***L*-FUZZY (K, E) -SOFT QUASI-UNIFORM SPACES INDUCED BY *L*-FUZZY (K, E) -SOFT PRE-PROXIMITIES**

A.A. Ramadan¹, Yong Chan Kim^{2 §}

¹Mathematics Department

Faculty of Science

Beni-Suef University

Beni-Suef, EGYPT

²Department of Mathematics

Gangneung-Wonju University

Gangneung, Gangwondo, 25457, KOREA

Abstract: In this paper, we introduce the notion of *L*-fuzzy (K, E) -soft pre-uniform spaces induced by *L*-fuzzy (K, E) -soft pre-proximity spaces in a strictly two-sided commutative quantale having an order reversing involution. We investigate the relations between *L*-fuzzy (K, E) -soft pre-uniform spaces and *L*-fuzzy (K, E) -soft pre-proximity spaces.

We give their examples.

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1. Introduction

Molodtsov [10,11] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Many researchers have contributed towards the algebraic structure of soft set theory [1-7]. Shabir and Naz [16] initiated the

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url: www.acadpubl.eu

[§]Correspondence author

study of soft topological spaces. Aygünoglu et.al [4] introduced the concept of soft topology in a sense of Šostak [9]. Cetkin et.al [5,6] studied soft proximities and discuss their properties. Ramadan et al. [12,13] introduced the L -fuzzy (K, E) -soft topological structures, L -fuzzy (K, E) -soft pre-uniformities and soft L -fuzzy topogenous orders in a strictly two-sided commutative quantale.

In this paper, we introduce the notion of L -fuzzy (K, E) -soft pre-uniform spaces induced by L -fuzzy (K, E) -soft pre-proximity spaces in a strictly two-sided commutative quantale having an order reversing involution. We investigate the relations between L -fuzzy (K, E) -soft pre-uniform spaces and L -fuzzy (K, E) -soft pre-proximity spaces. We give their examples.

2. Preliminaries

Let $L = (L, \leq, \vee, \wedge, 0, 1)$ be a completely distributive lattice with the least element 0 and the greatest element 1 in L .

Definition 2.1. [8,9] A complete lattice $(L, \leq, \odot)$ is called a strictly two-sided commutative quantale (stsc-quantale, for short) iff it satisfies the following properties.

(L1) $(L, \odot)$ is a commutative semigroup,

(L2) $x = x \odot 1$, for each $x \in L$ and 1 is the universal upper bound,

(L3) $\odot$ is distributive over arbitrary joins, i.e. $(\bigvee_i x_i) \odot y = \bigvee_i (x_i \odot y)$.

There exists an implication operator $\rightarrow$ defined as

$$x \rightarrow y = \bigvee \{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies $(x \odot z) \leq y$ iff $z \leq (x \rightarrow y)$.

In this paper, we always assume that $(L, \leq, \odot, \rightarrow, \oplus, *)$ is a stsc-quantales with an order reversing involution $*$ which is defined by $x \oplus y = (x^* \odot y^*)^*$, $x^* = x \rightarrow 0$ unless otherwise specified.

Remark 2.2. Every completely distributive lattice $(L, \leq, \wedge, \vee, *)$ with order reversing involution $*$ is a stsc-quantale $(L, \leq, \odot, \oplus, *)$ with a strong negation $*$ where $\odot = \wedge$ and $\oplus = \vee$.

Lemma 2.3. [8,9] For each $x, y, z, x_i, y_i, w \in L$, we have the following properties.

(1) $1 \rightarrow x = x$, $0 \odot x = 0$,

(2) If $y \leq z$, then $x \odot y \leq x \odot z$, $x \oplus y \leq x \oplus z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$,

- (3) $x \leq y$ iff $x \rightarrow y = 1$.
- (4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,
- (5) $x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i)$,
- (6) $(\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)$,
- (7) $x \rightarrow (\bigvee_i y_i) \geq \bigvee_i (x \rightarrow y_i)$,
- (8) $(\bigwedge_i x_i) \rightarrow y \geq \bigvee_i (x_i \rightarrow y)$,
- (9) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
- (10) $x \odot y = (x \rightarrow y^*)^*$ and $x \oplus y = x^* \rightarrow y$,
- (11) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \odot w)$,
- (12) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,
- (13) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \oplus z) \rightarrow (y \oplus w)$.
- (14) $x \rightarrow y = y^* \rightarrow x^*$.
- (15) $(x \vee y) \odot (z \vee w) \leq (x \vee z) \vee (y \odot w) \leq (x \oplus z) \vee (y \odot w)$.
- (16) $\bigvee_{i \in \Gamma} x_i \rightarrow \bigvee_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)$ and $\bigwedge_{i \in \Gamma} x_i \rightarrow \bigwedge_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)$,
- (17) $(x \odot y) \odot (z \oplus w) \leq (x \odot z) \oplus (y \odot w)$.

Throughout this paper, X refers to an initial universe, E and K are the sets of all parameters for X , and L^X is the set of all L -fuzzy sets on X .

Definition 2.4. [4] A map f is called an L -fuzzy soft set on X , where f is a mapping from E into L^X , i.e., $f_e := f(e)$ is an L -fuzzy set on X , for each $e \in E$. The family of all L -fuzzy soft sets on X is denoted by $(L^X)^E$. Let f and g be two L -fuzzy soft sets on X .

(1) f is an L -fuzzy soft subset of g and we write $f \sqsubseteq g$ if $f_e \leq g_e$, for each $e \in E$. f and g are equal if $f \sqsubseteq g$ and $g \sqsubseteq f$.

(2) The intersection of f and g is an L -fuzzy soft set $h = f \sqcap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

(3) The union of f and g is an L -fuzzy soft set $h = f \sqcup g$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(4) An L -fuzzy soft set $h = f \odot g$ is defined as $h_e = f_e \odot g_e$, for each $e \in E$.

(5) An L -fuzzy soft set $h = f \oplus g$ is defined as $h_e = f_e \oplus g_e$, for each $e \in E$.

(6) The complement of an L -fuzzy soft sets on X is denoted by f^* , where $f^* : E \rightarrow L^X$ is a mapping given by $f_e^* = (f_e)^*$, for each $e \in E$.

(7) f is called a null L -fuzzy soft set and is denoted by 0_X , if $f_e(x) = 0$, for each $e \in E$, $x \in X$.

(8) f is called an absolute L -fuzzy soft set and is denoted by 1_X , if $f_e(x) = 1$, for each $e \in E$, $x \in X$ and $(1_x)_e(x) = 1$.

Definition 2.5. [4] Let $\varphi : X \rightarrow Y$ and $\psi : E \rightarrow K$ be two mappings, where E and K are parameters sets for the crisp sets X and Y , respectively.

Then $\varphi_\psi : (X, E) \rightarrow (Y, K)$ is called a fuzzy soft mapping. Let f and g be two fuzzy soft sets over X and Y , respectively and let φ_ψ be a fuzzy soft mapping from (X, E) into (Y, K) .

(1) The image of f under the fuzzy soft mapping φ_ψ , denoted by $\varphi_\psi(f)$ is the fuzzy soft set on Y defined by

$$\varphi(f)_k(y) = \begin{cases} \bigvee_{\varphi(x)=y} \left(\bigvee_{\psi(e)=k} f_e(x) \right), & \text{if } x \in \varphi^{-1}(y) \\ 0, & \text{otherwise,} \end{cases}$$

$\forall k \in K, \forall y \in Y$.

(2) The pre-image of g under the fuzzy soft mapping φ_ψ , denoted by $\varphi_\psi^{-1}(g)$ is the fuzzy soft set on X defined by

$$\varphi_\psi^{-1}(g)_e(x) = g_{\psi(e)}(\varphi(x)), \forall e \in E, \forall x \in X.$$

Definition 2.6. [13] An L -fuzzy (K, E) -soft pre-uniformity is a mapping $\mathcal{U} : K \rightarrow L^{(L^{X \times X})^E}$ which satisfies the following conditions .

(SU1) There exists $u \in (L^{X \times X})^E$ such that $\mathcal{U}_k(u) = 1$.

(SU2) If $v \sqsubseteq u$, then $\mathcal{U}_k(v) \leq \mathcal{U}_k(u)$.

(SU3) For every $u, v \in (L^{X \times X})^E$, $\mathcal{U}_k(u \odot v) \geq \mathcal{U}_k(u) \odot \mathcal{U}_k(v)$.

(SU4) If $\mathcal{U}_k(u) \neq 0$ then $\top_\Delta \sqsubseteq u$ where, for each $e \in E$,

$$(\top_\Delta)_e(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

The pair $(X, \mathcal{U})$ is called an L -fuzzy (K, E) -soft pre-uniform space.

An L -fuzzy (K, E) -soft pre-uniformity is an L -fuzzy (K, E) -soft quasi-uniformity if

(UQ) $\mathcal{U}_k(u) \leq \bigvee \{ \mathcal{U}_k(v) \odot \mathcal{U}_k(w) \mid v \circ w \sqsubseteq u \}$, where

$$v_e \circ w_e(x, z) = \bigvee_{y \in X} v_e(x, y) \odot w_e(y, z),$$

An L -fuzzy (K, E) -soft quasi-uniform space $(X, \mathcal{U})$ is said to be an L -fuzzy (K, E) -soft uniform space if

(U) $\mathcal{U}_k(u) \leq \mathcal{U}_k(u^{-1})$, where $(u^{-1})_e(x, y) = u_e(y, x)$ for each $k \in K$ and $u \in (L^{X \times X})^E$.

Let $(X, \mathcal{U}^1)$ be an L -fuzzy (K_1, E_1) -soft pre-uniform space and $(Y, \mathcal{U}^2)$ be an L -fuzzy (K_2, E_2) -soft pre-uniform space. Let $\varphi : X \rightarrow Y$, $\psi : E_1 \rightarrow E_2$ and $\eta : K_1 \rightarrow K_2$ be mappings. Then $\varphi_{\psi, \eta}$ from $(X, \mathcal{U}^1)$ into $(Y, \mathcal{U}^2)$ is called L -fuzzy soft uniformly continuous if

$$\mathcal{U}_{\eta(k)}^2(v) \leq \mathcal{U}_k^1((\varphi \times \varphi)_\psi^{-1}(v)) \quad \forall v \in (L^{Y \times Y})^{E_2}, k \in K_1.$$

Remark 2.7. Let $(X, \mathcal{U})$ be an L -fuzzy (K, E) -soft uniform space.

(1) By (SU1) and (SU2), we have $\mathcal{U}_k(1_{X \times X}) = 1$ because $u \sqsubseteq 1_{X \times X}$ for all $u \in (L^{X \times X})^E$.

(2) Since $\mathcal{U}_k(u) \leq \mathcal{U}_k(u^{-1}) \leq \mathcal{U}_k((u^{-1})^{-1}) = \mathcal{U}_k(u)$, then $\mathcal{U}_k(u) = \mathcal{U}_k(u^{-1})$.

Definition 2.8. [13] A mapping $\delta : K \rightarrow L^{(L^X)^E \times (L^X)^E}$ ($\delta_k = \delta(k) : (L^X)^E \times (L^X)^E \rightarrow L$) is called an L -fuzzy (K, E) -soft pre-proximity on X if it satisfies the following axioms.

(SP1) $\delta_k(0_X, 1_X) = \delta_k(1_X, 0_X) = 0$.

(SP2) If $\delta_k(f, g) \neq 1$, then $f \sqsubseteq g^*$.

(SP3) If $f_1 \sqsubseteq f_2$ and $g_1 \sqsubseteq g_2$, then $\delta_k(f_1, g_1) \leq \delta_k(f_2, g_2)$.

(SP4) $\delta_k(f_1 \odot f_2, g_1 \oplus g_2) \leq \delta_k(f_1, g_1) \oplus \delta_k(f_2, g_2)$.

The pair (X, δ) is called an L -fuzzy (K, E) -soft pre-proximity space.

An L -fuzzy (K, E) -soft pre-proximity is called an L -fuzzy (K, E) -soft quasi-proximity on X if

(PQ) $\delta_k(f, g) \geq \bigwedge_h \{\delta_k(f, h) \oplus \delta_k(h^*, g)\}$.

An L -fuzzy (K, E) -soft quasi-proximity is called an L -fuzzy (K, E) -soft proximity on X if

(SP) $\delta_k(f, g) = \delta_k(g, f)$.

Let (X, δ^1) be an L -fuzzy (K_1, E_1) -soft quasi proximity space and (Y, δ^2) be an L -fuzzy (K_2, E_2) -soft pre-proximity space. Let $\varphi : X \rightarrow Y$, $\psi : E_1 \rightarrow E_2$ and $\eta : K_1 \rightarrow K_2$ be mappings. Then $\varphi_{\psi, \eta}$ from (X, δ^1) into (Y, δ^2) is called L -fuzzy soft proximally continuous if $\delta_k^1(\varphi_{\psi}^{-1}(f), \varphi_{\psi}^{-1}(g)) \leq \delta_{\eta(k)}^2(f, g) \quad \forall f, g \in (L^Y)^{E_2}, k \in K_1$.

Theorem 2.9. [13] Let $(X, \mathcal{U})$ be an L -fuzzy (K, E) -soft pre-uniform space. Define a mapping $\delta^{\mathcal{U}} : K \rightarrow L^{(L^X)^E \times (L^X)^E}$ by

$$\delta_k^{\mathcal{U}}(f, g) = \bigwedge \{\mathcal{U}_k^*(u) \mid u[f] \sqsubseteq g^*\},$$

where $u_e[f_e](x) = \bigvee_{y \in X} (f_e(y) \odot u_e(y, x))$, $\forall x \in X, \forall e \in E, \forall u \in (L^{X \times X})^E$ and $f \in (L^X)^E$. Then we have the following properties.

(1) $\delta^{\mathcal{U}}$ is an L -fuzzy (K, E) -soft pre-proximity space.

(2) If $(X, \mathcal{U})$ is an L -fuzzy (K, E) -soft quasi-uniform space, then $\delta^{\mathcal{U}}$ is L -fuzzy (K, E) -soft quasi-proximity space.

Theorem 2.10. [13] Let $(X, \mathcal{U})$ be an L -fuzzy (K_1, E_1) -soft quasi-uniform space and $(Y, \mathcal{V})$ be an L -fuzzy (K_2, E_2) -soft quasi-uniform space, $\phi : X \rightarrow Y$, $\psi : E_1 \rightarrow E_2$ and $\eta : K_1 \rightarrow K_2$ are functions. If $\phi_{\psi, \eta}$ from $(X, \mathcal{U})$ into $(Y, \mathcal{V})$

is L -fuzzy soft uniformly continuous, then $\phi_{\psi,\eta} : (X, \delta^{\mathcal{U}}) \rightarrow (Y, \delta^{\mathcal{V}})$ is L -fuzzy soft proximally continuous.

3. L -Fuzzy (K, E) -Soft Pre-Uniformities Induced by L -Fuzzy (K, E) -Soft Pre-Proximities

Lemma 3.1. [12] For every $f, g \in (L^X)^E$, we define $u_{f,g}, u_{f,g}^{-1} \in (L^{X \times X})^E$ by

$$(u_{f,g})_e(x, y) = f_e(x) \rightarrow g_e(y) \quad \forall e \in E,$$

$$(u_{f,g}^{-1})_e(x, y) = (u_{f,g})_e(y, x).$$

Then we have the following statements

- (1) $1_{X \times X} = u_{0_X, 0_X} = u_{1_X, 1_X}$.
- (2) If $f_1 \sqsubseteq f_2$ and $g_1 \sqsubseteq g_2$, then $u_{f_2, g_1} \sqsubseteq u_{f_1, g_2}$.
- (3) If $f \sqsubseteq g$, then $1_{\Delta} \sqsubseteq u_{f, g}$.
- (4) For every $u_{g, h} \in (L^{X \times X})^E$ and $f \in (L^X)^E$, we have $u_{h, g} \odot u_{f, h} \sqsubseteq u_{f, g}$.
- (5) $u_{f_1, g_1} \odot u_{f_2, g_2} \sqsubseteq u_{f_1 \odot f_2, g_1 \odot g_2}$.
- (6) $u_{f_1, g_1} \odot u_{f_2, g_2} \sqsubseteq u_{f_1 \oplus f_2, g_1 \oplus g_2}$.
- (7) $u_{f, g}^{-1} = u_{g^*, f^*}$.
- (8) $u_{f_1 \odot f_2, g_1 \odot g_2}^{-1} = u_{g_1^* \oplus g_2^*, f_1^* \oplus f_2^*}$,
- (9) $u_{f_1 \oplus f_2, g_1 \oplus g_2}^{-1} = u_{g_1^* \odot g_2^*, f_1^* \odot f_2^*}$.
- (10) $u[f] \sqsubseteq g$ iff $u \sqsubseteq u_{f, g}$.
- (11) $u_{f, g} = \bigvee \{u \in (L^{X \times X})^E \mid u[f] \sqsubseteq g\}$.
- (12) $u_{f, g}[f] \sqsubseteq g$ and $u_{f, f}[f] = f$.

In the following theorem, we obtain an L -fuzzy (K, E) -soft pre-uniform space from an L -fuzzy (K, E) -soft pre-proximity space.

Theorem 3.2. Let (X, δ) be an L -fuzzy (K, E) -soft pre-proximity space. Define $\mathcal{U}^\delta : K \rightarrow L^{(L^{X \times X})^E}$ by

$$\mathcal{U}_k^\delta(u) = \bigvee \{ \odot_{i=1}^n \delta_k(f_i, g_i)^* \mid \odot_{i=1}^n u_{f_i, g_i^*} \sqsubseteq u \},$$

where $\bigvee$ is taken over every finite family $\{u_{f_i, g_i} \mid i = 1, 2, 3, \dots, n\}$. Then

- (1) $(\mathcal{U}^\delta)_k(u_{f, g}) = \delta_k^*(f, g)$.
- (2) $\mathcal{U}^\delta$ is an L -fuzzy (K, E) -soft pre-uniformity on X ,

(3) If (X, δ) is an L -fuzzy (K, E) -soft quasi-proximity space, then $\mathcal{U}^\delta$ is an L -fuzzy (K, E) -soft quasi-uniformity on X .

(4) $\delta^{\mathcal{U}^\delta} = \delta$.

(5) If (X, δ) is an L -fuzzy (K, E) -soft proximity space, then $\mathcal{U}^\delta(u) = \mathcal{U}^\delta(u^{-1})$.

Proof. (1) From Lemma 3.1(5), since $\odot_{i=1}^n u_{f_i, g_i} \sqsubseteq u_{\odot_{i=1}^n f_i, \oplus_{i=1}^n g_i}$, by (SP4), we have $\delta_k^*(\odot_{i=1}^n f_i, \oplus_{i=1}^n g_i) \geq \odot_{i=1}^n \delta_k^*(f_i, g_i)$. It follows $\mathcal{U}_k^\delta(u_{f, g}) = \delta_k^*(f, g)$.

(2) (SU1) Since $\delta_k(0_X, 1_X)^* = \delta_k(1_X, 0_X)^* = 1$, there exists $1_{X \times X} = u_{0_X, 0_X} = u_{1_X, 1_X} \in (L^{X \times X})^E$. It follows $\mathcal{U}_k^\delta(1_{X \times X}) = 1$.

(SU2) It is trivial from the definition of $\mathcal{U}^\delta$.

(SU3) For every $u, v \in (L^{X \times X})^E$, each two families $\{u_{f_i, g_i} \mid \odot_{i=1}^n u_{f_i, g_i}^* \sqsubseteq u\}$ and $\{u_{h_j, w_j} \mid \odot_{j=1}^k u_{h_j, w_j}^* \sqsubseteq v\}$, we have

$$\begin{aligned} & \mathcal{U}_k^\delta(u) \odot \mathcal{U}_k^\delta(v) \\ &= (\bigvee \{\odot_{i=1}^n \delta_k(f_i, g_i)^* \mid \odot_{i=1}^n u_{f_i, g_i}^* \sqsubseteq u\}) \\ & \odot (\bigvee \{\odot_{j=1}^k \delta_k(h_j, w_j)^* \mid \odot_{j=1}^k u_{h_j, w_j}^* \sqsubseteq v\}) \\ &\leq \bigvee \{\odot_{i=1}^n \delta_k(f_i, g_i)^* \odot \odot_{j=1}^k \delta_k(h_j, w_j)^* \mid \\ & \quad \odot_{i=1}^n u_{f_i, g_i}^* \sqsubseteq u, \odot_{j=1}^k u_{h_j, w_j}^* \sqsubseteq v\} \\ &\leq \bigvee \{\odot_{i=1}^n \delta_k(f_i, g_i)^* \odot \odot_{j=1}^k \delta_k(h_j, w_j)^* \mid \\ & \quad \odot_{i=1}^n u_{f_i, g_i}^* \odot \odot_{j=1}^k u_{h_j, w_j}^* \sqsubseteq u \odot v\} \\ &\leq \mathcal{U}_k^\delta(u \odot v). \end{aligned}$$

(SU4) If $\mathcal{U}_k(u) \neq 0$, there exists a family $\{u_{f_i, g_i} \mid \odot_{i=1}^m u_{f_i, g_i}^* \sqsubseteq u\}$ such that $\odot_{j=1}^m \delta_k^*(f_i, g_i) \neq 0$. Since $\delta_k(f_i, g_i) \neq 1$, for $i = 1, 2, \dots, m$, then $f_i \sqsubseteq g_i^*$ for $i = 1, 2, \dots, m$, i.e. $1_\Delta \sqsubseteq u_{f_i, g_i}^*$. Thus $1_\Delta \sqsubseteq \odot_{i=1}^m u_{f_i, g_i}^* \sqsubseteq u$.

(3) (UQ) Suppose there exists $u \in (L^{X \times X})^E$ such that

$$\bigvee \{\mathcal{U}_k^\delta(v) \odot \mathcal{U}_k^\delta(w) \mid v \odot w \sqsubseteq u\} \not\sqsubseteq \mathcal{U}_k^\delta(u).$$

Put $t = \bigvee \{\mathcal{U}_k^\delta(v) \odot \mathcal{U}_k^\delta(w) \mid v \odot w \sqsubseteq u\}$. From the Definition of $\mathcal{U}_k^\delta(u)$, there exists family $\{u_{f_i, g_i} \mid \odot_{i=1}^m u_{f_i, g_i}^* \sqsubseteq u\}$ such that

$$t \not\sqsubseteq \odot_{i=1}^m \delta_k(f_i, g_i)^*.$$

Since $\delta_k(f_i, g_i) \geq \bigwedge_{h_i} \{\delta_k(f, h) \oplus \delta_k(h^*, g)\}$.

$$t \not\sqsubseteq \odot_{i=1}^m \{\bigvee_{h_i \in (L^X)^E} \{\delta_k(f_i, h_i)^* \odot \delta_k(h_i^*, g_i)^*\}\}.$$

Since L is a stsc-quantal, there exists $h_i \in (L^X)^E$ such that

$$t \not\geq \odot_{i=1}^m (\delta_k(f_i, h_i)^* \odot \delta_k(h_i^*, g_i)^*).$$

On the other hand, put $v_i = u_{h_i^*, g_i^*}, w_i = u_{f_i, h_i^*}$, then from Lemma 3.1 (4), it satisfies $v_i \circ w_i \sqsubseteq u_{h_i^*, g_i^*} \circ u_{f_i, h_i^*} \sqsubseteq u_{f_i, g_i^*}$, by (1),

$$\mathcal{U}_k^\delta(v_i) = \delta_k(h_i^*, g_i)^*, \mathcal{U}_k^\delta(w_i) = \delta_k(f_i, h_i)^*.$$

Let $v = \odot_{i=1}^m v_i$ and $w = \odot_{i=1}^m w_i$ be given. Since $v_i \circ w_i \sqsubseteq u_{f_i, g_i^*}$, for each $i = 1, 2, 3, \dots, m$, we have $(\odot_{i=1}^m v_i) \circ (\odot_{i=1}^m w_i) = \odot_{i=1}^m (v_i \circ w_i) \sqsubseteq \odot_{i=1}^m u_{f_i, g_i^*} \sqsubseteq u$.

Then we have $v \circ w \sqsubseteq u$ and $\mathcal{U}_k^\delta(v) \geq \odot_{i=1}^m \mathcal{U}_k^\delta(v_i)$ and $\mathcal{U}_k^\delta(w) \geq \odot_{i=1}^m \mathcal{U}_k^\delta(w_i)$. Thus,

$$\begin{aligned} t &= \bigvee \{ \mathcal{U}_k^\delta(v) \odot \mathcal{U}_k^\delta(w) \mid v \circ w \sqsubseteq u \} \\ &\geq \mathcal{U}_k^\delta(v) \odot \mathcal{U}_k^\delta(w) \geq (\odot_{i=1}^m \mathcal{U}_k^\delta(v_i)) \odot (\odot_{i=1}^m \mathcal{U}_k^\delta(w_i)) \\ &\geq \odot_{i=1}^m (\delta_k(h_i^*, g_i)^* \odot \delta_k(f_i, h_i)^*). \end{aligned}$$

It is a contradiction. Then $\mathcal{U}^\delta$ is an L -fuzzy (K, E) -soft quasi-uniformity on X .

(4) Since $u[f] \sqsubseteq g^*$, by Lemma 3.1(10), $u \sqsubseteq u_{f, g^*}$. Hence, $\delta_k^{\mathcal{U}^\delta}(f, g) = \bigwedge \{ (\mathcal{U}_k^\delta)^*(u) \mid u[f] \sqsubseteq g^* \} = (\mathcal{U}_k^\delta)^*(u_{f, g^*}) = \delta_k(f, g)$.

(5)

$$\begin{aligned} \mathcal{U}_k^\delta(u) &= \bigvee \{ \odot_{i=1}^n \delta_k(f_i, g_i)^* \mid \odot_{i=1}^n u_{f_i, g_i^*} \sqsubseteq u \} \\ &= \bigvee \{ \odot_{i=1}^n \delta_k(g_i, f_i)^* \mid \odot_{i=1}^n u_{f_i, g_i^*}^{-1} \sqsubseteq u^{-1} \} \\ &= \bigvee \{ \odot_{i=1}^n \delta_k(g_i, f_i)^* \mid \odot_{i=1}^n u_{g_i, f_i^*} \sqsubseteq u^{-1} \} \\ &= \mathcal{U}_k^\delta(u^{-1}). \end{aligned}$$

Definition 3.3. An L -fuzzy (K, E) -soft pre-uniform structure $\mathcal{U}$ on X is said to be compatible with an L -fuzzy (K, E) -soft pre-proximity δ on X if $\delta^\mathcal{U} = \delta$. The class $\prod(\delta)$ denotes the family of all L -fuzzy (K, E) -soft pre-uniform structure which are compatible with a given L -fuzzy (K, E) -soft pre-proximity structure δ .

Theorem 3.4. Let δ be an L -fuzzy (K, E) -soft pre-proximity on X and the L -fuzzy (K, E) -soft pre-proximity $\delta^{\mathcal{U}^\delta}$ induced by $\mathcal{U}^\delta$. Then

- (1) $\delta^{\mathcal{U}^\delta} = \delta$, that is, $\mathcal{U}^\delta \in \prod(\delta)$,
- (2) $\mathcal{U}^\delta$ is the coarsest member of $\prod(\delta)$, i.e., $\mathcal{U}^\delta \leq \mathcal{U}$.

Proof. (1) is easily proved from Theorem 3.2(4).

(2) Let $\mathcal{U}$ be an arbitrary member of $\prod(\delta)$. We will show that $\mathcal{U}_k^\delta(u) \leq \mathcal{U}_k(u)$, for all $u \in (L^{X \times X})^E$.

Suppose that there exists $u \in (L^{X \times X})^E$ such that

$$\mathcal{U}_k^\delta(u) \not\leq \mathcal{U}_k(u).$$

By the definition of $\mathcal{U}_k^\delta$, there exists a family $\{u_{f_i, g_i} \mid \odot_{i=1}^m u_{f_i, g_i^*} \sqsubseteq u\}$ such that

$$\odot_{j=1}^m \delta_k(f_i, g_i) \not\leq \mathcal{U}_k(u).$$

Since $\mathcal{U}^\delta \in \prod(\delta)$ from (1), that is, $\delta_k(f_i, g_i) = \delta_k^{\mathcal{U}^\delta}(f_i, g_i)$ for each $i = 1, 2, \dots, m$, by the definition of $\delta_k^{\mathcal{U}^\delta}$, there exists $v_i \in (L^{X \times X})^E$ with $v_i[f_i] \sqsubseteq g_i^*$ such that

$$\odot_{j=1}^m \mathcal{U}_k^\delta(v_i) \not\leq \mathcal{U}_k(u).$$

On the other hand, put $v = \odot_{i=1}^m v_i$. Since $v_i[f_i] \sqsubseteq g_i^*$, by the definition of u_{f_i, g_i^*} , we have $v_i \sqsubseteq u_{f_i, g_i^*}$. It follows that

$$v = \odot_{i=1}^m v_i \sqsubseteq \odot_{i=1}^m u_{f_i, g_i^*} \sqsubseteq u.$$

Hence, $\mathcal{U}_k(u) \geq \mathcal{U}_k^\delta(\odot_{i=1}^m u_{f_i, g_i}) \geq \mathcal{U}_k^\delta(v) \geq \odot_{i=1}^m \mathcal{U}_k^\delta(v_i)$. It is a contradiction.

Theorem 3.5. Let (X, δ^X) and (Y, δ^Y) be L -fuzzy (K_1, E_1) and (K_2, E_2) soft pre-proximity spaces. Let $\phi : X \rightarrow Y$, $\psi : E_1 \rightarrow E_2$ and $\eta : K_1 \rightarrow K_2$ are functions. If $\phi_{\psi, \eta} : (X, \delta_X) \rightarrow (X, \delta_X)$ is L -fuzzy soft proximally continuous, then $\phi_{\psi, \eta} : (X, \mathcal{U}^{\delta^X}) \rightarrow (Y, \mathcal{U}^{\delta^Y})$ is L -fuzzy soft uniformly continuous.

Proof.

$$\begin{aligned} ((\phi \times \phi)_\psi^{-1}(u_{f_i, g_i}))_e(x, y) &= (u_{f_i, g_i})_{\psi(e)}(\phi(x), \phi(y)) \\ &= u_{\phi_\psi^{-1}(f_i)_e, \phi_\psi^{-1}(g_i)_e}(x, y) \end{aligned}$$

we have

$$\begin{aligned} \mathcal{U}_{\eta(k)}^{\delta^Y}(u) &= \bigvee \{ \odot_{i=1}^n (\delta_{\eta(k)})^*(f_i, g_i) \mid \odot_{i=1}^n u_{f_i, g_i} \sqsubseteq u \} \\ &\leq \bigvee \{ \odot_{i=1}^n (\delta_{\eta(k)})^*(f_i, g_i) \mid (\phi \times \phi)_\psi^{-1}(\odot_{i=1}^n u_{f_i, g_i}) \sqsubseteq (\phi \times \phi)_\psi^{-1}(u) \} \\ &\leq \bigvee \{ \odot_{i=1}^n (\delta_{\eta(k)})^*(f_i, g_i) \mid \odot_{i=1}^n (\phi \times \phi)_\psi^{-1}(u_{f_i, g_i}) \sqsubseteq (\phi \times \phi)_\psi^{-1}(u) \} \\ &\leq \bigvee \{ \odot_{i=1}^n \delta_k^*(\phi_\psi^{-1}(f_i), \phi_\psi^{-1}(g_i)) \mid \odot_{i=1}^n (u_{\phi_\psi^{-1}(f_i), \phi_\psi^{-1}(g_i)}) \sqsubseteq (\phi \times \phi)_\psi^{-1}(u) \} \\ &= \mathcal{U}_k^{\delta^X}((\phi \times \phi)_\psi^{-1}(u)). \end{aligned}$$

Hence, $\phi_{\psi, \eta} : (X, \mathcal{U}^{\delta^X}) \rightarrow (Y, \mathcal{U}^{\delta^Y})$ is L -fuzzy soft uniformly continuous.

Example 3.6. Let $X = \{h_i \mid i = \{1, 2, 3\}\}$ with h_i =house and $E = \{e, b\}$ with e =expensive, b = beautiful. Define a binary operation $\odot$ on $[0, 1]$ by

$$x \odot y = \max\{0, x + y - 1\}, \quad x \rightarrow y = \min\{1 - x + y, 1\}$$

$$x \oplus y = \min\{1, x + y\}, \quad x^* = 1 - x$$

Then $([0, 1], \wedge, \rightarrow, 0, 1)$ is a stsc-quantale.

(1) Put $l, m \in (L^X)^E$ such that

$$\begin{aligned} l_e(h_1) &= 0.3, l_e(h_2) = 0.5, l_e(h_3) = 0.6 \\ l_b(h_1) &= 0.6, l_b(h_2) = 0.3, l_b(h_3) = 0.6 \\ m_e(h_1) &= 0.4, m_e(h_2) = 0.3, m_e(h_3) = 0.5 \\ m_b(h_1) &= 0.2, m_b(h_2) = 0.6, m_b(h_3) = 0.6 \end{aligned}$$

We define a $[0, 1]$ -fuzzy (E, E) -soft preproximity $\delta : E \rightarrow [0, 1]^{([0, 1]^X)^E \times ([0, 1]^X)^E}$ as follows

$$\delta_e(f, g) = \begin{cases} 0, & \text{if } f = 0_X \text{ or } g = 0_X, \\ 0.4, & \text{if } f \sqsubseteq l \sqsubseteq g^*, \\ 0.7, & \text{if } 0_X \neq f \sqsubseteq l \odot l \sqsubseteq g^*, l \not\sqsubseteq g^* \\ 1, & \text{otherwise,} \end{cases}$$

$$\delta_b(f, g) = \begin{cases} 0, & \text{if } f = 0_X \text{ or } g = 0_X, \\ 0.5, & \text{if } 0_X \neq f \sqsubseteq l \sqsubseteq g^* \neq 1_X, \\ 1, & \text{otherwise,} \end{cases}$$

We obtain $u_{l,l^*}, u_{l \odot l, (l \odot l)^*}, u_{m,m^*} \in ([0, 1]^{X \times X})^E$ as follows

$$u_{l,l^*}(e) = \begin{pmatrix} 1 & 1 & 1 \\ 0.8 & 1 & 1 \\ 0.7 & 0.9 & 1 \end{pmatrix}, \quad u_{l,l^*}(b) = \begin{pmatrix} 1 & 0.7 & 1 \\ 1 & 1 & 1 \\ 1 & 0.7 & 1 \end{pmatrix}$$

$$u_{l \odot l, (l \odot l)^*}(e) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0.8 & 0.8 & 1 \end{pmatrix}, \quad u_{l \odot l, (l \odot l)^*}(b) = \begin{pmatrix} 1 & 0.8 & 1 \\ 1 & 1 & 1 \\ 1 & 0.8 & 1 \end{pmatrix}$$

$$u_{m,m^*}(e) = \begin{pmatrix} 1 & 0.9 & 1 \\ 1 & 1 & 1 \\ 0.9 & 0.8 & 1 \end{pmatrix}, \quad u_{m,m^*}(b) = \begin{pmatrix} 1 & 1 & 1 \\ 0.6 & 1 & 1 \\ 0.6 & 1 & 1 \end{pmatrix}$$

By Theorem 3.2, we obtain a $[0, 1]$ -fuzzy soft (E, E) - pre-uniformity $\mathcal{U}^\delta : E \rightarrow [0, 1]^{([0, 1]^{X \times X})^E}$ as follows

$$\mathcal{U}_e^\delta(u) = \begin{cases} 1, & \text{if } u = 1_{X \times X}, \\ 0.6, & \text{if } u_{l, l^*} \sqsubseteq u \neq 1_{X \times X}, \\ 0.3, & \text{if } u_{l \odot l, (l \odot l)^*} \leq u \not\sqsupseteq u_{l, l}, \\ 0.2, & \text{if } u_{l, l^*} \odot u_{l, l^*} \sqsubseteq u \not\sqsupseteq u_{l \odot l, l \odot l}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{U}_b^\delta(u) = \begin{cases} 1, & \text{if } u = 1_{X \times X}, \\ 0.5, & \text{if } u_{m, m^*} \sqsubseteq u \neq 1_{X \times X}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $u_{l, l^*}[l] = l^*$ and $u_{l \odot l, l^* \odot l^*}[l \odot l] = l^* \odot l^*$ from Lemma 3.1 (12), by Theorems 2.9 and 3.2(4), we have $\delta^{\mathcal{U}^\delta} = \delta$, that is, $\mathcal{U}^\delta \in \Pi(\delta)$.

(2) Put $v, v \odot v, w \in ([0, 1]^{X \times X})^E$ as

$$\begin{aligned} v_e &= \begin{pmatrix} 1 & 0.6 & 0.5 \\ 0.3 & 1 & 0.5 \\ 0.4 & 0.6 & 1 \end{pmatrix} \quad v_b = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.7 & 1 & 0.5 \\ 0.6 & 0.6 & 1 \end{pmatrix} \\ (v \odot v)_e &= \begin{pmatrix} 1 & 0.2 & 0 \\ 0 & 1 & 0 \\ 0 & 0.2 & 1 \end{pmatrix} \quad (v \odot v)_b = \begin{pmatrix} 1 & 0 & 0 \\ 0.4 & 1 & 0 \\ 0.2 & 0.2 & 1 \end{pmatrix} \\ w_e &= \begin{pmatrix} 1 & 0.4 & 0.5 \\ 0.4 & 1 & 0.5 \\ 0.4 & 0.6 & 1 \end{pmatrix} \quad v_b = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.3 & 1 & 0.5 \\ 0.2 & 0.3 & 1 \end{pmatrix} \end{aligned}$$

We define $\mathcal{U} : E \rightarrow [0, 1]^{([0, 1]^{X \times X})^E}$ as follows:

$$\mathcal{U}_e(u) = \begin{cases} 1, & \text{if } u = 1_{Y \times Y} \\ 0.6, & \text{if } v \sqsubseteq u \neq 1_{Y \times Y}, \\ 0.3, & \text{if } v \odot v \sqsubseteq u \not\sqsupseteq v, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{U}_b(u) = \begin{cases} 1, & \text{if } u = 1_{Y \times Y} \\ 0.5, & \text{if } w \sqsubseteq u \neq 1_{Y \times Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{U}$ is a $[0, 1]$ -fuzzy (E, E) -soft pre-uniformity on X (ref. [13]). From Theorem 2.9, we obtain a $[0, 1]$ -fuzzy (E, E) -soft pre-proximity $\delta^{\mathcal{U}} : E \rightarrow$

$[0, 1]^{([0,1]^X)^E \times ([0,1]^X)^E}$ as follows

$$\delta_e^{\mathcal{U}}(f, g) = \begin{cases} 0, & \text{if } [1_{X \times X}](f) \sqsubseteq g^* \not\supseteq v[f], \\ 0.4, & \text{if } v[f] \sqsubseteq g^* \not\supseteq (v \odot v)[f], \\ 0.7, & \text{if } (v \odot v)[f] \sqsubseteq g^* \\ 1, & \text{otherwise,} \end{cases}$$

$$\delta_b^{\mathcal{U}}(f, g) = \begin{cases} 0, & \text{if } [1_{X \times X}](f) \sqsubseteq g^* \not\supseteq w[f], \\ 0.5, & \text{if } w[f] \sqsubseteq g^*, \\ 1, & \text{otherwise,} \end{cases}$$

By Theorem 3.2, we obtain $[0, 1]$ -fuzzy (E, E) soft pre-uniformity $\mathcal{U}^{\delta^{\mathcal{U}}} : E \rightarrow [0, 1]^{([0,1]^{X \times X})^E}$ as follows:

$$\mathcal{U}_e^{\delta^{\mathcal{U}}}(u) = \begin{cases} 1, & \text{if } u = 1_{X \times X} \\ 0.6, & \text{if } u_{v[f], (v[f])^*} \sqsubseteq u \neq 1_{Y \times Y}, \\ 0.3, & \text{if } u_{(v \odot v)[f], ((v \odot v)[f])^*} \sqsubseteq u \not\supseteq u_{v[f], (v[f])^*}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{U}_b^{\delta^{\mathcal{U}}}(u) = \begin{cases} 1, & \text{if } u = 1_{X \times X} \\ 0.5, & \text{if } u_{w[f], (w[f])^*} \sqsubseteq u \neq 1_{Y \times Y}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $v \leq u_{v[f], (v[f])^*}$ for $v \in \{v, v \odot v, w\}$, we have $\mathcal{U}^{\delta^{\mathcal{U}}} \leq \mathcal{U}$.

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