THE EULER FUNCTION GRAPH $G(\phi(n))$

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Abstract: The aim of this paper was to introduce the graph associated with the Euler's totient function and study its properties.

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1. Introduction

Euler totient function, also known as Euler's Phi function or simply Phi function, $\phi(n)$, for any natural number n in Number theory context represent the number of positive integer less than or equals n and relatively prime to n . For example, $\phi(1) = 1$; $\phi(7) = 6$. Moreover, $\phi(n)$ is even for $n \geq 3$. Pomerance [6] defines the divisor graph to any non empty set S of positive integers, where divisor graph $G(S)$ has vertex set S and any two vertices i and j are adjacent iff $\gcd(i, j) = \min(i, j)$. Certainly, $1 \leq \gcd(i, j)$. If equality of this considered, it makes him to define a *relatively prime graph* $RP(S)$ of S having vertex set as S and any two vertices are adjacent iff they are co-primes. Further he showed that every graph is a *relatively prime graph*.

J. Baskar Babujee[1] investigated prime labelling for Euler's Phi function $\phi(n)$ and proved that a maximal number of edges in a simple vertex prime labeling graph with n vertices is $\sum_{k=2}^n \phi(k)$. Cayley graph associated with the

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Euler totient function, called Euler totient Cayley graph, whose vertex set V is given by $Z_n = \{0, 1, 2, \dots, n-1\}$ and the edge set is $E = \{(x, y)/x - y \in S \text{ or } y - x \in S\}$ and is denoted by (Z_n, ϕ) , where S denote the set of all positive integers less than n and relatively prime to n . Moreover, Madhavi. L[3] proved that Euler totient function graph to be connected, Hamiltonian and Eulerian (for $n \geq 3$). M. Manjuri and B. Maheshwari[[4],[5]] proved the clique and matching domination number of $G(Z_n, \phi)$ for a prime is 1 and for prime powers to be 2. K.J. Sangeetha, B. Maheshwari[7] determined minimum edge cover, minimum edge dominating sets, edge covering number and edge domination number of $G(Z_n, \phi)$.

In this present paper, Euler function graph, a newer category of graph was introduced. Further, connectivity, bipartitionness, completeness, Euler function subgraph was also discussed.

2. Euler Function $G(\phi(n))$ Graph

Now, Let us define a Euler function graph,

Definition 2.1. For any natural number n , Euler function graph $G(\phi(n))$ is a simple (V, E) graph such that $V(G(\phi(n))) = \{a/g.c.d(a, n) = 1 \text{ and } a < n\}$ and $E(G(\phi(n))) = \{am/g.c.d(a, m) = 1 \text{ and } a < m \text{ or } m < a\}$.

Example 2.2. The Euler function graph for $n=1$ to 9 are

Remark 2.3. 1. The order of $G(\phi(n))$ is given by $n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \dots (1 - \frac{1}{p_k})$, where n can be expressed as a decomposition of positive integers as

$$n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}.$$

2. For any odd prime p , the order of $G(\phi(p))$ is exactly $(p-1)$ only and it is even.
3. Moreover, the maximum degree for $G(\phi(n))$, $\Delta = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \dots (1 - \frac{1}{p_k}) - 1$ and if p is odd prime, then the maximum degree is $p-2$.
4. Every Euler function graph is Relatively prime graph but the converse need not be true.

Theorem 2.4. For any positive integer n , the Euler function graph $G(\phi(n))$ is connected.

Proof. Clearly, $1 \in V(G(\phi(n)))$ for any natural number n and every integer is relatively prime to 1. This means the vertex 1 is adjacent to all other remaining vertices in $G(\phi(n))$. $\square$

Corollary 2.5. Line graph of Euler function graph, $L(G(\phi(n)))$ is also connected.

Theorem 2.6. $G(\phi(n))$ is complete iff every pair of vertices of $G(\phi(n))$ are relatively prime.

Proof. **Necessary part** Assume that atleast one pair of vertices which are not relatively prime (say) v_1, v_2 . Clearly, v_1 and v_2 are not adjacent in $G(\phi(n))$ and hence the resulting graph is not connected. **Sufficient part** It is obvious that if every pair of vertices of $G(\phi(n))$ are relatively prime, the graph is always connected. $\square$

Theorem 2.7. $G(\phi(n))$ is not Eulerian for any number n .

Proof. Clearly, for any $n \geq 3$, $\phi(n)$ is even and therefore there are even number of vertices in $G(\phi(n))$. Also, the vertex 1 is adjacent to all other vertices. This implies that $d_{G(\phi(n))}(1) = \text{odd no.}$ and hence the proof. $\square$

Definition 2.8. A subgraph H of a Euler function graph $G(\phi(n))$ is called a Euler function subgraph if H is itself a Euler function graph.

Example 2.9. Here H_1, H_2 are Euler function subgraph $G(\phi(3)), G(\phi(6))$ of a Euler function graph $G(\phi(5))$.

Remark 2.10. It is clear that every subgraph of $G(\phi(n))$ is neither a Euler function graph nor a relatively prime graph.

Theorem 2.11. If every pair of vertices in $G(\phi(n))$ are not relatively prime, then $G(\phi(n))$ is a bipartite graph.

Proof. Clearly, the vertex 1 is only element in any one of the partition (say) X and all other remaining vertices, which are not relatively prime to each other is in partition Y. The resulting graph is a star $K_{1,p-1}$, where p denotes the order of $G(\phi(n))$ and hence bipartite. $\square$

Remark 2.12. The following are some observations in $G(\phi(n))$

1. The Dominating number for $G(\phi(n))$ is 1 with minimal dominating set $\{1\}$.
2. The maximum no of edges in $G(\phi(p))$ cannot exceed $\binom{p-1}{2}$ or $\frac{(p-1)(p-2)}{2}$, for any prime p .

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