COUETTE FLOW OF A GAS BY ELZAKI TRANSFORM

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Abstract: Unsteady couette flow of a gas between two parallel infinite plates have been studied in this paper. We proposed a new transform technique called Elzaki transform. Analytical solution is obtained for the equation of motion by this method. Although the Elzaki transform have a close connection with the Laplace transform, Elzaki transform is easy to employ and has got several advantages over Laplace transform method.

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1. Introduction

Couette flow is frequently used in Physics and Engineering. Couette flow between parallel plates is a classical problem that has an important applications in power generators and pumps etc. Several investigators have been done this type of flow. But still it seems that not much work have been done. The Taylor Couette flow using DNS/SUV method was studied by Tuluszka-Szhilko et al [1]. Haider Zaman et al [2] discussed the Couette flow problem for an unsteady MHD fourth - grade fluid with Hall currents. Couette flow of a binary mixture of rigid sphere gases described by the linearized Boltzman equation was discussed by Gracia et al [3]. Risso et al[4] analysed the dilute gas Cou-

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ette flow theory and molecular dynamics simulation. Debashish Sircar et al [5] discussed the flow and heat transfer in a micro-channel gas-liquid couette flow. The unsteady and steady velocity profile for couette flow was presented and solved exactly using the numerical method by Rihan [6]. Tarig Elzaki et al [7] studied the fundamental properties and application of Elzaki transform. The connection between Laplace and Elzaki transform was discussed by Tarig Elzaki [8]. Hayat et al [9] used Laplace transform method to determine the analytical solutions of couette flows of a second grade fluid. Erdogan et al [10] solved unsteady couette flow for viscous fluid by Laplace transform method. Vennila [11] studied Elzaki Transform for Rotating Fluid.

2. Mathematical Analysis

Consider the motion of a fluid between two infinite parallel plates. At time $t < 0$, both the fluid and the plates are assumed to be at rest. At $t = 0$, the lower plate begins to move in its own plane with a velocity proportional to U and the upper plate at a distance l apart, remains fixed.

We choose a Cartesian coordinate system of axes with x-axis along the lower plate, y-axis perpendicular to it at z-axis lying on the plate.

The equation of motion can be written as

$$\begin{aligned}\frac{\partial u}{\partial t} + (u \cdot \nabla)u &= -\frac{1}{\rho} \text{grad} p + \nu \nabla^2 u \\ \text{div}(u) &= 0\end{aligned}\quad (1)$$

Equation (1) reduced to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad (2)$$

Boundary conditions are

$$\begin{cases} u = U, & y = 0 \text{ for } t > 0 \\ u = 0 \text{ at } y = l & \text{for } t > 0 \\ u = 0 \text{ in } 0 \leq y \leq l & \text{for } t \leq 0 \end{cases} \quad (3)$$

Let $\overline{T}(y, v)$ be the Elzaki transform of $u(y, t)$, (i.e.) $E[u(y, t)] = \overline{T}(y, v)$.

Then taking the Elzaki transform on both side of equation (2), we have

$$\frac{d^2 \overline{T}}{dy^2} - \frac{1}{\nu} \left\{ \frac{\overline{T}(y, v)}{v} - vu(y, 0) \right\} = 0 \quad (4)$$

Using condition equation (3) in equation (4), we obtain

$$\frac{d^2 \bar{T}}{dy^2} - \frac{1}{\nu v} \bar{T}(y, v) = 0 \quad (5)$$

On solving the above equation, we get

$$\bar{T}(y, v) = C_1 e^{(\frac{1}{\sqrt{\nu v}})y} + C_2 e^{-(\frac{1}{\sqrt{\nu v}})y} \quad (6)$$

Applying Elzaki transform of the boundary conditions, we obtain

$$\bar{T}(0, v) = Uv^2, \quad \bar{T}(l, v) = 0, \quad \bar{T}(y, 0) = 0 \quad (7)$$

Apply equation (7) in equation (6), we get

$$C_1 = \frac{Uv^2}{1 - e^{\frac{2l}{\sqrt{\nu v}}}}; \quad C_2 = -\frac{Uv^2 e^{\frac{l}{\sqrt{\nu v}}}}{1 - e^{\frac{2l}{\sqrt{\nu v}}}} \quad (8)$$

Substituting these values in equation ((6)), we obtain

$$\bar{T}(y, v) = -Uv^2 \left[\frac{e^{\frac{-l}{\sqrt{\nu v}}} e^{\frac{l}{\sqrt{\nu v}} y}}{e^{\frac{l}{\sqrt{\nu v}}} - e^{\frac{-l}{\sqrt{\nu v}}}} \right] + Uv^2 \left[\frac{e^{\frac{l}{\sqrt{\nu v}}} e^{-(\frac{l}{\sqrt{\nu v}})y}}{e^{\frac{l}{\sqrt{\nu v}}} - e^{\frac{-l}{\sqrt{\nu v}}}} \right] \quad (9)$$

$$\bar{T}(y, v) = Uv^2 \left[\frac{\sinh \left(\frac{l-y}{\sqrt{\nu v}} \right)}{\sinh \left(\frac{l}{\sqrt{\nu v}} \right)} \right] \quad (10)$$

$$u(y, t) = E^{-1} \left\{ Uv^2 \left[\frac{\sinh \left(\frac{l-y}{\sqrt{\nu v}} \right)}{\sinh \left(\frac{l}{\sqrt{\nu v}} \right)} \right] \right\} \quad (11)$$

The inverse transform E^{-1} of the right hand side terms in equation (8) may be determined by recalling a standard result as [7,8]

$$E^{-1}(\bar{T}(y, v)) = \sum \text{residue of } \left[e^{sT} s \bar{T} \left(y, \frac{1}{s} \right) \right] \quad (12)$$

(i.e.) $\bar{T} \left(y, \frac{1}{s} \right)$ has pole at $s = 0$ and other pole at $s = \frac{-n^2 \pi^2 \nu}{l^2}$, $n = 0, \pm 1, \pm 2, \dots$.

Now we calculate residues one by one.

Residue of $e^{sT} s \bar{T} \left(y, \frac{1}{s} \right)$ at $s = 0$ is

$$\lim_{s \rightarrow 0} \frac{d}{ds} \left\{ \frac{(s-0)^2 e^{st} s \sinh \left(\sqrt{\frac{s}{\nu}} (l-y) \right)}{s^2 \sinh \left(\sqrt{\frac{s}{\nu}} l \right)} \right\} = \frac{l-y}{l}.$$

Residue of $e^{sT} s \bar{T} \left(y, \frac{1}{s} \right)$ at $s = s_n$ is

$$\begin{aligned} & \lim_{s \rightarrow s_n} \frac{s e^{st} \sinh \left(\sqrt{\frac{s}{\nu}} (l-y) \right)}{s^2 \frac{d}{ds} \left[\sinh \left(\sqrt{\frac{s}{\nu}} l \right) \right]} \\ &= \frac{e^{-\left(\frac{n^2 \pi^2 \nu}{l^2}\right)t} \sinh \left(\frac{n\pi i}{l} (l-y) \right) 2\sqrt{\nu}}{\frac{\ln \pi i \sqrt{\nu}}{l} \cosh \left(\frac{n\pi i}{l} \frac{\sqrt{\nu}}{\sqrt{\nu}} l \right)} = - \frac{2e^{-\left(\frac{n^2 \pi^2 \nu}{l^2}\right)T} \sin \frac{n\pi y}{l}}{n\pi}. \end{aligned}$$

Putting all these residues in (11), we receive

$$u(y, t) = U \left\{ \frac{l-y}{l} + \frac{\sum_{n=1}^{\infty} (-2) e^{-\left(\frac{n^2 \pi^2 \nu}{l^2}\right)t} \sin \frac{n\pi y}{l}}{n\pi} \right\}.$$

3. Conclusion

In this paper analytical solution was obtained for the equation of motion for unsteady couette flow between two parallel infinite plates one of them was stationary and the other plate moved with constant velocity by Elzaki transform method.

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