THE EULER FUNCTION GRAPH $G(\phi(n))$

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Abstract: The aim of this paper was to introduce the graph associated with the Euler’s totient function and study its properties.

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1. Introduction

Euler totient function, also known as Euler’s Phi function or simply Phi function, $\phi(n)$, for any natural number $n$ in Number theory context represent the number of positive integer less than or equals $n$ and relatively prime to $n$. For example, $\phi(1) = 1; \phi(7) = 6$. Moreover, $\phi(n)$ is even for $n \geq 3$. Pomerance [6] defines the divisor graph to any non empty set $S$ of positive integers, where divisor graph $G(S)$ has vertex set $S$ and any two vertices $i$ and $j$ are adjacent iff $gcd(i, j) = min(i, j)$. Certainly, $1 \leq gcd(i, j)$. If equality of this considered, it makes him to define a relatively prime graph $RP(S)$ of $S$ having vertex set as $S$ and any two vertices are adjacent iff they are co-primes. Further he showed that every graph is a relatively prime graph.

J. Baskar Babujee[1] investigated prime labelling for Euler’s Phi function $\phi(n)$ and proved that a maximal number of edges in a simple vertex prime labeling graph with $n$ vertices is $\sum_{k=2}^{n} \phi(n)$. Cayley graph associated with the
Euler totient function, called Euler totient Cayley graph, whose vertex set \( V \) is given by \( Z_n = \{0, 1, 2, \ldots, n - 1\} \) and the edge set is \( E = \{(x, y) / x - y \in S \text{ or } y - x \in S \} \) and is denoted by \((Z_n, \phi)\), where \( S \) denote the set of all positive integers less than \( n \) and relatively prime to \( n \). Moreover, Madhavi L[3] proved that Euler totient function graph to be connected, Hamiltonian and Eulerian for \( n \geq 3 \). M. Manjuri and B. Maheshwari[4,5] proved the clique and matching domination number of \( G(Z_n, \phi) \) for a prime is 1 and for prime powers to be 2. K.J. Sangeetha, B. Maheswari[7] determined minimum edge cover, minimum edge dominating sets, edge covering number and edge domination number of \( G(Z_n, \phi) \).

In this present paper, Euler function graph, a newer category of graph was introduced. Further, connectivity, bipartitionness, completeness, Euler function subgraph was also discussed.

2. Euler Function \( G(\phi(n)) \) Graph

Now, Let us define a Euler function graph,

**Definition 2.1.** For any natural number \( n \), Euler function graph \( G(\phi(n)) \) is a simple \((V, E)\) graph such that \( V(G(\phi(n))) = \{a / g.c.d(a, n) = 1 \text{ and } a < n\} \) and \( E(G(\phi(n))) = \{am / g.c.d(a, m) = 1 \text{ and } a < m \text{ or } m < a\} \).

**Example 2.2.** The Euler function graph for \( n = 1 \) to 9 are

![Graphs](image)

**Remark 2.3.** 1. The order of \( G(\phi(n)) \) is given by \( n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_k}) \), where \( n \) can be expressed as a decomposition of positive integers as
\[ n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}. \]

2. For any odd prime \( p \), the order of \( G(\phi(p)) \) is exactly \((p - 1)\) only and it is even.

3. Moreover, the maximum degree for \( G(\phi(n)) \) , \( \Delta = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) - 1 \) and if \( p \) is odd prime, then the maximum degree is \( p - 2 \).

4. Every Euler function graph is Relatively prime graph but the converse need not be true.

**Theorem 2.4.** For any positive integer \( n \), the Euler function graph \( G(\phi(n)) \) is connected.

**Proof.** Clearly, \( 1 \in V(G(\phi(n))) \) for any natural number \( n \) and every integer is relatively prime to 1. This means the vertex 1 is adjacent to all other remaining vertices in \( G(\phi(n)) \). \( \square \)

**Corollary 2.5.** Line graph of Euler function graph, \( L(G(\phi(n))) \) is also connected.

**Theorem 2.6.** \( G(\phi(n)) \) is complete iff every pair of vertices of \( G(\phi(n)) \) are relatively prime.

**Proof.** **Necessary part** Assume that atleast one pair of vertices which are not relatively prime(say)\( v_1, v_2 \). Clearly, \( v_1 \) and \( v_2 \) are not adjacent in \( G(\phi(n)) \) and hence the resulting graph is not connected. **Sufficient part** It is obvious that if every pair of vertices of \( G(\phi(n)) \) are relatively prime, the graph is always connected. \( \square \)

**Theorem 2.7.** \( G(\phi(n)) \) is not Eulerian for any number \( n \).

**Proof.** Clearly, for any \( n \geq 3 \), \( \phi(n) \) is even and therefore there are even number of vertices in \( G(\phi(n)) \). Also, the vertex 1 is adjacent to all other vertices. This implies that \( d_{G(\phi(n))}(1) = \text{odd no.} \) and hence the proof. \( \square \)

**Definition 2.8.** A subgraph \( H \) of a Euler function graph \( G(\phi(n)) \) is called a Euler function subgraph if \( H \) is itself a Euler function graph.

**Example 2.9.** Here \( H_1, H_2 \) are Euler function subgraph \( G(\phi(3)), G(\phi(6)) \) of a Euler function graph \( G(\phi(5)) \).
Remark 2.10. It is clear that every subgraph of $G(\phi(n))$ is neither an Euler function graph nor a relatively prime graph.

Theorem 2.11. If every pair of vertices in $G(\phi(n))$ are not relatively prime, then $G(\phi(n))$ is a bipartite graph.

Proof. Clearly, the vertex 1 is only element in any one of the partition (say) $X$ and all other remaining vertices, which are not relatively prime to each other is in partition $Y$. The resulting graph is a star $K_{1,p-1}$, where $p$ denotes the order of $G(\phi(n))$ and hence bipartite.

Remark 2.12. The following are some observations in $G(\phi(n))$
1. The Dominating number for $G(\phi(n))$ is 1 with minimal dominating set $\{1\}$.
2. The maximum no of edges in $G(\phi(p))$ cannot exceed $\left(\frac{p-1}{2}\right)$ or $\frac{(p-1)(p-2)}{2}$, for any prime $p$.

References