APPROXIMATE QUADRATIC MAPPINGS IN MODULAR SPACES

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Abstract: In this article, we investigate an alternative generalized Hyers–Ulam stability theorem of a modified quadratic functional equation in a modular space X_ρ using Δ_3 -condition without the Fatou property on the modular function ρ .

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1. Introduction

In 1940, S.M. Ulam [12] raised the question concerning the stability of group homomorphisms: Let G be a group and let G' be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G \rightarrow G'$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism

$$F : G \rightarrow G' \text{ with } d(f(x), F(x)) < \varepsilon,$$

for all $x \in G$? The case of approximately additive mappings was solved by

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D.H. Hyers [5] under the assumption that X and Y are Banach spaces. A generalization of Hyers' theorem was provided by Th.M. Rassias [8] in 1978 and by P. Găvruta [4] in 1994.

The equation $f(x+y) + f(x-y) = 2[f(x) + f(y)]$, which may be originated from the important parallelogram equality $\|x+y\|^2 + \|x-y\|^2 = 2[\|x\|^2 + \|y\|^2]$ in inner product spaces, is called a quadratic functional equation, and every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers–Ulam stability problem for the quadratic functional equation was proved by F. Skof [11] for a mapping $f : X \rightarrow Y$, where X is a normed space and Y is a Banach space. S. Czerwik [3] proved the Hyers–Ulam stability of the quadratic functional equation with the sum of powers of norms in the sense of Th. M. Rassias approach using direct method. J.M. Rassias [6] proved the Hyers–Ulam stability of the quadratic functional equation with the product of powers of norms using direct method. On the other hand, C. Borelli and G.L. Forti [2] have proved the generalized Hyers–Ulam stability theorem of the quadratic functional equation whose quadratic difference was controlled by general functions with regularity conditions.

Next, we introduce to recall some basic definitions and remarks of modular spaces with modular functions, which are primitive notions corresponding to norms or metrics, as in the followings.

Definition 1. Let χ be a linear space.

(a) A function $\rho : \chi \rightarrow [0, \infty]$ is called a modular if for arbitrary $x, y \in \chi$,

- (1) $\rho(x) = 0$ if and only if $x = 0$,
- (2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,
- (3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,

(b) alternatively, if (3) is replaced by

- (3)' $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,

then we say that ρ is a convex modular.

A modular ρ defines a corresponding modular space, i.e., the linear space χ_ρ given by

$$\chi_\rho = \{x \in \chi : \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0\}.$$

Let ρ be a convex modular. Then, the modular space χ_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\}.$$

A modular function ρ is said to satisfy the Δ_2 -condition if there exists $\kappa > 0$ such that $\rho(2x) \leq \kappa\rho(x)$ for all $x \in \chi_\rho$.

Here, if the modular ρ satisfies the Δ_2 -condition, we observe that $\kappa \geq 1$ for nontrivial modular ρ , and that $\kappa \geq 2$ for nontrivial convex modular ρ . Similarly, a modular function ρ is said to satisfy the Δ_3 -condition if there exists $\kappa > 0$ such that $\rho(3x) \leq \kappa\rho(x)$ for all $x \in \chi_\rho$.

Remark 2. (a) If ρ is a modular on χ , we note that $\rho(tx)$ is an increasing function in $t \geq 0$ for each fixed $x \in \chi$, that is, $\rho(ax) \leq \rho(bx)$ whenever $0 \leq a < b$. (b) If ρ is a convex modular on χ and $|\alpha| \leq 1$, then $\rho(\alpha x) \leq |\alpha|\rho(x)$ for all $x \in \chi$. Moreover, we see that $\rho(\alpha x) \leq \alpha\rho(x)$ for all $x \in \chi$, provided $0 < \alpha \leq 1$, more generally, $\rho(\sum_{i=1}^n \alpha_i x_i) \leq \sum_{i=1}^n \alpha_i \rho(x_i)$ for all $x_i \in \chi$ and $\alpha_i \geq 0$ ($i = 1, \dots, n$) whenever $0 < \sum_{i=1}^n \alpha_i := \alpha \leq 1$. In fact, since $\sum_{i=1}^n (\frac{\alpha_i}{\alpha}) = 1$,

$$\begin{aligned} \rho\left(\sum_{i=1}^n \alpha_i x_i\right) &= \rho\left(\alpha \left(\sum_{i=1}^n \frac{\alpha_i}{\alpha} x_i\right)\right) \\ &\leq \alpha \rho\left(\sum_{i=1}^n \frac{\alpha_i}{\alpha} x_i\right) \leq \alpha \sum_{i=1}^n \frac{\alpha_i}{\alpha} \rho(x_i) = \sum_{i=1}^n \alpha_i \rho(x_i) \end{aligned}$$

for all $x_i \in \chi$.

Definition 3. Let χ_ρ be a modular space and let $\{x_n\}$ be a sequence in χ_ρ . Then,

- (1) $\{x_n\}$ is ρ -convergent to $x \in \chi_\rho$ and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$.
- (2) $\{x_n\}$ is called ρ -Cauchy in χ_ρ if $\rho(x_n - x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.
- (3) A subset K of χ_ρ is called ρ -complete if and only if any ρ -Cauchy sequence is ρ -convergent to an element in K .

They say that the modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n \rightarrow \infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ -convergent to x .

Concerning the stability of functional equations, G. Sadeghi [10] has proved generalized Hyers–Ulam stability via the fixed point method of a generalized Jensen functional equation $f(rx + sy) = rg(x) + sh(y)$ in convex modular spaces with the Fatou property satisfying the Δ_2 -condition with $0 < \kappa \leq 2$. In the paper [13], the authors have presented the generalized Hyers–Ulam stability of quadratic functional equations via the extensive studies of fixed point

theory in the framework of modular spaces whose modulars are convex, lower semicontinuous but do not satisfy any relatives of Δ_2 -conditions. Recently, A. Zivari-Kazempour and M. Eshaghi Gordji [14] have determined the general solution of the quadratic functional equation

$$\begin{aligned} f(x+2y) + f(y+2z) + f(z+2x) \\ = 2f(x+y+z) + 3[f(x) + f(y) + f(z)], \end{aligned} \quad (1)$$

which is equivalent to the quadratic functional equation

$$f(x+y) + f(x-y) = 2[f(x) + f(y)],$$

and then they have proved its generalized Hyers–Ulam stability in normed spaces. In this article, we first present generalized Hyers–Ulam stability via direct method of the equation (1) in modular spaces without using the Fatou property and Δ_3 -conditions, and then investigate alternatively generalized Hyers–Ulam stability via direct method of the equation (1) in modular spaces using necessarily Δ_3 -conditions without the Fatou property.

2. Generalized Hyers–Ulam Stability of Equation (1)

Before making up the main subject, we use the following abbreviation for notational convenience :

$$\begin{aligned} Df(x, y, z) &:= f(x+2y) + f(y+2z) + f(z+2x) \\ &\quad - 2f(x+y+z) - 3[f(x) + f(y) + f(z)] \end{aligned}$$

for all x, y, z in a linear space X . In the following theorem, we first present a generalized Hyers–Ulam stability via direct method of the equation (1) in modular spaces without using the Fatou property and Δ_3 -condition.

Theorem 4. *Let X be a linear space and χ_ρ a ρ -complete convex modular space. Suppose that a mapping $f : X \rightarrow \chi_\rho$ satisfies*

$$\rho(Df(x, y, z)) \leq \phi(x, y, z) \quad (2)$$

and $\phi : X^3 \rightarrow [0, \infty)$ is a mapping such that

$$\Phi(x, y, z) := \sum_{j=0}^{\infty} \frac{\phi(3^j x, 3^j y, 3^j z)}{9^j} < \infty \quad (3)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $F_1 : X \rightarrow \chi_\rho$ defined by $F_1(x) := \lim_{n \rightarrow \infty} \frac{f(3^n x)}{9^n}$, $x \in X$, which satisfies the equation (1) and

$$\rho(f(x) - F_1(x)) \leq \frac{1}{9}\Phi(x, x, x) \quad (4)$$

for all $x \in X$.

Proof. Letting $y = z := x$ in (2), we obtain

$$\rho(Df(x, x, x)) = \rho(f(3x) - 9f(x)) \leq \phi(x, x, x), \quad (5)$$

and so

$$\rho(f(x) - \frac{f(3x)}{9}) \leq \frac{1}{9}\rho(f(3x) - 9f(x)) \leq \frac{1}{9}\phi(x, x, x)$$

for all $x \in X$. By induction on n , one can prove the following functional inequality

$$\rho(f(x) - \frac{f(3^n x)}{9^n}) \leq \sum_{j=0}^{n-1} \frac{\phi(3^j x, 3^j x, 3^j x)}{9^{j+1}} \quad (6)$$

for all $x \in X$.

Now, replacing x by $3^m x$ in (6), we have

$$\rho(\frac{f(3^m x)}{9^m} - \frac{f(3^{n+m} x)}{9^{n+m}}) \leq \frac{1}{9} \sum_{j=m}^{n+m-1} \frac{\phi(3^j x, 3^j x, 3^j x)}{9^j} \quad (7)$$

which converges to zero as $m \rightarrow \infty$ by the assumption (3). Thus the above inequality implies that the sequence $\{\frac{f(3^n x)}{9^n}\}$ is ρ -Cauchy for all $x \in X$ and so it is convergent in χ_ρ since the space χ_ρ is ρ -complete. Thus, we may define a mapping $F_1 : X \rightarrow \chi_\rho$ as

$$F_1(x) := \rho - \lim_{n \rightarrow \infty} \frac{f(3^n x)}{9^n}, \quad (x \in X),$$

which implies

$$\lim_{n \rightarrow \infty} \rho(\frac{f(3^n x)}{9^n} - F_1(x)) = 0, \quad (x \in X).$$

Now, we claim the mapping F_1 satisfies the equation (1). Setting $(x, y, z) := (3^n x, 3^n y, 3^n z)$ in (2), and dividing the resulting inequality by 9^n , we get

$$\rho(\frac{Df(3^n x, 3^n y, 3^n z)}{9^n}) \leq \frac{\rho(Df(3^n x, 3^n y, 3^n z))}{9^n} \leq \frac{\phi(3^n x, 3^n y, 3^n z)}{9^n}$$

for all $x, y, z \in X$. Thus, it follows from the property $\rho(\alpha u) \leq \alpha \rho(u)$, ($0 < \alpha \leq 1$, $u \in \chi_\rho$) that

$$\begin{aligned}
& \rho\left(\frac{1}{15}DF_1(x, y, z)\right) \\
& \leq \rho\left(\frac{1}{15}DF_1(x, y, z) - \frac{Df(3^n x, 3^n y, 3^n z)}{15 \cdot 9^n} + \frac{Df(3^n x, 3^n y, 3^n z)}{15 \cdot 9^n}\right) \\
& \leq \frac{1}{15}\rho\left(F_1(x + 2y) - \frac{f(3^n(x + 2y))}{9^n}\right) + \frac{1}{15}\rho\left(F_1(y + 2z) - \frac{f(3^n(y + 2z))}{9^n}\right) \\
& + \frac{1}{15}\rho\left(F_1(z + 2x) - \frac{f(3^n(z + 2x))}{9^n}\right) + \frac{2}{15}\rho\left(F_1(x + y + z) - \frac{f(3^n(x + y + z))}{9^n}\right) \\
& + \frac{3}{15}\rho\left(F_1(x) - \frac{f(3^n x)}{3^n}\right) + \frac{3}{15}\rho\left(F_1(y) - \frac{f(3^n y)}{3^n}\right) + \frac{3}{15}\rho\left(F_1(z) - \frac{f(3^n z)}{3^n}\right) \\
& + \frac{1}{15}\rho\left(\frac{Df(3^n x, 3^n y, 3^n z)}{9^n}\right)
\end{aligned}$$

for all $x, y, z \in X$ and all positive integers n . Taking the limit as $n \rightarrow \infty$, one obtains $\rho(\frac{1}{15}DF_1(x, y, z)) = 0$, and so $DF_1(x, y, z) = 0$ for all $x, y, z \in X$. Hence F_1 satisfies the equation (1) and so it is quadratic.

On the other hand, since $\sum_{i=0}^n \frac{1}{9^{i+1}} + \frac{1}{9} \leq 1$ for all $n \in \mathbb{N}$, it follows from (5) and the convexity of the modular ρ that

$$\begin{aligned}
& \rho(f(x) - F_1(x)) \\
& = \rho\left(-\sum_{i=0}^n \frac{1}{9^{i+1}}Df(3^i x, 3^i x, 3^i x) + \frac{f(3^{n+1}x)}{9^{n+1}} - \frac{F_1(3x)}{9}\right) \\
& \leq \sum_{i=0}^n \frac{1}{9^{i+1}}\rho(Df(3^i x, 3^i x, 3^i x)) + \frac{1}{9}\rho\left(\frac{f(3^n \cdot 3x)}{9^n} - F_1(3x)\right) \\
& \leq \sum_{i=0}^n \frac{1}{9^{i+1}}\phi(3^i x, 3^i x, 3^i x) + \frac{1}{9}\rho\left(\frac{f(3^n \cdot 3x)}{9^n} - F_1(3x)\right),
\end{aligned}$$

without applying the Fatou property of the modular ρ for all $x \in X$ and all $n \in \mathbb{N}$, from which we obtain the approximation of f by the quadratic mapping F_1 as follows

$$\rho(f(x) - F_1(x)) \leq \sum_{i=0}^{\infty} \frac{1}{9^{i+1}}\phi(3^i x, 3^i x, 3^i x) = \frac{1}{9}\Phi(x, x, x)$$

for all $x \in X$ by taking $n \rightarrow \infty$ in the last inequality.

To show the uniqueness of F_1 , we assume that there exists a quadratic mapping $G_1 : X \rightarrow \chi_\rho$ which satisfies the inequality

$$\rho(f(x) - G_1(x)) \leq \sum_{j=0}^{\infty} \frac{\phi(3^j x, 3^j x, 3^j x)}{9^{j+1}}$$

for all $x \in X$, but suppose $F_1(x_0) \neq G_1(x_0)$ for some $x_0 \in X$. Then there exists a positive constant $\varepsilon > 0$ such that $\varepsilon < \rho(F_1(x_0) - G_1(x_0))$. For such given $\varepsilon > 0$, it follows from (3) that there is a positive integer $n_0 \in \mathbb{N}$ such that $\sum_{j=n_0}^{\infty} \frac{\phi(3^j x_0, 3^j x_0, 3^j x_0)}{9^{j+1}} < \frac{\varepsilon}{2}$. Since F_1 and G_1 are quadratic mappings, we see from the equality $F_1(3^{n_0} x_0) = 9^{n_0} F_1(x_0)$ and $G_1(3^{n_0} x_0) = 9^{n_0} G_1(x_0)$ that

$$\begin{aligned} \varepsilon &< \rho(F_1(x_0) - G_1(x_0)) \\ &= \rho\left(\frac{F_1(3^{n_0} x_0) - f(3^{n_0} x_0)}{9^{n_0}} + \frac{f(3^{n_0} x_0) - G_1(3^{n_0} x_0)}{9^{n_0}}\right) \\ &\leq \frac{1}{9^{n_0}} \rho\left(F_1(3^{n_0} x_0) - f(3^{n_0} x_0)\right) + \frac{1}{9^{n_0}} \rho\left(f(3^{n_0} x_0) - G_1(3^{n_0} x_0)\right) \\ &\leq \frac{2}{9^{n_0}} \sum_{j=0}^{\infty} \frac{\phi(3^{j+n_0} x_0, 3^{j+n_0} x_0, 3^{j+n_0} x_0)}{9^{j+1}} \\ &\leq 2 \sum_{j=n_0}^{\infty} \frac{\phi(3^j x_0, 3^j x_0, 3^j x_0)}{9^{j+1}} < \varepsilon, \end{aligned}$$

which leads a contradiction. Hence the mapping F_1 is a unique quadratic mapping near f satisfying the approximation (4) in the modular space χ_ρ . $\square$

As corollaries of Theorem 4, we obtain the following stability results of the equation (1), which generalize stability results in normed spaces.

Corollary 5. *Suppose X is a linear space and χ_ρ is a ρ -complete convex modular space. Let $f : X \rightarrow \chi_\rho$ be a mapping satisfying*

$$\rho(Df(x, y, z)) \leq \varepsilon$$

for some $\varepsilon > 0$ and for all $x, y, z \in X$. Then there exists a unique quadratic mapping $F_1 : X \rightarrow \chi_\rho$ which satisfies (1) and

$$\rho(f(x) - F_1(x)) \leq \frac{\varepsilon}{8} \quad (x \in X). \quad (8)$$

Corollary 6. Suppose X is a normed space and χ_ρ is a ρ -complete convex modular space. For given real numbers $\varepsilon_i > 0$ and $r_i < 2$ ($i = 1, 2, 3$), if a mapping $f : X \rightarrow \chi_\rho$ satisfies

$$\rho(Df(x, y, z)) \leq \varepsilon_1 \|x\|^{r_1} + \varepsilon_2 \|y\|^{r_2} + \varepsilon_3 \|z\|^{r_3}$$

for all $x, y, z \in X$, then there exists a unique quadratic mapping $F_1 : X \rightarrow \chi_\rho$ such that

$$\rho(f(x) - F_1(x)) \leq \frac{\varepsilon_1 \|x\|^{r_1}}{3^2 - 3^{r_1}} + \frac{\varepsilon_2 \|x\|^{r_2}}{3^2 - 3^{r_2}} + \frac{\varepsilon_3 \|x\|^{r_3}}{3^2 - 3^{r_3}}$$

for all $x \in X$, where $x \neq 0$, if $r_i < 0$.

Corollary 7. Suppose X is a normed space and χ_ρ is a ρ -complete convex modular space. For given real numbers $\theta > 0$ and r_i ($i = 1, 2, 3$), with $r < 2$, $r := r_1 + r_2 + r_3$, if a mapping $f : X \rightarrow \chi_\rho$ satisfies

$$\rho(Df(x, y, z)) \leq \theta \|x\|^{r_1} \|y\|^{r_2} \|z\|^{r_3}$$

for all $x, y, z \in X$, then there exists a unique quadratic mapping $F_1 : X \rightarrow \chi_\rho$ such that

$$\rho(f(x) - F_1(x)) \leq \frac{\theta}{3^2 - 3^r} \|x\|^r, \quad (x \in X),$$

where $x \neq 0$, if $r < 0$.

We remark that a modular function ρ satisfies the Δ_3 -condition if there exists $\kappa > 0$ such that $\rho(3x) \leq \kappa \rho(x)$ for all $x \in \chi_\rho$, where we note $\kappa \geq 3$ for nontrivial convex modular ρ with Δ_3 -condition. So, we apply $\rho(9x) \leq M\rho(x)$ for all $x \in \chi_\rho$, where $M := \kappa^2 \geq 9$ to the following theorem.

Theorem 8. Suppose X is a linear space and χ_ρ is a ρ -complete convex modular space with Δ_3 -condition. If there exists a function $\varphi : X^3 \rightarrow [0, \infty)$ for which a mapping $f : X \rightarrow \chi_\rho$ satisfies

$$\rho(Df(x, y, z)) \leq \varphi(x, y, z), \quad \text{and} \quad (9)$$

$$\sum_{j=1}^{\infty} \frac{M^{2j}}{9^j} \varphi\left(\frac{x}{3^j}, \frac{y}{3^j}, \frac{z}{3^j}\right) := \Psi(x, y, z) < \infty \quad (10)$$

for all $x, y, z \in X$, then there exists a unique quadratic mapping $F_2 : X \rightarrow \chi_\rho$, defined as $F_2(x) = \lim_{n \rightarrow \infty} 9^n f(\frac{x}{3^n})$, $x \in X$, which satisfies the equation (1) and

$$\rho(f(x) - F_2(x)) \leq \frac{1}{9} \Psi(x, x, x) \quad (11)$$

for all $x \in X$.

Proof. We see from (5) that

$$\rho(f(x) - 9f(\frac{x}{3})) \leq \varphi(\frac{x}{3}, \frac{x}{3}, \frac{x}{3})$$

for all $x \in X$. Thus, by the convexity of the modular ρ one has

$$\begin{aligned} \rho(f(x) - 9^2f(\frac{x}{3^2})) &\leq \frac{1}{9}\rho\left(9f(x) - 9^2f(\frac{x}{3})\right) + \frac{1}{9}\rho\left(9^2f(\frac{x}{3}) - 9^3f(\frac{x}{3^2})\right) \\ &\leq \frac{M}{9}\varphi(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}) + \frac{M^2}{9}\varphi(\frac{x}{3^2}, \frac{x}{3^2}, \frac{x}{3^2}) \end{aligned}$$

for all $x \in X$. Then it follows by induction on $n > 1$ that

$$\begin{aligned} \rho(f(x) - 9^n f(\frac{x}{3^n})) & \tag{12} \\ &\leq \sum_{j=1}^{n-1} \frac{M^{2j-1}}{9^j} \varphi(\frac{x}{3^j}, \frac{x}{3^j}, \frac{x}{3^j}) + \frac{M^{2(n-1)}}{9^{n-1}} \varphi(\frac{x}{3^n}, \frac{x}{3^n}, \frac{x}{3^n}) \end{aligned}$$

for all $x \in X$. In fact, it is true for $n = 2$. Assume that the inequality (12) holds true for n . Thus, using the convexity of the modular ρ , we deduce

$$\begin{aligned} \rho(f(x) - 9^{n+1} f(\frac{x}{3^{n+1}})) & \tag{13} \\ &= \rho\left(\frac{1}{9}(9f(x) - 9^2f(\frac{x}{3})) + \frac{1}{9}(9^2f(\frac{x}{3}) - 9^{n+2}f(\frac{x}{3^{n+1}}))\right) \\ &\leq \frac{M}{9}\rho\left(f(x) - 9f(\frac{x}{3})\right) + \frac{M^2}{9}\rho\left(f(\frac{x}{3}) - 9^n f(\frac{x}{3^{n+1}})\right) \\ &\leq \frac{M}{9}\varphi(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}) + \frac{M^2}{9} \sum_{j=1}^{n-1} \frac{M^{2j-1}}{9^j} \varphi(\frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}) \\ &\quad + \frac{M^2}{9} \frac{M^{2(n-1)}}{9^{n-1}} \varphi(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}) \\ &= \sum_{j=1}^n \frac{M^{2j-1}}{9^j} \varphi(\frac{x}{3^j}, \frac{x}{3^j}, \frac{x}{3^j}) + \frac{M^{2n}}{9^n} \varphi(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}), \end{aligned}$$

which proves (12) for $n + 1$. Now, replacing x by $3^{-m}x$ in (12), we have

$$\begin{aligned} &\rho\left(9^m f(\frac{x}{3^m}) - 9^{n+m} f(\frac{x}{3^{n+m}})\right) \\ &\leq M^m \rho\left(f(\frac{x}{3^m}) - 9^n f(\frac{x}{3^{n+m}})\right) \end{aligned}$$

$$\begin{aligned}
&\leq M^m \sum_{j=1}^{n-1} \frac{M^{2j-1}}{9^j} \varphi\left(\frac{x}{3^{j+m}}, \frac{x}{3^{j+m}}, \frac{x}{3^{j+m}}\right) + M^m \frac{M^{2(n-1)}}{9^{n-1}} \varphi\left(\frac{x}{3^{n+m}}, \frac{x}{3^{n+m}}, \frac{x}{3^{n+m}}\right) \\
&\leq \frac{9^m}{M^m} \sum_{j=m+1}^{n+m-1} \frac{M^{2j-1}}{9^j} \varphi\left(\frac{x}{3^j}, \frac{x}{3^j}, \frac{x}{3^j}\right) + \frac{9^m}{M^m} \frac{M^{2(n+m-1)}}{9^{n+m-1}} \varphi\left(\frac{x}{3^{n+m}}, \frac{x}{3^{n+m}}, \frac{x}{3^{n+m}}\right)
\end{aligned}$$

which converges to zero as $m \rightarrow \infty$ by the assumption (10). Thus, the sequence $\{9^n f(\frac{x}{3^n})\}$ is ρ -Cauchy for all $x \in X$ and so it is ρ -convergent in χ_ρ since the space χ_ρ is ρ -complete. Thus, we may define a mapping $F_2 : X \rightarrow \chi_\rho$ as

$$F_2(x) := \rho - \lim_{n \rightarrow \infty} 9^n f\left(\frac{x}{3^n}\right), \quad (x \in X), \quad (14)$$

which implies

$$\lim_{n \rightarrow \infty} \rho(9^n f\left(\frac{x}{3^n}\right) - F_2(x)) = 0, \quad (x \in X).$$

Thus, by the Δ_3 -condition without using the Fatou property, we can see the following inequality

$$\begin{aligned}
\rho(f(x) - F_2(x)) &\leq \frac{1}{9} \rho\left(9f(x) - 9^{n+1}f\left(\frac{x}{3^n}\right) + 9^{n+1}f\left(\frac{x}{3^n}\right) - 9F_2(x)\right) \\
&\leq \frac{M}{9} \rho\left(f(x) - 9^n f\left(\frac{x}{3^n}\right)\right) + \frac{M}{9} \rho\left(9^n f\left(\frac{x}{3^n}\right) - F_2(x)\right) \\
&\leq \frac{M}{9} \sum_{j=1}^{n-1} \frac{M^{2j-1}}{9^j} \varphi\left(\frac{x}{3^j}, \frac{x}{3^j}, \frac{x}{3^j}\right) + \frac{M}{9} \frac{M^{2(n-1)}}{9^{n-1}} \varphi\left(\frac{x}{3^n}, \frac{x}{3^n}, \frac{x}{3^n}\right) \\
&\quad + \frac{M}{9} \rho\left(9^n f\left(\frac{x}{3^n}\right) - F_2(x)\right), \quad (\forall n \in \mathbb{N}) \\
&\leq \frac{1}{9} \sum_{j=1}^{\infty} \frac{M^{2j}}{9^j} \varphi\left(\frac{x}{3^j}, \frac{x}{3^j}, \frac{x}{3^j}\right)
\end{aligned}$$

by taking $n \rightarrow \infty$. Now, we claim the mapping F_2 satisfies the equation (1). Setting $(x, y, z) := (3^{-n}x, 3^{-n}y, 3^{-n}z)$ in (9), and multiplying the resulting inequality by 9^n , we get

$$\begin{aligned}
\rho(9^n Df(3^{-n}x, 3^{-n}y, 3^{-n}z)) &\leq M^n \varphi(3^{-n}x, 3^{-n}y, 3^{-n}z) \\
&\leq \frac{M^{2n}}{9^n} \varphi(3^{-n}x, 3^{-n}y, 3^{-n}z),
\end{aligned}$$

which tends to zero as $n \rightarrow \infty$ for all $x, y, z \in X$. Thus, it follows from the property $\rho(\alpha u) \leq \alpha \rho(u)$, $(0 < \alpha \leq 1, u \in \chi_\rho)$ that

$$\rho\left(\frac{1}{15} DF_2(x, y, z)\right)$$

$$\begin{aligned}
&= \rho\left(\frac{1}{15}DF_2(x, y, z) - \frac{1}{15}9^n Df\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) + \frac{1}{15}9^n Df\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right)\right) \\
&\leq \frac{1}{15}\rho\left(F_2(x+2y) - 9^n f\left(\frac{x+2y}{3^n}\right)\right) + \frac{1}{15}\rho\left(F_2(y+2z) - 9^n f\left(\frac{y+2z}{3^n}\right)\right) \\
&\quad + \frac{1}{15}\rho\left(F_2(z+2x) - 9^n f\left(\frac{z+2x}{3^n}\right)\right) + \frac{2}{15}\rho\left(F_2(x+y+z) - 9^n f\left(\frac{x+y+z}{3^n}\right)\right) \\
&\quad + \frac{3}{15}\rho\left(F_2(x) - 9^n f\left(\frac{x}{3^n}\right)\right) + \frac{3}{15}\rho\left(F_2(y) - 9^n f\left(\frac{y}{3^n}\right)\right) + \frac{3}{15}\rho\left(F_2(z) - 9^n f\left(\frac{z}{3^n}\right)\right) \\
&\quad + \frac{1}{15}\rho\left(9^n Df\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right)\right)
\end{aligned}$$

for all $x, y, z \in X$. Taking the limit as $n \rightarrow \infty$, one obtains

$$\rho\left(\frac{1}{15}DF_2(x, y, z)\right) = 0,$$

and thus $DF_2(x, y, z) = 0$ for all $x, y, z \in X$. Hence F_2 satisfies the equation (1), and so it is quadratic.

To show the uniqueness of F_2 , we assume that there exists a quadratic mapping $G_2 : X \rightarrow \chi_\rho$ which satisfies the inequality

$$\rho(f(x) - G_2(x)) \leq \frac{1}{9} \sum_{j=1}^{\infty} \frac{M^{2j}}{9^j} \varphi\left(\frac{x}{3^j}, \frac{x}{3^j}, \frac{x}{3^j}\right)$$

for all $x \in X$. Since F_2 and G_2 are quadratic mappings, we see from the equality $9^n F_2(3^{-n}x) = F_2(x)$ and $9^n G_2(3^{-n}x) = G_2(x)$ that

$$\begin{aligned}
\rho(G_2(x) - F_2(x)) &= \rho\left(\frac{9^{n+1}}{9}\left(G_2\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right) + \frac{9^{n+1}}{9}\left(f\left(\frac{x}{3^n}\right) - F_2\left(\frac{x}{3^n}\right)\right)\right) \\
&\leq \frac{M^{n+1}}{9}\rho\left(G_2\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right) + \frac{M^{n+1}}{9}\rho\left(f\left(\frac{x}{3^n}\right) - F_2\left(\frac{x}{3^n}\right)\right) \\
&\leq \frac{2M^{n+1}}{9^2} \sum_{j=1}^{\infty} \frac{M^{2j}}{9^j} \varphi\left(\frac{x}{3^{j+n}}, \frac{x}{3^{j+n}}, \frac{x}{3^{j+n}}\right) \\
&\leq \frac{2M}{9^2} \frac{9^n}{M^n} \sum_{j=1}^{\infty} \frac{M^{2(j+n)}}{9^{j+n}} \varphi\left(\frac{x}{3^{j+n}}, \frac{x}{3^{j+n}}, \frac{x}{3^{j+n}}\right)
\end{aligned}$$

which tends to zero as $n \rightarrow \infty$ for all $x \in X$. Hence the mapping F_2 is a unique quadratic mapping satisfying (11). $\square$

Remark 9. In Theorem 8, if χ_ρ is a Banach space with norm ρ , and so $\rho(3x) = 3\rho(x)$, $\kappa := 3$, then we see from (9) and (10) that there exists a unique

quadratic mapping $F_2 : X \rightarrow \chi_\rho$, defined as $F_2(x) = \lim_{n \rightarrow \infty} 9^n f(\frac{x}{3^n})$, $x \in X$, which satisfies the equation (1) and

$$\rho(f(x) - F_2(x)) \leq \frac{1}{9} \sum_{j=1}^{\infty} 9^j \varphi\left(\frac{x}{3^j}, \frac{x}{3^j}, \frac{x}{3^j}\right)$$

for all $x \in X$, which is an implicit alternative stability result in the paper [14].

As corollaries of Theorem 8, we obtain the following stability results of the equation (1), which generalize stability results in normed spaces.

Corollary 10. *Suppose X is a normed space and χ_ρ is a ρ -complete convex modular space with Δ_3 -condition. For given real numbers $\varepsilon_i > 0$ and $r_i > 0$ subject to $\kappa^4 < 3^{2+r_i}$ ($i = 1, 2, 3$), if a mapping $f : X \rightarrow \chi_\rho$ satisfies*

$$\rho(Df(x, y, z)) \leq \varepsilon_1 \|x\|^{r_1} + \varepsilon_2 \|y\|^{r_2} + \varepsilon_3 \|z\|^{r_3}$$

for all $x, y, z \in X$, then there exists a unique quadratic mapping $F_2 : X \rightarrow \chi_\rho$ such that

$$\rho(f(x) - F_2(x)) \leq \frac{\kappa^4 \varepsilon_1 \|x\|^{r_1}}{9(3^{2+r_1} - \kappa^4)} + \frac{\kappa^4 \varepsilon_2 \|x\|^{r_2}}{9(3^{2+r_2} - \kappa^4)} + \frac{\kappa^4 \varepsilon_3 \|x\|^{r_3}}{9(3^{2+r_3} - \kappa^4)}$$

for all $x \in X$.

Corollary 11. *Suppose X is a vector space and χ_ρ is a ρ -complete convex modular space with Δ_3 -condition. For given real numbers $\theta > 0$ and $r_i > 0$ ($i = 1, 2, 3$), subject to $\kappa^4 < 3^{2+r}$, where $r := r_1 + r_2 + r_3$, if a mapping $f : X \rightarrow \chi_\rho$ satisfies*

$$\rho(Df(x, y, z)) \leq \theta \|x\|^{r_1} \|y\|^{r_2} \|z\|^{r_3}$$

for all $x, y, z \in X$, then there exists a unique quadratic mapping $F_2 : X \rightarrow \chi_\rho$ such that

$$\rho(f(x) - F(x)) \leq \frac{\kappa^4 \theta}{9(3^{2+r} - \kappa^4)} \|x\|^r$$

for all $x \in X$.

Remark 12. In Corollary 11, if $f(0) = 0$ and $r_i > 0$ for some $i = 1, 2, 3$, then we find that $Df(x, y, 0) = 0$ for the case $r_3 > 0$ without loss of generality, and so f is itself quadratic in this case.

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