$$

$$+ \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \left\{ \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{p^i} - \overline{C}_{4p^i} + \overline{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2} \overline{C}_{\lambda p^i} \right. \\ \left. - \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2} \overline{C}_{\nu p^i} - \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2} \overline{C}_{\chi p^i} \right\}.$$

Here β is a 4-th root of unity in F and T_{i+j} , P_{i+j} and R_{i+j} are given by:

For $p \equiv 3 \pmod{4}$

$$\sum_{t=0}^{n-j-1} \frac{1}{P_{i+j}} T_{i+j}^2 + \sum_{t=0}^{n-j-1} \frac{1}{P_{i+j}} P_{i+j}^2 = p^n + p^{n-1},$$

$$\sum_{t=0}^{n-j-1} \frac{2}{P_{i+j}} T_{i+j} P_{i+j} = (-1)^{n-1} (p^n - p^{n-1}),$$

$$R_{n-1} = \sqrt{p^{2n-1}}, \text{ and for all } j \leq n-2, R_j = 0.$$

For $p \equiv 1 \pmod{4}$

$$\sum_{t=0}^{n-j-1} \frac{2}{P_{i+j}} T_{i+j} P_{i+j} = p^n + p^{n-1},$$

$$\sum_{t=0}^{n-j-1} \frac{1}{P_{i+j}} \{T_{i+j}^2 + P_{i+j}^2\} = p^{n-1} - p^n,$$

$$R_{n-1} = \sqrt{-p^{2n-1}}, \text{ and for all } j \leq n-2, R_j = 0.$$

Proof. By the definition

$$\theta_s(x) = \frac{1}{8p^n} [\varepsilon_0^s \overline{C}_0 + \varepsilon_{p^n}^s \overline{C}_{p^n} + \varepsilon_{2p^n}^s \overline{C}_{2p^n} + \varepsilon_{4p^n}^s \overline{C}_{4p^n} + \varepsilon_{\nu p^n}^s \overline{C}_{\nu p^n} \\ + \sum_{i=0}^{n-1} \{ \varepsilon_{p^i}^s \overline{C}_{p^i} + \varepsilon_{2p^i}^s \overline{C}_{2p^i} + \varepsilon_{4p^i}^s \overline{C}_{4p^i} + \varepsilon_{8p^i}^s \overline{C}_{8p^i} + \varepsilon_{\lambda p^i}^s \overline{C}_{\lambda p^i} + \varepsilon_{\mu p^i}^s \overline{C}_{\mu p^i} + \varepsilon_{\nu p^i}^s \overline{C}_{\nu p^i} \\ + \varepsilon_{\chi p^i}^s \overline{C}_{\chi p^i} \}]. \quad (3.1)$$

To evaluate $\theta_0(x)$, take $s = 0$, then $\varepsilon_k^0 = \sum_{s \in \Omega_0} \alpha^{-ks} = \alpha^{-k \cdot 0} = 1$ for all $0 \leq k \leq 8p^n - 1$. Therefore,

$$\theta_0(x) = \frac{1}{8p^n} [\overline{C}_0 + \overline{C}_{p^n} + \overline{C}_{2p^n} + \overline{C}_{4p^n} + \overline{C}_{\nu p^n} + \sum_{i=0}^{n-1} \{\overline{C}_{p^i} + \overline{C}_{2p^i} + \overline{C}_{4p^i} + \overline{C}_{8p^i} + \overline{C}_{\lambda p^i} + \overline{C}_{\mu p^i} + \overline{C}_{\nu p^i} + \overline{C}_{\chi p^i}\}].$$

For the evaluation of $\theta_{p^n}(x)$, take $s = p^n$ so we have to compute $\varepsilon_k^{p^n}$ for $k = 0, p^n, 2p^n, 4p^n, \nu p^n, p^i, 2p^i, 4p^i, 8p^i, \lambda p^i, \mu p^i, \nu p^i, \chi p^i$. Since $-\Omega_{p^n} = \Omega_{p^n}$. So

$$\varepsilon_k^{p^n} = \sum_{s \in \Omega_{p^n}} \alpha^{-ks} = \sum_{s \in \Omega_{p^n}} \alpha^{ks} = \alpha^{kp^n} + \alpha^{kp^n q}.$$

Therefore

$$\begin{aligned} \varepsilon_0^{p^n} &= -\varepsilon_{4p^n}^{p^n} = -\varepsilon_{4p^i}^{p^n} = \varepsilon_{8p^i}^{p^n} = 2, \\ \varepsilon_{p^n}^{p^n} &= -\varepsilon_{\nu p^n}^{p^n} = (\alpha^{p^{2n}} + \alpha^{7p^{2n}}) \cdot \varepsilon_{2p^n}^{p^n} = \varepsilon_{2p^i}^{p^n} = \varepsilon_{\mu p^i}^{p^n} = 0 \end{aligned}$$

(using Lemma 3.15),

$$\varepsilon_{p^i}^{p^n} = \varepsilon_{\nu p^i}^{p^n} = (\alpha^{p^{n+i}} + \alpha^{7p^{n+i}}), \quad \varepsilon_{\lambda p^i}^{p^n} = -\varepsilon_{\chi p^i}^{p^n} = (\alpha^{p^{n+i}} - \alpha^{7p^{n+i}}) \beta^{p^{n+i}}.$$

From (3.1) we get

$$\begin{aligned} \theta_{p^n}(x) &= \frac{1}{8p^n} [2\overline{C}_0 + (\alpha^{p^{2n}} + \alpha^{7p^{2n}})\overline{C}_{p^n} - 2\overline{C}_{4p^n} - (\alpha^{p^{2n}} + \alpha^{7p^{2n}})\overline{C}_{\nu p^n} + \sum_{i=0}^{n-1} \{(\alpha^{p^{n+i}} + \alpha^{7p^{n+i}})\overline{C}_{p^i} - 2\overline{C}_{4p^i} + 2\overline{C}_{8p^i} + (\alpha^{p^{n+i}} - \alpha^{7p^{n+i}})\beta^{p^{n+i}}\overline{C}_{\lambda p^i} - (\alpha^{p^{n+i}} + \alpha^{7p^{n+i}})\overline{C}_{\nu p^i} \\ &\quad - (\alpha^{p^{n+i}} - \alpha^{7p^{n+i}})\beta^{p^{n+i}}\overline{C}_{\chi p^i}\}]. \end{aligned}$$

Similarly, $\theta_{2p^n}(x)$, $\theta_{4p^n}(x)$, $\theta_{\nu p^n}(x)$, can be obtained by using the lemma 3.6 and lemma 3.15.

Further to evaluate $\theta_{p^j}(x)$, take $s = p^j$ so we have to compute $\varepsilon_k^{p^j}$ for $k = 0, p^n, 2p^n, 4p^n, \nu p^n, p^i, 2p^i, 4p^i, 8p^i, \lambda p^i, \mu p^i, \nu p^i, \chi p^i$.

Case I. If $p \equiv 3 \pmod{4}$.

Since in this case $-\Omega_{p^j} = \Omega_{p^j}$, using remark 3.4. So, $\varepsilon_k^{p^j} = \sum_{s \in \Omega_{p^j}} \alpha^{-ks} =$

$\sum_{s \in \Omega_{p^j}} \alpha^{ks}$. Therefore, using lemma 3.11 – 3.15,

$$\begin{aligned} \varepsilon_0^{p^j} &= -\varepsilon_{4p^n}^{p^j} = \phi(p^{n-j}), & \varepsilon_{p^n}^{p^j} &= -\varepsilon_{\nu p^n}^{p^j} = \frac{\phi(p^{n-j})}{2} (\alpha^{p^{n+j}} + \alpha^{7p^{n+j}}), \\ \varepsilon_{2p^n}^{p^j} &= \sum_{s \in \Omega_{p^j}} \alpha^{2ps} = 0, \end{aligned}$$

$$\begin{aligned}
\varepsilon_{p^i}^{p^j} &= -\varepsilon_{\nu p^i}^{p^j} = \begin{cases} \frac{\phi(p^{n-j})}{2}(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}}), & \text{if } i+j \geq n, \\ \frac{1}{p^j}T_{i+j}, & \text{if } i+j \leq n-1. \end{cases} \\
\varepsilon_{2p^i}^{p^j} &= -\varepsilon_{\mu p^i}^{p^j} = \begin{cases} 0, & \text{if } i+j \geq n, \\ \frac{1}{p^j}R_{i+j}, & \text{if } i+j \leq n-1. \end{cases} \\
\varepsilon_{4p^i}^{p^j} &= \sum_{s \in \Omega_{p^j}} \alpha^{4p^i s} = \begin{cases} 0, & \text{if } i+j < n, \\ p^{n-j-1}, & \text{if } i+j = n-1, \\ -\phi(p^{n-j}), & \text{if } i+j \geq n. \end{cases} \\
\varepsilon_{8p^i}^{p^j} &= \sum_{s \in \Omega_{p^j}} \alpha^{8p^i s} = \begin{cases} 0, & \text{if } i+j < n, \\ -p^{n-j-1}, & \text{if } i+j = n-1, \\ \phi(p^{n-j}), & \text{if } i+j \geq n. \end{cases} \\
\varepsilon_{\lambda p^i}^{p^j} &= -\varepsilon_{\chi p^i}^{p^j} = \begin{cases} \frac{\phi(p^{n-j})}{2}(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}, & \text{if } i+j \geq n, \\ -\frac{1}{p^j}P_{i+j}, & \text{if } i+j \leq n-1. \end{cases}
\end{aligned}$$

Hence

$$\begin{aligned}
\theta_{p^j}(x) &= \frac{1}{8p^n}[\phi(p^{n-j})\{\overline{C}_0 + \frac{(\alpha^{p^{n+j}} + \alpha^{7p^{n+j}})}{2}\overline{C}_{p^n} - \overline{C}_{4p^n} \\
&\quad - \frac{(\alpha^{p^{n+j}} + \alpha^{7p^{n+j}})}{2}\overline{C}_{\nu p^n}\} + p^{n-j-1}\{\overline{C}_{4p^{n-j-1}} \\
&\quad - \overline{C}_{8p^{n-j-1}}\} + \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{T_{i+j}\overline{C}_{p^i} + R_{i+j}\overline{C}_{2p^i} - P_{i+j}\overline{C}_{\lambda p^i} - R_{i+j}\overline{C}_{\mu p^i} - T_{i+j}\overline{C}_{\nu p^i} \\
&\quad + P_{i+j}\overline{C}_{\chi p^i}\} + \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \left\{ \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2}\overline{C}_{p^i} - \overline{C}_{4p^i} + \overline{C}_{8p^i} \right. \\
&\quad \left. + \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2}\overline{C}_{\lambda p^i} - \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2}\overline{C}_{\nu p^i} - \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2}\overline{C}_{\chi p^i} \right\}.
\end{aligned}$$

Case II. If $p \equiv 1 \pmod{4}$.

Since $-\Omega_{p^j} = \Omega_{\chi p^j}$, using remark 3.4. So, we have

$$\varepsilon_k^{p^j} = \sum_{s \in \Omega_{p^j}} \alpha^{-ks} = \sum_{s \in \Omega_{\chi p^j}} \alpha^{ks} = \sum_{s \in \Omega_{p^j}} \alpha^{\chi ks}.$$

Therefore, using Lemmas 3.11-3.15, we receive

$$\begin{aligned}
\varepsilon_0^{p^j} &= -\varepsilon_{4p^n}^{p^j} = \phi(p^{n-j}), \\
\varepsilon_{p^n}^{p^j} &= -\varepsilon_{\nu p^n}^{p^j} = -\frac{\phi(p^{n-j})}{2}(\alpha^{p^{n+j}} - \alpha^{7p^{n+j}})\beta^{p^{n+j}},
\end{aligned}$$

$$\varepsilon_{2p^n}^{p^j} = \sum_{s \in \Omega_{p^j}} \alpha^{2\chi p^n s} = - \sum_{s \in \Omega_{p^j}} \alpha^{2p^n s} = 0,$$

$$\varepsilon_{p^i}^{p^j} = -\varepsilon_{\nu p^i}^{p^j} = \begin{cases} -\frac{\phi(p^{n-j})}{2}(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{n+j}}, & \text{if } i+j \geq n, \\ \frac{1}{p^j}P_{i+j}, & \text{if } i+j \leq n-1. \end{cases}$$

$$\varepsilon_{2p^i}^{p^j} = -\varepsilon_{\mu p^i}^{p^j} = \begin{cases} 0, & \text{if } i+j \geq n, \\ -\frac{1}{p^j}R_{i+j}, & \text{if } i+j \leq n-1. \end{cases}$$

$$\varepsilon_{4p^i}^{p^j} = \sum_{s \in \Omega_{p^j}} \alpha^{4\chi p^i s} = \sum_{s \in \Omega_{p^j}} \alpha^{4p^i s} = \begin{cases} 0, & \text{if } i+j < n, \\ p^{n-j-1}, & \text{if } i+j = n-1, \\ -\phi(p^{n-j}), & \text{if } i+j \geq n. \end{cases}$$

$$\varepsilon_{8p^i}^{p^j} = \sum_{s \in \Omega_{p^j}} \alpha^{8\chi p^i s} = \sum_{s \in \Omega_{p^j}} \alpha^{8p^i s} = \begin{cases} 0, & \text{if } i+j < n, \\ -p^{n-j-1}, & \text{if } i+j = n-1, \\ \phi(p^{n-j}), & \text{if } i+j \geq n. \end{cases}$$

$$\varepsilon_{\lambda p^i}^{p^j} = -\varepsilon_{\chi p^i}^{p^j} = \begin{cases} -\frac{\phi(p^{n-j})}{2}(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}}), & \text{if } i+j \geq n, \\ -\frac{1}{p^j}T_{i+j}, & \text{if } i+j \leq n-1. \end{cases}$$

Hence

$$\begin{aligned} \theta_{p^j}(x) = & \frac{1}{8p^n}[\phi(p^{n-j})\{\overline{C}_0 - \frac{(\alpha^{p^{n+j}} - \alpha^{7p^{n+j}})\beta^{p^{i+j}}}{2}\overline{C}_{p^n} - \overline{C}_{4p^n} \\ & + \frac{(\alpha^{p^{n+j}} - \alpha^{7p^{n+j}})\beta^{p^{i+j}}}{2}\overline{C}_{\nu p^n}\} \\ & + p^{n-j-1}\{\overline{C}_{4p^{n-j-1}} - \overline{C}_{8p^{n-j-1}}\} + \frac{1}{p^j} \sum_{i=0}^{n-j-1} \{P_{i+j}\overline{C}_{p^i} \\ & - R_{i+j}\overline{C}_{2p^i} - T_{i+j}\overline{C}_{\lambda p^i} + R_{i+j}\overline{C}_{\mu p^i} - P_{i+j}\overline{C}_{\nu p^i} + T_{i+j}\overline{C}_{\chi p^i}\} \\ & - \phi(p^{n-j}) \sum_{i=n-j}^{n-1} \{ \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2}\overline{C}_{p^i} + \overline{C}_{4p^i} - \overline{C}_{8p^i} + \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2}\overline{C}_{\lambda p^i} \\ & - \frac{(\alpha^{p^{i+j}} - \alpha^{7p^{i+j}})\beta^{p^{i+j}}}{2}\overline{C}_{\nu p^i} - \frac{(\alpha^{p^{i+j}} + \alpha^{7p^{i+j}})}{2}\overline{C}_{\chi p^i} \}]. \end{aligned}$$

Similarly, using lemma 3.7 – 3.8 and lemma 3.11 – 3.15, we can evaluate expression for $\theta_{2p^j}, \theta_{4p^j}, \theta_{8p^j}, \theta_{\lambda p^j}, \theta_{\mu p^j}, \theta_{\nu p^j}, \theta_{\chi p^j}$.

Further, the relations for R_j, T_j , and P_j can be obtained by using the relations $\theta_{2p^j}(\alpha^{2p^j}) = 1, \theta_{p^j}(\alpha^{p^j}) = 1, \theta_{p^j}(\alpha^{\lambda p^j}) = 0$ and lemma 3.11 – 3.15. $\square$

4. Dimension and Generating Polynomial

If α is primitive $8p^n$ th root of unity, then $m_s(x) = \prod_{s \in \Omega_s} (x - \alpha^s)$.

Remark. If $m_s(x)$ denote the minimal polynomial for α^s , $s \in \Omega_s$, then the generating polynomial for cyclic code of length $8p^n$ corresponding to the cyclotomic coset Ω_s is $\frac{x^{8p^n} - 1}{m_s(x)}$ and the dimension of minimal cyclic code M_s is equal to the cardinality of the class Ω_s [5].

Thus, the dimensions of the codes $M_0, M_{2p^n}, M_{4p^n}, M_{4p^j}$, and M_{8p^j} , are 1, 2, 1, $\phi(p^{n-j})$ and $\phi(p^{n-j})$ respectively.

Theorem 4.1. (i) The generating polynomials for the codes $M_0, M_{2p^n}, M_{4p^n}, M_{4p^j}$, and M_{8p^j} , are

$$\begin{aligned} & (1 + x + x^2 + \dots + x^{8p^n-1}), (x^6 - x^4 + x^2 - 1)(1 + x^8 + \dots + x^{8(p^n-1)}), \\ & (x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1)(1 + x^8 + \dots + x^{8(p^n-1)}), \\ & (x^{p^{n-j-1}} + 1)(x^{p^{n-j}} - 1)(x^{2p^{n-j}} + 1)(x^{4p^{n-j}} + 1)(1 + x^8 + \dots + x^{8p^{n-j}(p^j-1)}), \\ & (x^{p^{n-j-1}} - 1)(x^{p^{n-j}} + 1)(x^{2p^{n-j}} + 1)(x^{4p^{n-j}} + 1)(1 + x^8 + \dots + x^{8p^{n-j}(p^j-1)}), \end{aligned}$$

respectively.

(ii) The generating polynomials for $M_{p^n} \oplus M_{\nu p^n}$ and $M_{p^j} \oplus M_{2p^j} \oplus M_{\lambda p^j} \oplus M_{\mu p^j} \oplus M_{\nu p^j} \oplus M_{\chi p^j}$, are

$$(x^4 - 1)(1 + x^8 + \dots + x^{8(p^{n-1})})$$

and

$$(x^{2p^{n-j-1}} + 1)(x^{4p^{n-j-1}} + 1)(x^{2p^{n-j}} - 1)(1 + x^8 + \dots + x^{8p^{n-j}(p^j-1)}),$$

respectively.

Proof. (i) The minimal polynomials for $\alpha^0, \alpha^{2p^n}, \alpha^{4p^n}, \alpha^{4p^j}, \alpha^{8p^j}$ are

$$(x - 1), (x^2 + 1), (x + 1),$$

$$\frac{(x^{p^{n-j}} + 1)}{(x^{p^{n-j-1}} + 1)}, \frac{(x^{p^{n-j}} - 1)}{(x^{p^{n-j-1}} - 1)},$$

respectively. By Remark 4.1, the corresponding generating polynomials are $(1 + x + x^2 + \dots + x^{8p^n-1}), (x^6 - x^4 + x^2 - 1)(1 + x^8 + \dots + x^{8(p^n-1)}),$

$$(x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1)(1 + x^8 + \dots + x^{8(p^n-1)}), \\ (x^{p^{n-j-1}} + 1)(x^{p^{n-j}} - 1)(x^{2p^{n-j}} + 1)(x^{4p^{n-j}} + 1)(1 + x^8 + \dots + x^{8p^{n-j}(p^j-1)}), \\ (x^{p^{n-j-1}} - 1)(x^{p^{n-j}} + 1)(x^{2p^{n-j}} + 1)(x^{4p^{n-j}} + 1)(1 + x^8 + \dots + x^{8p^{n-j}(p^j-1)}).$$

(ii) The product of minimal polynomials satisfied by α^{p^n} and $\alpha^{\nu p^n}$ is $(x^4 + 1)$. Therefore, the generating polynomial for $M_{p^n} \oplus M_{\nu p^n}$ is $(x^4 - 1)(1 + x^8 + \dots + x^{8(p^n-1)})$.

Also the product of minimal polynomials satisfied by $\alpha^{p^j}, \alpha^{2p^j}, \alpha^{\lambda p^j}, \alpha^{\mu p^j}, \alpha^{\nu p^j}$ and χ^{p^j} is

$$\frac{(x^{2p^{n-j}} + 1)(x^{4p^{n-j}} + 1)}{(x^{2p^{n-j-1}} + 1)(x^{4p^{n-j-1}} + 1)}.$$

Therefore, the corresponding generating polynomial for $M_{p^j} \oplus M_{2p^j} \oplus M_{\lambda p^j} \oplus M_{\mu p^j} \oplus M_{\nu p^j} \oplus M_{\chi p^j}$, is $(x^{2p^{n-j-1}} + 1)(x^{4p^{n-j-1}} + 1)(x^{2p^{n-j}} - 1)(1 + x^8 + \dots + x^{8p^{n-j}(p^j-1)})$. $\square$

5. Minimum Distance

Here, we find the minimum distance of the minimal cyclic code M_s of length $8p^n$, generated by the primitive idempotent θ_s :

Lemma 5.1. *If l is a cyclic code of length m generated by $g(x)$ and its minimum distance is d , then the code $\bar{l}$ of length mk generated by $g(x)(1 + x^m + x^{2m} + \dots + x^{(k-1)m})$ is a repetition code of l repeated k times and its minimum distance is dk [2].*

Theorem 5.2. *The minimum distance of the codes M_0, M_{2p^n}, M_{4p^n} are $8p^n, 4p^n, 8p^n$ respectively and for the codes M_{p^n} and $M_{\nu p^n}$ are greater than or equal to $2p^n$.*

Proof. Since the generating polynomial for the code M_0 is $(1 + x + x^2 + \dots + x^{8p^n-1})$, which is itself a polynomial of length $8p^n$, hence its minimum distance is $8p^n$.

The minimum distance of the cyclic code M_{2p^n} with generating polynomial $(x^6 - x^4 + x^2 - 1)(1 + x^8 + \dots + x^{8(p^n-1)})$ is $4p^n$ as it is repetition of the cyclic code of length 8 with generating polynomial $(x^6 - x^4 + x^2 - 1)$ whose minimum distance is 4, repeated p^n times.

In a similar way, the minimum distance of the cyclic code M_{4p^n} with generating polynomial $(x^7 - x^6 + x^5 - x^4 - x^3 + x^2 - x + 1)(1 + x^8 + \dots + x^{8(p^n-1)})$ is

$8p^n$ as it is repetition of the cyclic code of length 8 with generating polynomial $(x^7 - x^6 + x^5 - x^4 - x^3 + x^2 - x + 1)$ whose minimum distance is 8, repeated p^n times.

Since the generating polynomial for the cyclic codes M_{p^n} and $M_{\nu p^n}$ is $(x^4 - 1)(1 + x^8 + \dots + x^{8(p^n-1)})$, therefore, if we take a code l of length 8 generated by the polynomial $(x^4 - 1)$, then the minimum distance of this code is 2. Since the cyclic code $M(p^n, \nu p^n)$ generated by the polynomial $(x^4 - 1)(1 + x^8 + \dots + x^{8(p^n-1)})$ is a repetition code of the code l , repeated p^n times. Hence its minimum distance is $2p^n$. The codes corresponding to Ω_p^n and $\Omega_{\nu p^n}$ are subcodes of above code so their minimum distances are greater than or equal to $2p^n$. $\square$

Theorem 5.3. For $0 \leq j \leq n-1$, the minimum distance of the cyclic codes M_{4p^j} and M_{8p^j} are $16p^j$ and for the codes $M_{p^j}, M_{2p^j}, M_{\lambda p^j}, M_{\mu p^j}, M_{\nu p^j}$ and $M_{\chi p^j}$ are greater than equal to $8p^j$.

Proof. Consider the cyclic codes M_{4p^i} . Since the generating polynomial of the cyclic code of length $8p^n$ is $(x^{p^{n-j}} - 1)(x^{p^{n-j-1}} + 1)(x^{2p^{n-i}} + 1)(x^{4p^{n-i}} + 1)(1 + x^{8p^{n-i}} + \dots + x^{8p^{n-j}(p^j-1)})$,

therefore, if we take a cyclic code C of length p^{n-j} generated by the polynomial $(x^{p^{n-j-1}} + 1)$, then the minimal distance of this code is 2. Now consider the cyclic code C_1 of length $2p^{n-j}$ generated by the polynomial $(x^{p^{n-j-1}} + 1)(x^{p^{n-j}} - 1)$, and then minimum distance of this code is 4, as it is 2 time repetition of the code C . Further, the minimum distance of the code C_2 of length $4p^{n-j}$ generated by the polynomial $(x^{p^{n-j-1}} + 1)(x^{p^{n-j}} - 1)(x^{2p^{n-j}} + 1)$ is 8, as it is 2 time repetition of the code C_1 . Hence, the minimum distance of the code C_3 of length $8p^{n-j}$ generated by the polynomial $(x^{p^{n-j-1}} + 1)(x^{p^{n-j}} - 1)(x^{2p^{n-j}} + 1)(x^{4p^{n-j}} + 1)$, and then minimum distance of this code is 16, as it is 2 time repetition of the code C_2 . Since the cyclic code of length $8p^n$ generated by the polynomial $(x^{p^{n-j-1}} + 1)(x^{p^{n-j}} - 1)(x^{2p^{n-j}} + 1)(x^{4p^{n-j}} + 1)(1 + x^{8p^{n-j}} + \dots + x^{8p^{n-j}(p^j-1)})$ is a repetition code of the code C_3 , repeated p^j times. Hence its minimum distance is $16p^j$.

Similarly, the minimum distance of the cyclic code M_{8p^i} of length $8p^n$ with generating polynomial $(x^{p^{n-j-1}} - 1)(x^{p^{n-j}} + 1)(x^{2p^{n-j}} + 1)(x^{4p^{n-j}} + 1)(1 + x^{8p^{n-j}} + \dots + x^{8p^{n-j}(p^j-1)})$ is also $16p^j$.

Since the product of generating polynomial for the cyclic codes $M_{p^i}, M_{2p^i}, M_{\lambda p^i}, M_{\mu p^i}, M_{\nu p^i}$ and $M_{\chi p^i}$ is $(x^{2p^{n-j-1}} + 1)(x^{2p^{n-j}} - 1)(x^{4p^{n-j-1}} + 1)(1 + x^{8p^{n-j}} + \dots + x^{8p^{n-j}(p^j-1)})$, therefore, if we take a code C of length $8p^{n-j}$ generated by the polynomial

$$(x^{2p^{n-j-1}} + 1)(x^{2p^{n-j}} - 1)(x^{4p^{n-j-1}} + 1),$$

then the minimum distance of this code is 8. Since the cyclic code C_1 of length $8p^n$ generated by the polynomial

$$(x^{2p^{n-j-1}} + 1)(x^{2p^{n-j}} - 1)(x^{4p^{n-j-1}} + 1)(1 + x^{8p^{n-j}} + \dots x^{8p^{n-j}(p^j-1)})$$

is a repetition code of the code C , repeated p^j times. Hence its minimum distance is $8p^j$.

The codes corresponding to Ω_{p^j} and $\Omega_{\lambda p^j}$ are the sub codes of above codes so, their minimum distances are greater than or equal to $8p^j$. $\square$

6. Example

This section presents example based on the theoretical results obtained in the previous sections.

Example 6.1. cyclic code of length 40.

Take $p = 5, n = 1, q = 7$. Then, the q-cyclotomic cosets modulo 40 are

$$\Omega_0 = \{0\}, \Omega_1 = \{1, 7, 9, 23\}, \Omega_2 = \{2, 6, 14, 18\}, \Omega_4 = \{4, 12, 28, 36\}, \Omega_5 = \{5, 35\},$$

$$\Omega_8 = \{8, 16, 24, 32\}, \Omega_{10} = \{10, 30\}, \Omega_{11} = \{11, 13, 19, 37\}, \Omega_{20} = \{20\}, \Omega_{21} = \{3, 21, 27, 29\},$$

$$\Omega_{22} = \{22, 26, 34, 38\}, \Omega_{25} = \{15, 25\}, \Omega_{31} = \{17, 31, 33, 39\}$$

Also, $T_0 = 3, R_0 = 3, P_0 = 1$ and the corresponding primitive idempotents in $\frac{GF(7)[x]}{\langle x^{40}-1 \rangle}$ are

$$\Omega_0(x) = \frac{1}{40}[\overline{C}_0 + \overline{C}_5 + \overline{C}_{10} + \overline{C}_{20} + \overline{C}_{25} + \overline{C}_1 + \overline{C}_2 + \overline{C}_4 + \overline{C}_8 + \overline{C}_{11} + \overline{C}_{22} + \overline{C}_{21} + \overline{C}_{31}]$$

$$\Omega_1(x) = \frac{1}{40}[4\overline{C}_0 + \overline{C}_1 + 4\overline{C}_2 + \overline{C}_4 + 6\overline{C}_5 + 6\overline{C}_8 + 4\overline{C}_{11} + 3\overline{C}_{20} + 6\overline{C}_{21} + 3\overline{C}_{22} + \overline{C}_{25} + 3\overline{C}_{31}]$$

$$\Omega_2(x) = \frac{1}{40}[4\overline{C}_0 + 4\overline{C}_1 + \overline{C}_2 + 6\overline{C}_4 + 6\overline{C}_8 + 3\overline{C}_{10} + 3\overline{C}_{11} + 4\overline{C}_{20} + 4\overline{C}_{21} + \overline{C}_{22} + 3\overline{C}_{31}]$$

$$\Omega_4(x) = \frac{1}{40}[4\overline{C}_0 + \overline{C}_1 + 6\overline{C}_2 + 6\overline{C}_4 + 3\overline{C}_5 + 6\overline{C}_8 + 4\overline{C}_{10} + \overline{C}_{11} + 4\overline{C}_{20} + \overline{C}_{21} + 6\overline{C}_{22} + 3\overline{C}_{25} + \overline{C}_{31}]$$

$$\Omega_5(x) = \frac{1}{40}[2\overline{C}_0 + 4\overline{C}_1 + 5\overline{C}_4 + 3\overline{C}_5 + 2\overline{C}_8 + 4\overline{C}_{11} + 5\overline{C}_{20} + 3\overline{C}_{21} + 4\overline{C}_{25} + 3\overline{C}_{31}]$$

$$\Omega_8(x) = \frac{1}{40}[4\overline{C}_0 + 6\overline{C}_1 + 6\overline{C}_2 + 6\overline{C}_4 + 3\overline{C}_5 + 6\overline{C}_8 + 4\overline{C}_{10} + 6\overline{C}_{11} + 4\overline{C}_{20} + 6\overline{C}_{21} + 6\overline{C}_{22} + 3\overline{C}_{25} + 6\overline{C}_{31}]$$

$$\Omega_{10}(x) = \frac{1}{20}[\overline{C}_0 + 6\overline{C}_2 + \overline{C}_4 + \overline{C}_8 + 6\overline{C}_{10} + \overline{C}_{20} + 6\overline{C}_{22}]$$

$$\Omega_{11}(x) = \frac{1}{40}[4\overline{C}_0 + 4\overline{C}_1 + 3\overline{C}_2 + \overline{C}_4 + 6\overline{C}_5 + 6\overline{C}_8 + \overline{C}_{11} + 3\overline{C}_{20} + \overline{C}_{21} + 4\overline{C}_{22} + \overline{C}_{25} + \overline{C}_{31}]$$

$$\Omega_{20}(x) = \frac{1}{40}[\overline{C}_0 + 6\overline{C}_1 + \overline{C}_2 + \overline{C}_4 + 6\overline{C}_5 + \overline{C}_8 + \overline{C}_{10} + 6\overline{C}_{11} + \overline{C}_{20} + 6\overline{C}_{21} + \overline{C}_{22} + 6\overline{C}_{25} + 6\overline{C}_{31}]$$

$$\begin{aligned}
\Omega_{21}(x) &= \frac{1}{40}[4\overline{C}_0 + 6\overline{C}_1 + 4\overline{C}_2 + \overline{C}_4 + \overline{C}_5 + 6\overline{C}_8 + 3\overline{C}_{11} + 3\overline{C}_{20} + \overline{C}_{21} + 3\overline{C}_{22} + \\
&\quad 6\overline{C}_{25} + 4\overline{C}_{31}] \\
\Omega_{22}(x) &= \frac{1}{40}[4\overline{C}_0 + 3\overline{C}_1 + \overline{C}_2 + 6\overline{C}_4 + 6\overline{C}_8 + 3\overline{C}_{10} + 4\overline{C}_{11} + 4\overline{C}_{20} + 3\overline{C}_{21} + \\
&\quad \overline{C}_{22} + 4\overline{C}_{31}] \\
\Omega_{25}(x) &= \frac{1}{40}[2\overline{C}_0 + 3\overline{C}_1 + 5\overline{C}_4 + 4\overline{C}_5 + 2\overline{C}_8 + 3\overline{C}_{11} + 5\overline{C}_{20} + 4\overline{C}_{21} + 3\overline{C}_{25} + 4\overline{C}_{31}] \\
\Omega_{31}(x) &= \frac{1}{40}[4\overline{C}_0 + 3\overline{C}_1 + 3\overline{C}_2 + \overline{C}_4 + \overline{C}_5 + 6\overline{C}_8 + 6\overline{C}_{11} + 3\overline{C}_{20} + 4\overline{C}_{21} + \\
&\quad 4\overline{C}_{22} + 6\overline{C}_{25} + \overline{C}_{31}]
\end{aligned}$$

Minimal polynomials of $\alpha^0, \alpha^1, \alpha^2, \alpha^4, \alpha^5, \alpha^8, \alpha^{10}, \alpha^{11}, \alpha^{20}, \alpha^{21}, \alpha^{22}, \alpha^{25}$ and α^{31} are

$x-1, x^4+x^3-x^2+3x+1, x^4-3x^3-3x^2+3x+1, x^4-x^3+x^2-x+1, x^2-4x+1, x^4+x^3+x^2+x+1, x^2+1, x^4+4x^3-x^2-x+1, x+1, x^4-x^3-x^2+4x+1, x^4+3x^3-3x^2-3x+1, x^2-3x+1$ and $x^4-4x^3-x^2+x+1$ respectively.

The minimal codes $M_0, M_1, M_2, M_4, M_5, M_8, M_{10}, M_{11}, M_{20}, M_{21}, M_{22}, M_{25}$ and M_{31} of length 40 are as follows:

Code	Dimension	Minimum Distance	Generating Polynomial
M_0	1	40	$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{17} + x^{18} + x^{19} + x^{20} + x^{21} + x^{22} + x^{23} + x^{24} + x^{25} + x^{26} + x^{27} + x^{28} + x^{29} + x^{30} + x^{31} + x^{32} + x^{33} + x^{34} + x^{35} + x^{36} + x^{37} + x^{38} + x^{39}$
M_1	4	$8 \leq d \leq 32$	$6 + 3x + 4x^2 + 6x^3 + 5x^4 + 5x^5 + x^6 + 5x^7 + 4x^8 + x^9 + 2x^{10} + 4x^{12} + 6x^{12} + 5x^{14} + x^{15} + 6x^{16} + x^{20} + 4x^{21} + 3x^{22} + x^{23} + 2x^{24} + 2x^{25} + 6x^{26} + 2x^{27} + 3x^{28} + 6x^{29} + 5x^{30} + 3x^{32} + x^{33} + 2x^{34} + 6x^{35} + x^{36}$
M_2	4	$8 \leq d \leq 24$	$6 + 3x + 2x^2 + 2x^4 + 4x^5 + 6x^6 + x^{10} + 4x^{11} + 5x^{12} + 5x^{14} + 3x^{15} + x^{16} + 6x^{20} + 3x^{21} + 2x^{22} + 2x^{24} + 2x^{25} + 6x^{26} + x^{30} + 4x^{31} + 5x^{32} + 5x^{34} + 3x^{35} + x^{36}$
M_4	4	16	$-1 - x + x^5 + x^6 - x^{10} - x^{11} + x^{15} + x^{16} - x^{20} - x^{21} + x^{25} + x^{26} - x^{30} - x^{31} + x^{35} + x^{36}$
M_5	2	30	$6 + 3x + 6x^2 + x^4 + 4x^5 + x^6 + 6x^8 + 3x^9 + 6x^{10} + x^{12} + 4x^{13} + x^{14} + 6x^{16} + 3x^{17} + 6x^{18} + x^{20} + 4x^{21} + x^{22} + 6x^{24} + 3x^{25} + 6x^{26} + x^{28} + 4x^{29} + x^{30} + 6x^{32} + 3x^{33} + 6x^{34} + x^{36} + 4x^{37} + x^{38}$
M_8	4	16	$-1 + x - x^5 + x^6 - x^{10} + x^{11} - x^{15} + x^{16} - x^{20} + x^{21} - x^{25} + x^{26} - x^{30} + x^{31} - x^{35} + x^{36}$
M_{10}	2	20	$-1 + x^2 - x^4 + x^6 - x^8 + x^{10} - x^{12} + x^{14} - x^{16} + x^{18} - x^{20} + x^{22} - x^{24} + x^{26} - x^{28} + x^{30} - x^{32} + x^{34} - x^{36} + x^{38}$
M_{11}	4	$8 \leq d \leq 32$	$6 + 6x + 5x^2 + x^3 + 4x^4 + 2x^6 + 6x^7 + 4x^8 + 2x^9 + x^{10} + 2x^{11} + 5x^{12} + x^{13} + 4x^{14} + 4x^{15} + 6x^{16} + x^{20} + x^{21} + 2x^{22} + 6x^{23} + 3x^{24} + 5x^{26} + x^{27} + 3x^{28} + 5x^{29} + 6x^{30} + 5x^{31} + 2x^{32} + 6x^{33} + 3x^{34} + 3x^{35} + x^{36}$
M_{20}	1	40	$-1 + x - x^2 + x^3 - x^4 + x^5 - x^6 + x^7 - x^8 + x^9 - x^{10} + x^{11} - x^{12} + x^{13} - x^{14} + x^{15} - x^{16} + x^{17} - x^{18} + x^{19} - x^{20} + x^{21} - x^{22} + x^{23} - x^{24} + x^{25} - x^{26} + x^{27} - x^{28} + x^{29} - x^{30} + x^{31} - x^{32} + x^{33} - x^{34} + x^{35} - x^{36} + x^{37} - x^{38} + x^{39}$
M_{21}	4	$8 \leq d \leq 32$	$6 + 4x + 4x^2 + x^3 + 5x^4 + 2x^5 + x^6 + 2x^7 + 4x^8 + 6x^9 + 2x^{10} + 4x^{12} + x^{13} + 5x^{14} + 6x^{15} + 6x^{16} + x^{20} + 3x^{21} + 3x^{22} + 6x^{23} + 2x^{24} + 5x^{25} + 6x^{26} + 5x^{27} + 3x^{28} + x^{29} + 5x^{30} + 3x^{32} + 6x^{33} + 2x^{34} + x^{35} + x^{36}$

Code	Dimension	Minimum Distance	Generating Polynomial
M_{22}	4	$8 \leq d \leq 24$	$6+4x+2x^2+2x^4+3x^5+6x^6+x^{10}+3x^{11}+5x^{12}+5x^{14}+4x^{15}+x^{16}+6x^{20}+4x^{21}+2x^{22}+2x^{24}+3x^{25}+6x^{26}+x^{30}+3x^{31}+5x^{32}+5x^{34}+4x^{35}+x^{36}$
M_{25}	2	30	$6+4x+6x^2+x^4+3x^5+x^6+6x^8+4x^9+6x^{10}+x^{12}+3x^{13}+x^{14}+6x^{16}+4x^{17}+6x^{18}+x^{20}+3x^{21}+x^{22}+6x^{24}+4x^{25}+6x^{26}+x^{28}+3x^{29}+x^{30}+6x^{32}+4x^{33}+6x^{34}+x^{36}+3x^{37}+x^{38}$
M_{31}	4	$8 \leq d \leq 32$	$6+x+5x^2+6x^3+4x^4+2x^6+x^7+4x^8+5x^9+x^{10}+5x^{11}+5x^{12}+6x^{13}+4x^{14}+3x^{15}+6x^{16}+x^{20}+6x^{21}+2x^{22}+x^{23}+3x^{24}+5x^{26}+6x^{27}+3x^{28}+2x^{29}+6x^{30}+2x^{31}+2x^{32}+x^{33}+3x^{34}+4x^{35}+x^{36}$

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