NUMERICAL SOLUTION OF UNSTEADY FLOW BETWEEN PARALLEL PLATES USING HAAR WAVELET-QUASILINEARIZATION METHOD

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Abstract: The present study is concerned with the analysis of unsteady flow of viscous incompressible fluid between parallel rectangular and circular plates. The Navier-Stokes equations modelling the flow are transformed into a fourth order non-linear ordinary differential equation using a similarity transformation. The resulting equation is solved by a new efficient numerical scheme based on Haar wavelet-quasilinearization method. The solution representing pressure gradient is obtained for large values of Reynolds number. A comparison with numerical results generated by MATLAB routine `bvp4c` demonstrates that the proposed method is more efficient and promising technique for solving highly nonlinear two-point boundary value problems. The absolute error and rate of convergence of numerical results are computed to test the applicability and accuracy of the method.

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Key Words: Haar wavelets, quasilinearization, operational matrix, squeezing flow

1. Introduction

Unsteady flow in channels, pipes and fluid flow near an oscillating plate have been intensively studied over the past few years due to its practical importance in understanding several engineering and physiological flow problems. For in-

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stance, such flow of significance in lubrication is the squeezing flow between two parallel plates [1, 2, 3]. The problem of unsteady squeezing of viscous fluid between two plates occurs in unsteady loading. In the light of this application, many researchers have explored the effect of inertia on the bearing characteristics [4]. Chandrashekhara and Ramanaiah [5] investigated unsteady flow between two parallel rectangular and circular plates by considering all inertia terms and combined effect of inertia force is studied. Such problems admits similarity solution there by reducing unsteady Navier-Stokes equations to non-linear ordinary differential equation. The non-linear nature of the modelled equation in most cases precludes its exact solution. Therefore different numerical and semi-numerical methods have been applied for providing approximate solutions such as Finite difference method (FDM), Finite element method (FEM), Perturbation method [6], Adomian decomposition method (ADM) [7], Homotopy analysis method (HAM) [8] etc. The classical discretization methods are powerful tools for solving differential equation but method fails to give the solution when singularity exists. Similarly, semi numerical methods, like perturbation, homotopy analysis methods requires perturbation parameter or the solution sought out to confirm the so-called rule of solution expression and the rule of coefficient ergodicity. That is where wavelets play an important role in obtaining the approximate solutions of such problems.

Several researchers [9, 10, 11, 12, 13] have contributed to the present growth of mathematical analysis of different types of wavelets. In particular, Chen and Hasio [14] proposed the Haar wavelet which is mathematically the simplest among all the wavelet families. Following this pioneering work, Lepik [15, 16, 17, 18] introduced the numerical solution of differential equation using Haar Wavelet. A comprehensive survey on Haar wavelet methods are given in a paper by Hariharan and Kannan[19]. In a recent study, Kaur et al.[20] proposed a modified Haar wavelet method by introducing the quasilinearization approach to overcome the difficulties in solving the non-linear boundary value problems. Later, Saeed and Rehman [21] successfully used the Haar wavelet quasilinearization approach for the solution of non-linear fractional differential equations.

In the present study, we consider the problem of the unsteady flow of a viscous incompressible fluid between two parallel infinitely long rectangular plates and two parallel circular plates [5]. Such flow situations are frequently come across in lubrication and known as squeeze films. The objective of the paper is to discuss about flow structure for large Reynolds number α . In this paper, we employ an efficient numerical scheme based on Haar wavelet in conjunction with quasilinearization process to tackle a fourth order non-linear differential

Figure 1: Schematic diagram of the problem (parallel rectangular plates or circular plates).

equation that governs the above problem. In the earlier study [22], this problem was analysed by perturbation method which is restricting the solution for moderately large values of α . The proposed method does not pose any restriction on the parameter α .

This paper is organized as follows: In Section 2, formulation of the problem is given. Section 3 contains the review of essential features of Haar wavelets, while in Section 4 basic ideas and principles of quasilinearization are discussed. Section 5 is devoted to implementation of the scheme to the given problem and efficiency of the method is discussed. The obtained results are discussed in Section 6 and we conclude our work in Section 7.

2. Mathematical Formulation of the Problem

2.1. Two-Dimensional Flow

Consider the viscous incompressible fluid between two parallel infinitely long rectangular plates of width $2a$ separated by a small distance $h(t)$ as shown in Figure 1. It is assumed that the upper plate is moving downwards and the lower plate remains fixed [5]. Let u and w be the velocity components of the fluid in the x and z directions respectively. The two-dimensional governing equations of continuity and of motion are:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (2)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) \quad (3)$$

corresponding boundary conditions are

$$\begin{aligned} u &= w = 0 & \text{at } z = 0 \\ u &= 0, \quad w = \dot{h} & \text{at } z = h \end{aligned} \quad (4)$$

Introducing the similarity transformations [5]:

$$w = -\psi(z, t) = -\frac{\nu}{h} F(\eta) \quad (5)$$

$$u = x \frac{\partial \psi}{\partial z} = \frac{x\nu F'}{h^2} \quad (6)$$

where $\eta = \frac{z}{h(t)}$, we obtain

$$\frac{\partial p}{\partial x} = -\frac{\rho\nu^2 x}{h^4} (\alpha\eta F'' + 2\alpha F' + F'^2 - FF'' - F''') \quad (7)$$

$$\frac{\partial p}{\partial \eta} = -\frac{\rho\nu^2}{h^2} (-\alpha\eta F' - \alpha F + FF' + F'') \quad (8)$$

where $\alpha = -\frac{h\dot{h}}{\nu}$ is the local Reynolds number. Eliminating the pressure from (7) and (8), we obtain

$$F'''' + FF''' - \alpha\eta F''' - F'F'' - 3\alpha F'' = 0 \quad (9)$$

The relevant boundary conditions are

$$\begin{aligned} F &= F' = 0 & \text{at } \eta = 0 \\ F &= \alpha, \quad F' = 0 & \text{at } \eta = 1 \end{aligned} \quad (10)$$

2.2. Axisymmetric Flow

Axisymmetric flow between two parallel circular plates of radius a separated by a small distance $h(t)$ is shown in Figure 1. The upper plate moves towards the lower plate which is fixed. In cylindrical coordinates (x, θ, z) , with velocity components u and w in the radial and axial directions respectively [5]. The axisymmetric flow equations are:

$$\frac{\partial}{\partial x}(xu) + \frac{\partial}{\partial z}(xw) = 0 \quad (11)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{x^2} \right) \quad (12)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{x} \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial z^2} \right) \quad (13)$$

with the following boundary conditions

$$\begin{aligned} u &= w = 0 & \text{at } z &= 0 \\ u &= 0, \quad w = \dot{h} & \text{at } z &= h \end{aligned} \quad (14)$$

Using the similarity transformations [5]:

$$w = -\frac{2\nu}{h} F(\eta) \quad (15)$$

$$u = \frac{x\nu F'}{h^2} \quad (16)$$

where $\eta = \frac{z}{h(t)}$, the above equations reduces to:

$$F'''' + 2FF''' - \alpha\eta F''' - 3\alpha F'' = 0 \quad (17)$$

and the boundary conditions are

$$\begin{aligned} F &= F' = 0 & \text{at } \eta &= 0 \\ F &= \alpha/2, \quad F' = 0 & \text{at } \eta &= 1 \end{aligned} \quad (18)$$

3. Haar Wavelets

Wavelets are generally constructed through multiresolution analysis (MRA). Mathematically, MRA deals with the decomposition of whole function space into individual subspaces $V_j \subset V_{j+1}$ so that the space V_{j+1} consists of all rescaled functions in V_j . This means, a decomposition of each function into components of different scales so that an individual component of the original function f occurs in each subspace [23]. These components can describe finer and finer versions of the original function f .

Definition: A multiresolution analysis(MRA) is a sequence $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$, such that the following properties are satisfied:

- (a) The sequence is nested, i.e, for all $j \in \mathbb{Z}$, $V_j \subset V_{j+1}$
- (b) The spaces are related to each other by dyadic scaling, i.e, $f(x) \in V_j$ iff $f(\frac{x}{2^j}) \in V_0$ (or $f(x) \in V_0$ iff $f(2^j x) \in V_j$)
- (c) The union $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L_2(\mathbb{R})$, i.e, $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L_2(\mathbb{R})$
- (d) The intersection of the spaces is reduced to the null function, i.e, $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$
- (e) There exists a function $\phi \in V_0$ such that the set $\{\phi_{0,k} = \phi(x - k)\}_{k \in \mathbb{Z}}$ is a Riesz basis for V_0 .

The function ϕ is called the scaling function.

The Haar scaling function is defined as

$$h_1(x) = \begin{cases} 1, & x \in [0, 1) \\ 0, & \text{elsewhere} \end{cases}$$

which forms a square wave with unit magnitude. The Haar mother wavelet is obtained as the linear combination of the Haar scaling function:

$$\begin{aligned} h_2(x) &= h_1(2x) - h_1(2x - 1) \\ &= \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ -1, & \frac{1}{2} \leq x < 1 \\ 0, & \text{elsewhere} \end{cases} \end{aligned} \quad (19)$$

All the other subsequent members in Haar wavelet family are constructed from translation and dilation of $h_2(x)$:

$$\begin{aligned} h_i(x) &= h_2(2^j x - k) \\ &= \begin{cases} 1, & x \in [\xi_1(i), \xi_2(i)) \\ -1, & x \in [\xi_2(i), \xi_3(i)) \\ 0, & \text{elsewhere} \end{cases} \end{aligned} \quad (20)$$

where

$$\xi_1(i) = \frac{k}{m}, \quad \xi_2(i) = \frac{k + 0.5}{m}, \quad \xi_3(i) = \frac{k + 1}{m}$$

and $j = 0, 1, \dots, J$ stand for the dilation parameter. As j increases, the support decreases. The translation parameter $k = 0, 1, \dots, m - 1$ indicates the location of the particular square wave ($m = 2^j$) and $i = m + k + 1$ is the wavelet number.

The minimum value of i is 2 (i.e. $m = 1, k = 0$) and the maximum value of i is $2M$ where $M = 2^J$.

The integrals of the Haar functions are

$$\begin{aligned}
 p_{1,i}(x) &= \int_0^x h_i(t) dt, \\
 &= \begin{cases} x - \xi_1(i) & \text{for } x \in [\xi_1(i), \xi_2(i)), \\ \xi_3(i) - x & \text{for } x \in [\xi_2(i), \xi_3(i)), \\ 0 & \text{elsewhere} \end{cases} \\
 p_{2,i}(x) &= \int_0^x p_{1,i}(t) dt, \\
 &= \begin{cases} \frac{1}{2}(x - \xi_1(i))^2 & \text{for } x \in [\xi_1(i), \xi_2(i)), \\ \frac{1}{4m^2} - \frac{1}{2}(\xi_3(i) - x)^2 & \text{for } x \in [\xi_2(i), \xi_3(i)), \\ \frac{1}{4m^2} & \text{for } x \in [\xi_3(i), 1) \\ 0 & \text{elsewhere} \end{cases}
 \end{aligned}$$

Similarly,

$$p_{l+1,i}(x) = \int_0^x p_{l,i}(t) dt, \quad l = 2, 3, \dots$$

To apply the Haar wavelets for the numerical solutions one must put them into a discrete form using collocation method in which the number of collocation points are doubled at each iteration [18]. The collocation points are $x_l = (l - 0.5)\Delta x$, $l = 1, 2, \dots, 2M$, $\Delta x = 1/2M$.

The results can be formulated into matrix form as $H(i, l) = h_i(x_l)$, $P_\alpha(i, l) = p_{\alpha,i}(x_l)$, $\alpha = 1, 2, \dots$. These $2M \times 2M$ Haar matrices are sparse matrices and thus implementation of algorithms will be quite easier.

As the Haar wavelets forms a basis for $L^2[0, 1)$, any function $y(x)$ which is square integrable in the interval $[0, 1)$ can be decomposed as

$$y(x) = \sum_{i=1}^{\infty} a_i h_i(x), \quad (21)$$

where a_i are the Haar wavelet coefficients given as

$$a_i = 2^j \int_0^1 y(x) h_i(x) dx.$$

The series expansion of $y(x)$ can be terminated at finite terms if $y(x)$ is a piecewise constant or it may be approximated as a piecewise constant for each subinterval.

4. Quasilinearization

The concept of quasilinearization technique was introduced by Bellman and Kalaba [24] for obtaining numerical solutions of individual or system of nonlinear differential equations. In quasilinearization technique, the nonlinear differential equation is solved recursively by a series of linear differential equations. This method not only linearizes the nonlinear equation but also provides a sequence of functions that converges to the solution of the original nonlinear equation which is very essential in practice. To explain the quasilinearization process let us consider the nonlinear r^{th} order differential equation in the form

$$L^{(r)}y(x) = f(y(x), y^{(1)}(x), y^{(2)}(x), \dots, y^{(r-1)}(x), x) \quad (22)$$

with r boundary conditions

$$g_k(y(a), y^{(1)}(a), y^{(2)}(a), \dots, y^{(r-1)}(a)) = 0, \quad k = 1, \dots, l$$

and

$$g_k(y(b), y^{(1)}(b), y^{(2)}(b), \dots, y^{(r-1)}(b)) = 0, \quad k = l + 1, \dots, r.$$

Here f is nonlinear function of $y(x)$ and its $r - 1$ derivatives are $y^{(j)}(x)$, $j = 1, 2, \dots, r - 1$ and $L^{(r)}$ is the linear r^{th} order differential operator. The quasilinearization procedure determines the $(n + 1)^{\text{th}}$ iterative approximation $y_{n+1}(x)$ to the solution of r^{th} order nonlinear ordinary differential equation (22) as a solution of linear differential equation

$$\begin{aligned} L^{(r)}y_{n+1}(x) &= f(y_n(x), y_n^{(1)}(x), y_n^{(2)}(x), \dots, y_n^{(r-1)}(x), x) \\ &+ \sum_{j=0}^{r-1} (y_{n+1}^{(j)}(x) - y_n^{(j)}(x)) f_{y^{(j)}}(y_n(x), y_n^{(1)}(x), \dots, y_n^{(r-1)}(x), x) \end{aligned}$$

where $y_n^0(x) = y_n(x)$. The functions $f_{y^{(j)}} = \frac{\partial f}{\partial y^{(j)}}$ are functional derivatives of the functional

$$f(y(x), y^{(1)}(x), y^{(2)}(x), \dots, y^{(r-1)}(x), x) = 0$$

The general conditions for the convergence of the quasilinearization method for solving nonlinear ordinary differential equations derived in [25]. In this method, the zeroth approximation $y_0(x)$ can be obtained from physical or mathematical considerations of given problem.

5. Method of Solution

Equations (9) and (17) with the respective boundary conditions can be put in the general form as

$$F'''' + AFF''' - \alpha\eta F''' - BF'F'' - 3\alpha F'' = 0 \quad (23)$$

subject to conditions

$$\begin{aligned} F &= F' = 0 & \text{at } \eta = 0 \\ F &= C\alpha, \quad F' = 0 & \text{at } \eta = 1 \end{aligned} \quad (24)$$

where $A = 1, B = 1, C = 1$ corresponds to two-dimensional flow and $A = 2, B = 0, C = \frac{1}{2}$ for axisymmetric flow.

On implementing quasilinearization method, (23) reduces to the linearized form as

$$\begin{aligned} F_{n+1}'''' + (AF_n - \alpha\eta)F_{n+1}''' - (BF_n' + 3\alpha)F_{n+1}'' - BF_n''F_{n+1}' + AF_n'''F_{n+1} \\ = AF_nF_n''' - BF_n'F_n'' \end{aligned} \quad (25)$$

with corresponding boundary conditions

$$\begin{aligned} F_{n+1}(0) &= F_{n+1}'(0) = 0 \\ F_{n+1}(1) &= C\alpha, \quad F_{n+1}'(1) = 0 \end{aligned} \quad (26)$$

We seek the solution of (25) by using the approach recommended by Chen and Hsiao [14]. That is, instead of solution of the differential equation its highest derivative is expanded into the Haar series.

$$F_{n+1}''''(\eta) = \sum_{i=1}^{2M} a_i h_i(\eta) \quad (27)$$

This approximation is integrated while the boundary conditions are incorporated by using integration constants. We derive the lower order derivatives as follows:

$$F_{n+1}'''(\eta) = \sum_{i=1}^{2M} a_i p_{1,i}(\eta) + F_{n+1}^{(3)}(0) \quad (28)$$

$$F''_{n+1}(\eta) = \sum_{i=1}^{2M} a_i p_{2,i}(\eta) + F_{n+1}^{(2)}(0) + \eta F_{n+1}^{(3)}(0) \quad (29)$$

$$F'_{n+1}(\eta) = \sum_{i=1}^{2M} a_i p_{3,i}(\eta) + \eta F_{n+1}^{(2)}(0) + \frac{\eta^2}{2} F_{n+1}^{(3)}(0) \quad (30)$$

$$F_{n+1}(\eta) = \sum_{i=1}^{2M} a_i p_{4,i}(\eta) + \frac{\eta^2}{2} F_{n+1}^{(2)}(0) + \frac{\eta^3}{6} F_{n+1}^{(3)}(0) \quad (31)$$

The unknown terms $F_{n+1}^{(2)}(0)$, $F_{n+1}^{(3)}(0)$ in the above equations are evaluated and given by

$$F_{n+1}^{(2)}(0) = 6C\alpha + 2 \sum_{i=1}^{2M} a_i E_{2,i} - 6 \sum_{i=1}^{2M} a_i E_{3,i} \quad (32)$$

$$F_{n+1}^{(3)}(0) = -12C\alpha - 6 \sum_{i=1}^{2M} a_i E_{2,i} + 12 \sum_{i=1}^{2M} a_i E_{3,i} \quad (33)$$

where

$$E_{2,i} = \int_0^1 p_{2,i}(t) dt$$

$$E_{3,i} = \int_0^1 p_{3,i}(t) dt$$

Substituting (27)-(33) in (25) and simplifying we obtain the following system of $2M$ number of algebraic equations:

$$\begin{aligned} & \sum_{i=1}^{2M} a_i \left[h_i(\eta) + (AF_n(\eta) - \alpha\eta) \left(p_{1,i}(\eta) - 6E_{2,i} + 12E_{3,i} \right) - (BF'_n(\eta) + 3\alpha) \right. \\ & \left(p_{2,i}(\eta) + 2E_{2,i} - 6E_{3,i} - 6\eta E_{2,i} + 12\eta E_{3,i} \right) - BF''_n(\eta) \left(p_{3,i}(\eta) + 2\eta E_{2,i} - 6\eta E_{3,i} \right. \\ & \left. - 3\eta^2 E_{2,i} + 6\eta^2 E_{3,i} \right) + AF'''_n(\eta) \left(p_{4,i}(\eta) + \eta^2 E_{2,i} - 3\eta^2 E_{3,i} - \eta^3 E_{2,i} + 2\eta^3 E_{3,i} \right) \Big] \\ & = 18C\alpha^2 - 48C\alpha^2\eta + 12AC\alpha F_n(\eta) + 6BC\alpha F'_n(\eta) - 12BC\alpha\eta F'_n(\eta) \\ & + 6BC\alpha\eta F''_n(\eta) - 6BC\alpha\eta^2 F''_n(\eta) - BF'_n(\eta) F''_n(\eta) - 3AC\alpha\eta^2 F'''_n(\eta) \\ & + 2AC\alpha\eta^3 F'''_n(\eta) + AF_n(\eta) F'''_n(\eta) \end{aligned} \quad (34)$$

With the appropriate initial approximations, the above system is put into the matrix form by introducing the row vectors $\eta = [\eta(l)]$, $a = [a_i]$ and the following $2M \times 2M$ matrices: $H = [h_i(\eta)]$, $P_1 = [p_{1,i}(\eta_l)]$, $P_2 = [p_{2,i}(\eta_l)]$, $P_3 = [p_{3,i}(\eta_l)]$, $P_4 = [p_{4,i}(\eta_l)]$. Solving the system (34) using MATLAB, we obtain the wavelet solution for different

level of resolutions which are displayed in Tables 1 and 3.

The applicability and efficiency of the Haar wavelet-quasilinearization method is demonstrated by estimating the error in the solution. The order of convergence [26] for n th order ordinary differential equation using Haar wavelets is $O\left[\left(\frac{1}{2^{J+1}}\right)^2\right]$, where J is the resolution level. However, in the proposed study, the exact solution of the problem is not known and therefore we estimate the accuracy of the obtained approximate solution by using the following relation [16]

$$\Delta(v) = \left| \frac{S(v+1)}{S(v)} - 1 \right|$$

where $S(v) = \Delta\eta_{(v)} \sum_{i=1}^{2M} |F_{(v)}(\eta_i)|$ and $\Delta\eta_{(v)}$ is the step size of the v^{th} approximation. The convergence rate of the solution can be determined with the aid of the function

$$\sigma(v) = \frac{\Delta(v-1)}{\Delta(v)}, \quad v = 2, 3, \dots$$

The absolute relative error and the rate of convergence for different values of J are shown in Tables 2 and 4. We also solved this problem by using MATLAB routine `bvp4c`, which is based on an adaptive Lobatto quadrature scheme [27], for comparing our results. The solution shown in Table 1 and 3 was obtained with the relative error tolerance on the residuals, $\text{RelTol}=10^{-6}$.

6. Results and Discussion

In this work, the novelty of Haar wavelet-quasilinearization method (HWQLM) in obtaining solution for fourth order non-linear ordinary differential equation representing unsteady flow between parallel plates is presented. The Haar series (31) gives the solution of the governing problem. The present analysis enables in obtaining the converging solution for the given governing problem. We have computed pressure gradient representing the series (28) for much larger value of α . In this method we could able to find the solution for $\alpha \geq 300$ whereas in earlier findings [5, 22] they have obtained the solution only up to $\alpha = 4$ and 60 respectively due to its limited validity and slow convergence of the series. The advantage of this method is that it has no restriction on the parameter α . To check the accuracy of the HWQLM, comparison is made with the numerical solution obtained by MATLAB routine `bvp4c` (which fails to give converging solution for $\alpha > 180$) and an excellent agreement was found. This reveals that the proposed method is more efficient and superior to the traditional numerical methods for the solution of nonlinear higher order boundary value problems. Moreover, quasilinearization process significantly reduces the number of iterations to obtain required accuracy of the solution (only four iterations). Comparison between the proposed Haar wavelet method (at different resolution levels J), `bvp4c` and previous findings ([5, 22]) for pressure gradient, $F'''(0)$, at selected α values are shown in Table 1 and 3. We notice that pressure gradient increases considerably with increase in α in two-dimensional

Table 1: Comparison of pressure gradient ($-F'''(0)$) obtained using HWQLM, bvp4c with other methods (two-dimensional flow)

α	Pressure Gradient $(-F'''(0))$					
	HWQLM		bvp4c	Computer Ex- tended Series[22]	Brown's Method[22]	Numerical[5]
	$J = 8$	$J = 10$				
0	0.0	0.0	0.0	0.0	0.0	0.0
1	16.0729	16.0729	16.0729	16.07	16.07	16.15
3	72.0225	72.0225	72.0225	72.02	72.02	72.04
4	111.5549	111.5547	111.5547	111.55	102.63	111.43
5	158.6126	158.6123	158.6123	158.61	-	-
10	503.5355	503.5326	503.5324	503.53	-	-
20	1719.8287	1719.8060	1719.8045	1719.80	-	-
30	3614.2306	3614.1552	3614.1501	3614.14	-	-
40	6171.9866	6171.8101	6171.7983	6171.77	-	-
50	9384.5178	9384.1764	9384.1536	9384.11	-	-
60	13246.0351	13245.4498	13245.4107	13245.07	-	-
70	17752.2880	17751.3646	17751.3030	-	-	-
80	22899.9830	22898.6120	22898.5205	-	-	-
90	28686.4703	28684.5272	28684.3975	-	-	-
100	35109.5588	35106.9041	35106.7268	-	-	-
120	49858.3953	49853.8387	49853.5343	-	-	-
150	76717.2877	76708.4578	76707.8674	-	-	-
180	109233.1977	109218.0328	-	-	-	-
200	134042.2171	134021.4862	-	-	-	-
220	161350.4607	161322.9521	-	-	-	-
240	191153.4959	191117.8805	-	-	-	-
260	223447.4728	223402.3043	-	-	-	-
280	258229.0098	258172.7251	-	-	-	-
300	295495.1078	295426.0274	-	-	-	-
$\downarrow$						

flow and axisymmetric flow. Also, for higher wavelet resolution level J , the method gives more accurate solutions. To validate our results obtained, we performed error analysis [16] and analysed that by doubling the number of collocation points the error roughly decreases four times. This theoretical estimation demonstrates the convergence of the numerical solution obtained by proposed method, which are shown in Table 2 and 4 for different values of α .

Table 2: Error estimates Δ and convergence rate σ for two-dimensional flow case

J	$\alpha = 10$		$\alpha = 20$		$\alpha = 40$		$\alpha = 60$	
	Δ	σ	Δ	σ	Δ	σ	Δ	σ
4	1.78E-16	-	7.08E-13	-	1.26E-10	-	1.18E-09	-
5	1.78E-16	1.00	1.54E-13	4.59	2.13E-11	5.90	1.28E-10	9.25
6	0	-	3.77E-14	4.10	4.88E-12	4.37	2.71E-11	4.72
7	1.07E-15	0.00	9.41E-15	4.00	1.20E-12	4.08	6.51E-12	4.16
8	1.78E-15	0.60	1.42E-15	6.63	2.99E-13	4.00	1.61E-12	4.04
9	2.13E-15	0.83	3.55E-16	4.00	7.19E-14	4.15	4.04E-13	3.99

7. Conclusion

The study confirms that the proposed HWQLM converges rapidly to the solution of original non-linear problem and can be used to solve many other non-linear differential equations arising in fluid mechanics and non-linear science in general. Another significant advantage of this method is that its implementation does not depend on small parameters unlike other traditional perturbation methods.

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Table 3: Comparison of pressure gradient ($-F'''(0)$) obtained using HWQLM, bvp4c with other methods (axisymmetric flow)

α	Pressure Gradient ($-F'''(0)$)					
	HWQLM		bvp4c	Computer Ex- tended Series[22]	Brown's Method[22]	Numerical[5]
	$J = 8$	$J = 10$				
0	0.0	0.0	0.0	0.0	0.0	0.0
1	7.6599	7.6599	7.6599	7.659	7.659	7.72
3	32.7540	32.7540	32.7539	32.75	32.75	32.95
4	50.0826	50.0824	50.0824	50.08	50.08	50.69
5	70.5403	70.5399	70.5399	70.53	-	-
10	218.6203	218.6181	218.6179	218.61	-	-
20	735.3927	735.3770	735.3760	735.37	-	-
30	1536.4411	1536.3915	1536.3883	1536.38	-	-
40	2615.3567	2615.2436	2615.2361	2615.23	-	-
50	3968.3640	3968.1490	3968.1347	3968.12	-	-
60	5592.9152	5592.5510	5592.5267	5592.47	-	-
70	7487.1425	7486.5732	7486.5351	-	-	-
80	9649.6004	9648.7613	9648.7052	-	-	-
90	12079.1272	12077.9452	12077.8661	-	-	-
100	14774.7625	14773.1559	14773.0483	-	-	-
120	20961.2298	20958.4950	20958.3117	-	-	-
150	32219.7247	32214.4748	32214.1216	-	-	-
180	45841.5089	45832.5572	45831.9531			
200	56230.8109	56218.6226	-	-	-	-
220	67663.9076	67647.7918	-	-	-	-
240	80138.8651	80118.0665	-	-	-	-
260	93654.0054	93627.7045	-	-	-	-
280	108207.8556	108175.1693	-	-	-	-
300	123799.1107	123759.0928	-	-	-	-
$\downarrow$						

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Table 4: Error estimates Δ and convergence rate σ for axisymmetric flow case

J	$\alpha = 10$		$\alpha = 20$		$\alpha = 40$		$\alpha = 60$	
	Δ	σ	Δ	σ	Δ	σ	Δ	σ
4	5.68E-15	-	1.22E-12	-	7.93E-11	-	1.17E-09	-
5	1.78E-15	3.20	2.67E-13	4.57	1.30E-11	6.09	1.19E-10	9.90
6	3.55E-16	5.00	6.43E-14	4.15	2.97E-12	4.38	2.51E-11	4.73
7	1.42E-15	0.25	1.62E-14	3.98	7.28E-13	4.08	6.03E-12	4.16
8	1.07E-15	1.33	4.44E-15	3.64	1.79E-13	4.06	1.49E-12	4.04
9	1.42E-15	0.75	8.88E-16	5.00	4.60E-14	3.90	3.72E-13	4.01

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