Abstract: In this paper, we introduce the notion of $L$-fuzzy $(K, E)$-soft pre-uniform spaces induced by $L$-fuzzy $(K, E)$-soft pre-proximity spaces in a strictly two-sided commutative quantale having an order reversing involution. We investigate the relations between $L$-fuzzy $(K, E)$-soft pre-uniform spaces and $L$-fuzzy $(K, E)$-soft pre-proximity spaces.

We give their examples.

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1. Introduction

Molodtsov [10,11] introduced the soft set as a mathematical tool for dealing information as the uncertainty of data in engineering, physics, computer sciences and many other diverse field. Many researchers have contributed towards the algebraic structure of soft set theory [1-7]. Shabir and Naz [16] initiated the

In this paper, we introduce the notion of L-fuzzy \((K,E)\)-soft pre-uniform spaces induced by L-fuzzy \((K,E)\)-soft pre-proximity spaces in a strictly two-sided commutative quantale. We investigate the relations between L-fuzzy \((K,E)\)-soft pre-uniform spaces and L-fuzzy \((K,E)\)-soft pre-proximity spaces. We give their examples.

2. Preliminaries

Let \(L = (L, \leq, \lor, \land, 0, 1)\) be a completely distributive lattice with the least element 0 and the greatest element 1 in \(L\).

**Definition 2.1.** [8,9] A complete lattice \((L, \leq, \odot)\) is called a strictly two-sided commutative quantale (stsc-quantale, for short) iff it satisfies the following properties.

(L1) \((L, \odot)\) is a commutative semigroup,
(L2) \(x = x \odot 1\), for each \(x \in L\) and 1 is the universal upper bound,
(L3) \(\odot\) is distributive over arbitrary joins, i.e. \((\bigvee_i x_i) \odot y = \bigvee_i (x_i \odot y)\).

There exists an implication operator \(\rightarrow\) defined as

\[ x \rightarrow y = \bigvee\{z \in L | x \odot z \leq y\}. \]

Then it satisfies \((x \odot z) \leq y\) iff \(z \leq (x \rightarrow y)\).

In this paper, we always assume that \((L, \leq, \odot, \rightarrow, \oplus, *)\) is a stsc-quantales with an order reversing involution \(*\) which is defined by \(x \oplus y = (x^* \odot y^*)^*\), \(x^* = x \rightarrow 0\) unless otherwise specified.

**Remark 2.2.** Every completely distributive lattice \((L, \leq, \land, \lor, *)\) with order reversing involution \(*\) is a stsc-quantale \((L, \leq, \odot, \oplus, *)\) with a strong negation \(*\) where \(\odot = \land\) and \(\oplus = \lor\).

**Lemma 2.3.** [8,9] For each \(x, y, z, x_i, y_i, w \in L\), we have the following properties.

1. \(1 \rightarrow x = x\), \(0 \odot x = 0\),
2. If \(y \leq z\), then \(x \odot y \leq x \odot z\), \(x \oplus y \leq x \oplus z\), \(x \rightarrow y \leq x \rightarrow z\) and \(z \rightarrow x \leq y \rightarrow x\),
(3) $x \leq y$ iff $x \rightarrow y = 1$.
(4) $(\bigwedge_i y_i)^* = \bigvee_i y_i^*$, $(\bigvee_i y_i)^* = \bigwedge_i y_i^*$,
(5) $x \rightarrow (\bigwedge_i y_i) = \bigvee_i (x \rightarrow y_i)$,
(6) $(\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)$,
(7) $x \rightarrow (\bigvee_i y_i) \geq \bigvee_i (x \rightarrow y_i)$,
(8) $(\bigwedge_i x_i) \rightarrow y \geq \bigvee_i (x_i \rightarrow y)$,
(9) $(x \circ y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(10) $x \circ y = (x \rightarrow y)^*$ and $x \oplus y = x^* \rightarrow y$,
(11) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \circ z) \rightarrow (y \odot w)$,
(12) $x \rightarrow y \leq (x \circ z) \rightarrow (y \circ z)$ and $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$,
(13) $(x \rightarrow y) \odot (z \rightarrow w) \leq (x \odot z) \rightarrow (y \oplus w)$.
(14) $x \rightarrow y = y^* \rightarrow x^*$.
(15) $(x \lor y) \odot (z \lor w) \leq (x \lor z) \lor (y \odot w) \leq (x \oplus z) \lor (y \odot w)$.
(16) $\bigvee_{i \in \Gamma} x_i \rightarrow \bigvee_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)$ and $\bigwedge_{i \in \Gamma} x_i \rightarrow \bigwedge_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)$.
(17) $(x \circ y) \odot (z \lor w) \leq (x \circ z) \lor (y \odot w)$.

Throughout this paper, $X$ refers to an initial universe, $E$ and $K$ are the sets of all parameters for $X$, and $L^X$ is the set of all $L$-fuzzy sets on $X$.

**Definition 2.4.** [4] A map $f$ is called an $L$- fuzzy soft set on $X$, where $f$ is a mapping from $E$ into $L^X$, i.e., $f_e := f(e)$ is an $L$- fuzzy set on $X$, for each $e \in E$. The family of all $L$- fuzzy soft sets on $X$ is denoted by $(L^X)^E$. Let $f$ and $g$ be two $L$- fuzzy soft sets on $X$.

1. $f$ is an $L$-fuzzy soft subset of $g$ and we write $f \subseteq g$ if $f_e \leq g_e$, for each $e \in E$. $f$ and $g$ are equal if $f \subseteq g$ and $g \subseteq f$.
(2) The intersection of $f$ and $g$ is an $L$- fuzzy soft set $h = f \cap g$, where $h_e = f_e \land g_e$, for each $e \in E$.
(3) The union of $f$ and $g$ is an $L$- fuzzy soft set $h = f \cup g$, where $h_e = f_e \lor g_e$, for each $e \in E$.
(4) An $L$- fuzzy soft set $h = f \circ g$ is defined as $h_e = f_e \circ g_e$, for each $e \in E$.
(5) An $L$- fuzzy soft set $h = f \oplus g$ is defined as $h_e = f_e \oplus g_e$, for each $e \in E$.
(6) The complement of an $L$- fuzzy soft sets on $X$ is denoted by $f^*$, where $f^* : E \rightarrow L^X$ is a mapping given by $f^*_e = (f_e)^*$, for each $e \in E$.
(7) $f$ is called a null $L$- fuzzy soft set and is denoted by $0_X$, if $f_e(x) = 0$, for each $e \in E$, $x \in X$.
(8) $f$ is called an absolute $L$- fuzzy soft set and is denoted by $1_X$, if $f_e(x) = 1$, for each $e \in E$, $x \in X$ and $(1_x)_e(x) = 1$.

**Definition 2.5.** [4] Let $\varphi : X \rightarrow Y$ and $\psi : E \rightarrow K$ be two mappings, where $E$ and $K$ are parameters sets for the crisp sets $X$ and $Y$, respectively.
Then $\varphi_{\psi} : (X, E) \to (Y, K)$ is called a fuzzy soft mapping. Let $f$ and $g$ be two fuzzy soft sets over $X$ and $Y$, respectively and let $\varphi_{\psi}$ be a fuzzy soft mapping from $(X, E)$ into $(Y, K)$.

(1) The image of $f$ under the fuzzy soft mapping $\varphi_{\psi}$, denoted by $\varphi_{\psi}(f)$ is the fuzzy soft set on $Y$ defined by

$$
\varphi(f)_k(y) = \begin{cases} 
\bigvee_{\varphi(x)=y} \bigvee_{\psi(e)=k} f_e(x), & \text{if } x \in \varphi^{-1}(y) \\
0, & \text{otherwise,}
\end{cases}
$$

$\forall k \in K, \forall y \in Y$.

(2) The pre-image of $g$ under the fuzzy soft mapping $\varphi_{\psi}$, denoted by $\varphi_{\psi}^{-1}(g)$ is the fuzzy soft set on $X$ defined by

$$
\varphi_{\psi}^{-1}(g)_e(x) = g_{\psi(e)}(\varphi(x)), \forall e \in E, \forall x \in X.
$$

**Definition 2.6.** [13] An $L$-fuzzy $(K, E)$-soft pre-uniformity is a mapping $U : K \to L^{(L^{X\times X})^E}$ which satisfies the following conditions .

(SU1) There exists $u \in (L^{X\times X})^E$ such that $U_k(u) = 1$.

(SU2) If $v \subseteq u$, then $U_k(v) \leq U_k(u)$.

(SU3) For every $u, v \in (L^{X\times X})^E$, $U_k(u \circ v) \geq U_k(u) \circ U_k(v)$.

(SU4) If $U_k(u) \neq 0$ then $\top_{\triangle} \subseteq u$ where, for each $e \in E$,

$$
(\top_{\triangle})_e(x, y) = \begin{cases} 
1, & \text{if } x = y, \\
0, & \text{if } x \neq y.
\end{cases}
$$

The pair $(X, U)$ is called an $L$-fuzzy $(K, E)$-soft pre-uniform space.

An $L$-fuzzy $(K, E)$-soft pre-uniformity is an $L$-fuzzy $(K, E)$-soft quasi-uniformity if

(UQ) $U_k(u) \leq \bigvee \{U_k(v) \circ U_k(w) \mid v \circ w \subseteq u\}$, where

$$
v_e \circ w_e(x, z) = \bigvee_{y \in X} v_e(x, y) \circ w_e(y, z),
$$

An $L$-fuzzy $(K, E)$-soft quasi-uniform space $(X, U)$ is said to be an $L$-fuzzy $(K, E)$-soft uniform space if

(U) $U_k(u) \leq U_k(u^{-1})$, where $(u^{-1})_e(x, y) = u_e(y, x)$ for each $k \in K$ and $u \in (L^{X\times X})^E$.

Let $(X, U^1)$ be an $L$-fuzzy $(K_1, E_1)$-soft pre-uniform space and $(Y, U^2)$ be an $L$-fuzzy $(K_2, E_2)$-soft pre-uniform space. Let $\varphi : X \to Y$, $\psi : E_1 \to E_2$ and $\eta : K_1 \to K_2$ be mappings. Then $\varphi_{\psi, \eta}$ from $(X, U^1)$ into $(Y, U^2)$ is called $L$-fuzzy soft uniformly continuous if

$$
U^2_{\eta(k)}(v) \leq U^1_{k}( ((\varphi \times \varphi)^{-1})(v)) \forall v \in (L^Y)^{E_2}, k \in K_1.
$$
**Remark 2.7.** Let \((X, \mathcal{U})\) be an \(L\)-fuzzy \((K, E)\)-soft uniform space.

1. By (SU1) and (SU2), we have \(\mathcal{U}_k(1_{X \times X}) = 1\) because \(u \subseteq 1_{X \times X}\) for all \(u \in (L^X)^E\).
2. Since \(\mathcal{U}_k(u) \subseteq \mathcal{U}_k(u^{-1}) \subseteq \mathcal{U}_k((u^{-1})^{-1}) = \mathcal{U}_k(u)\), then \(\mathcal{U}_k(u) = \mathcal{U}_k(u^{-1})\).

**Definition 2.8.** [13] A mapping \(\delta : K \to L^{(L^X)^E \times (L^X)^E} (\delta_k = \delta(k) : L^X \times L^X \to L)\) is called an \(L\)-fuzzy \((K, E)\)-soft pre-proximity on \(X\) if it satisfies the following axioms.

1. \((\text{SP1})\) \(\delta_k(0_X, 1_X) = \delta_k(1_X, 0_X) = 0\).
2. \((\text{SP2})\) If \(\delta_k(f, g) \neq 1\), then \(f \subseteq g^*\).
3. \((\text{SP3})\) If \(f_1 \subseteq f_2\) and \(g_1 \subseteq g_2\), then \(\delta_k(f_1, g_1) \leq \delta_k(f_2, g_2)\).
4. \((\text{SP4})\) If \(\delta_k(f_1 \odot f_2, g_1 \oplus g_2) \leq \delta_k(f_1, g_1) \oplus \delta_k(f_2, g_2)\).

The pair \((X, \delta)\) is called an \(L\)-fuzzy \((K, E)\)-soft pre-proximity space.

An \(L\)-fuzzy \((K, E)\)-soft pre-proximity is called an \(L\)-fuzzy \((K, E)\)-soft quasi-proximity on \(X\) if

\[(\text{PQ})\] \(\delta_k(f, g) \geq \bigwedge_h \{\delta_k(f, h) \oplus \delta_k(h^*, g)\}\).

An \(L\)-fuzzy \((K, E)\)-soft quasi-proximity is called an \(L\)-fuzzy \((K, E)\)-soft quasi-proximity on \(X\) if

\[(\text{SP})\] \(\delta_k(f, g) = \delta_k(g, f)\).

Let \((X, \delta^1)\) be an \(L\)-fuzzy \((K_1, E_1)\)-soft quasi proximity space and \((Y, \delta^2)\) be an \(L\)-fuzzy \((K_2, E_2)\)-soft pre-proximity space. Let \(\varphi : X \to Y\), \(\psi : E_1 \to E_2\) and \(\eta : K_1 \to K_2\) be mappings. Then \(\varphi \psi \eta\) from \((X, \delta^1)\) into \((Y, \delta^2)\) is called \(L\)-fuzzy soft proximally continuous if \(\delta^1_k(\varphi^{-1}(f), \psi^{-1}(g)) \leq \delta^2_{\eta(k)}(f, g)\) \(\forall f, g \in (L^Y)^{E_2}, k \in K_1\).

**Theorem 2.9.** [13] Let \((X, \mathcal{U})\) be an \(L\)-fuzzy \((K, E)\)-soft pre-uniform space.

Define a mapping \(\delta^U : K \to L^{(L^X)^E \times (L^X)^E}\) by

\[\delta^U_k(f, g) = \bigwedge \{\mathcal{U}^c_k(u) \mid u[f] \subseteq g^*\},\]

where \(u_e[f_e(x)](x) = \bigvee_{y \in X} (f_e(y) \circ u_e(y, x))\), \(\forall x \in X, \forall e \in E, \forall u \in (L^X)^E, f \in (L^X)^E\). Then we have the following properties.

1. \(\delta^U\) is an \(L\)-fuzzy \((K, E)\)-soft pre-proximity space.
2. If \((X, \mathcal{U})\) is an \(L\)-fuzzy \((K, E)\)-soft quasi-uniform space, then \(\delta^U\) is \(L\)-fuzzy \((K, E)\)-soft quasi-proximity space.

**Theorem 2.10.** [13] Let \((X, \mathcal{U})\) be an \(L\)-fuzzy \((K_1, E_1)\)-soft quasi-uniform space and \((Y, \mathcal{V})\) be an \(L\)-fuzzy \((K_2, E_2)\)-soft quasi-uniform space, \(\phi : X \to Y, \psi : E_1 \to E_2\) and \(\eta : K_1 \to K_2\) are functions. If \(\phi \psi \eta\) from \((X, \mathcal{U})\) into \((X, \mathcal{V})\)
is $L$-fuzzy soft uniformly continuous, then $\phi_{\psi,\eta} : (X, \delta^L) \to (Y, \delta^Y)$ is $L$-fuzzy soft proximally continuous.

3. $L$-Fuzzy $(K, E)$-Soft Pre-Uniformities Induced by $L$-Fuzzy $(K, E)$-Soft Pre-Proximities

Lemma 3.1. [12] For every $f, g \in (L^X)^E$, we define $u_{f,g}, u_{f,g}^{-1} \in (L^X \times X)^E$ by

$$(u_{f,g})_e(x,y) = f_e(x) \to g_e(y) \quad \forall e \in E,$$

$$(u_{f,g}^{-1})_e(x,y) = (u_{f,g})_e(y,x).$$

Then we have the following statements

1. $1_{X \times X} = u_{0_0,0_X} = u_{1_X,1_X}$.
2. If $f_1 \sqsubseteq f_2$ and $g_1 \sqsubseteq g_2$, then $u_{f_2,g_1} \sqsubseteq u_{f_1,g_2}$.
3. If $f \sqsubseteq g$, then $1_{\Delta} \sqsubseteq u_{f,g}$.
4. For every $u_{g,h} \in (L^X \times X)^E$ and $f \in (L^X)^E$, we have $u_{h,g} \circ u_{f,h} \sqsubseteq u_{f,g}$.
5. $u_{f,g} \sqsubseteq u_{f,g} \circ u_{f,g} \sqsubseteq u_{f,g}$.
6. $u_{f,g} \sqsubseteq u_{f,g} \circ u_{f,g} \sqsubseteq u_{f,g}$.
7. $u_{f,g}^{-1} = u_{g^*} \circ f^*$.
8. $u_{f_1 \circ f_2 \circ g_1 \circ g_2} = u_{g_2^* \circ g_1^* \circ f_2^* \circ f_1^*}$.
9. $u_{f_1 \circ f_2 \circ g_1 \circ g_2} = u_{g_2^* \circ g_1^* \circ f_2^* \circ f_1^*}$.
10. $u[f] \sqsubseteq g$ iff $u \sqsubseteq u_{f,g}$.
11. $u_{f,g} = \bigvee \{ u \in (L^X \times X)^E | u[f] \sqsubseteq g \}$.
12. $u_{f,g}^* [f] \sqsubseteq g$ and $u_{f,g}^* [f] = f$.

In the following theorem, we obtain an $L$-fuzzy $(K, E)$-soft pre-uniform space from an $L$-fuzzy $(K, E)$-soft pre-proximity space.

Theorem 3.2. Let $(X, \delta)$ be an $L$-fuzzy $(K, E)$-soft pre-proximity space. Define $\mathcal{U}^\delta : K \to L(L^X \times X)^E$ by

$$\mathcal{U}^\delta_k(u) = \bigvee \{ \circ_{i=1}^n \delta_k(f_i, g_i)^* \mid \circ_{i=1}^n u_{f_i,g_i} \sqsubseteq u \},$$

where $\bigvee$ is taken over every finite family $\{ u_{f_i,g_i} \mid i = 1, 2, 3, ..., n \}$. Then

1. $\mathcal{U}^\delta_k(u_{f,g}) = \delta_k(f,g)$.
2. $\mathcal{U}^\delta$ is an $L$-fuzzy $(K, E)$-soft pre-uniformity on $X$. 


(3) If \((X, \delta)\) is an \(L\)-fuzzy \((K, E)\)-soft quasi-proximity space, then \(U^\delta\) is an \(L\)-fuzzy \((K, E)\)-soft quasi-uniformity on \(X\).

(4) \(\delta^u \circ \delta = \delta\).

(5) If \((X, \delta)\) is an \(L\)-fuzzy \((K, E)\)-soft proximity space, then \(U^\delta(u) = U^\delta(u^{-1})\).

**Proof.**

(1) From Lemma 3.1(5), since \(\bigcirc_{i=1}^n u_{f_i,g_i} \subseteq u \bigcirc_{i=1}^n f_i \bigcirc_{i=1}^n g_i\), by (SP4), we have \(\delta^*_k(\bigcirc_{i=1}^n f_i, \bigcirc_{i=1}^n g_i) \geq \bigcirc_{i=1}^n \delta^*_k(f_i, g_i)\). It follows \(U^\delta_k(u_{f,g}) = \delta^*_k(f,g)\).

(2) (SU1) Since \(\delta_k(0_X,1_X)^* = \delta_k(1_X,0_X)^* = 1\), there exists \(1_X \times X = u_{0_X,0_X} = u_{1_X,1_X} \in (L^X \times X)^E\). It follows \(U^\delta_k(1_X \times X) = 1\).

(SU2) It is trivial from the definition of \(U^\delta\).

(SU3) For every \(u,v \in (L^X \times X)^E\), each two families \(\{u_{f_i,g_i} | \bigcirc_{i=1}^n u_{f_i,g_i} \subseteq u\}\) and \(\{u_{h_j,w_j} | \bigcirc_{j=1}^m u_{h_j,w_j} \subseteq v\}\), we have

\[
U^\delta_k(u) \circ U^\delta_k(v) = \left( \bigvee \left\{ \bigcirc_{i=1}^n \delta_k(f_i,g_i)^* \big{|} \bigcirc_{i=1}^n u_{f_i,g_i} \subseteq u \right\} \right) \\
\cap \left( \bigvee \left\{ \bigcirc_{j=1}^m \delta_k(h_j,w_j)^* \big{|} \bigcirc_{j=1}^m u_{h_j,w_j} \subseteq v \right\} \right) \\
\leq \bigvee \left\{ \bigcirc_{i=1}^n \delta_k(f_i,g_i)^* \cap \bigcirc_{j=1}^m \delta_k(h_j,w_j)^* \big{|} \bigcirc_{i=1}^n u_{f_i,g_i} \subseteq u, \bigcirc_{j=1}^m u_{h_j,w_j} \subseteq v \right\} \\
\leq \bigvee \left\{ \bigcirc_{i=1}^n \delta_k(f_i,g_i)^* \cap \bigcirc_{j=1}^m \delta_k(h_j,w_j)^* \big{|} \bigcirc_{i=1}^n u_{f_i,g_i} \cap \bigcirc_{j=1}^m u_{h_j,w_j} \subseteq u \cap v \right\} \\
\leq U^\delta_k(u \circ v).
\]

(SU4) If \(U_k(u) \neq 0\), there exists a family \(\{u_{f_i,g_i} | \bigcirc_{i=1}^m u_{f_i,g_i} \subseteq u\}\) such that \(\bigcirc_{i=1}^m \delta^*_k(f_i,g_i) \neq 0\). Since \(\delta_k(f_i,g_i) \neq 1\), for \(i = 1, 2, ..., m\), then \(f_i \subseteq g_i^*\) for \(i = 1, 2, ..., m\), i.e. \(1 \triangle \bigcup u_{f_i,g_i^*}\). Thus \(1 \triangle \bigcup u_{f_i,g_i^*} \subseteq u\).

(3) (UQ) Suppose there exists \(u \in (L^X \times X)^E\) such that

\[
\bigvee \{U^\delta_k(v) \circ U^\delta_k(w) \big{|} v \circ w \subseteq u\} \neq U^\delta_k(u).
\]

Put \(t = \bigvee \{U^\delta_k(v) \circ U^\delta_k(w) \big{|} v \circ w \subseteq u\}\). From the Definition of \(U^\delta_k(u)\), there exists family \(\{u_{f_i,g_i} | \bigcirc_{i=1}^m u_{f_i,g_i} \subseteq u\}\) such that

\[
t \not\supseteq \bigcirc_{i=1}^m \delta_k(f_i,g_i)^*.
\]

Since \(\delta_k(f_i,g_i) \geq \bigwedge_{h_i} \{\delta_k(f,h) \oplus \delta_k(h^*,g)\}\),

\[
t \not\supseteq \bigcirc_{i=1}^m \{\bigvee_{h_i \in (L^X)^E} \{\delta_k(f_i,h_i)^* \oplus \delta_k(h_i^*,g_i)^*\}\}.
\]
Since \( L \) is a stsc-quantal, there exists \( h_i \in (L^X)^E \) such that
\[
\exists \bigoplus_{i=1}^m (\delta_k(f_i, h_i)^* \circ \delta_k(h_i^*, g_i)^*).
\]

On the other hand, put \( v_i = u_{h_i^*, g_i^*}, w_i = u_{f_i, h_i^*} \), then from Lemma 3.1 (4), it satisfies \( v_i \circ w_i \subseteq u_{h_i^*, g_i^*} \circ u_{f_i, h_i^*} \subseteq u_{f_i, g_i^*} \), by (1),
\[
U_k^\delta(v_i) = \delta_k(h_i^*, g_i)^*, \quad U_k^\delta(w_i) = \delta_k(f_i, h_i)^*.
\]

Let \( v = \bigoplus_{i=1}^m v_i \) and \( w = \bigoplus_{i=1}^m w_i \) be given. Since \( v_i \circ w_i \subseteq u_{f_i, g_i^*} \), for each \( i = 1, 2, 3, \ldots, m \), we have \( (\bigoplus_{i=1}^m v_i) \circ (\bigoplus_{i=1}^m w_i) = \bigoplus_{i=1}^m (v_i \circ w_i) \subseteq \bigoplus_{i=1}^m u_{f_i, g_i^*} \subseteq u \).

Then we have \( v \circ w \subseteq u \) and
\[
U_k^\delta(v) \geq \bigoplus_{i=1}^m U_k^\delta(v_i) \text{ and } U_k^\delta(w) \geq \bigoplus_{i=1}^m U_k^\delta(w_i).
\]
Thus,
\[
t = \bigvee\{U_k^\delta(v) \circ U_k^\delta(w) \mid v \circ w \subseteq u\}
\geq U_k^\delta(v) \circ U_k^\delta(w) \geq (\bigoplus_{i=1}^m U_k^\delta(v_i)) \circ (\bigoplus_{i=1}^m U_k^\delta(w_i))
\geq \bigoplus_{i=1}^m (\delta_k(h_i^*, g_i)^* \circ \delta_k(f_i, h_i)^*).
\]
It is a contradiction. Then \( U^\delta \) is an \( L \)-fuzzy \((K, E)\)-soft quasi-uniformity on \( X \).

(4) Since \( u[f] \subseteq g^* \), by Lemma 3.1(10), \( u \subseteq u_{f, g^*} \). Hence, \( \delta_k^\delta(f, g) = \bigwedge\{(U_k^\delta)^*(u) \mid u[f] \subseteq g^*\} = (U_k^\delta)^*(u_{f, g^*}) = \delta_k(f, g) \).

(5) \( U_k^\delta(u) = \bigvee\{\bigoplus_{i=1}^n \delta_k(f_i, g_i)^* \mid \bigoplus_{i=1}^n u_{f_i, g_i^*} \subseteq u\}
= \bigvee\{\bigoplus_{i=1}^n \delta_k(g_i, f_i)^* \mid \bigoplus_{i=1}^n u_{f_i, g_i^*} \subseteq u^{-1}\}
= \bigvee\{\bigoplus_{i=1}^n \delta_k(g_i, f_i)^* \mid \bigoplus_{i=1}^n u_{g_i, f_i^*} \subseteq u^{-1}\}
= U_k^\delta(u^{-1}).

**Definition 3.3.** An \( L \)-fuzzy \((K, E)\)-soft pre-uniform structure \( U \) on \( X \) is said to be compatible with an \( L \)-fuzzy \((K, E)\)-soft pre-proximity \( \delta \) on \( X \) if \( \delta^{U} = \delta \). The class \( \prod(\delta) \) denotes the family of all \( L \)-fuzzy \((K, E)\)-soft pre-uniform structure which are compatible with a given \( L \)-fuzzy \((K, E)\)-soft pre-proximity structure \( \delta \).

**Theorem 3.4.** Let \( \delta \) be an \( L \)-fuzzy \((K, E)\)-soft pre-proximity on \( X \) and the \( L \)-fuzzy \((K, E)\)-soft pre-proximity \( \delta^{U^\delta} \) induced by \( U^\delta \). Then
(1) \( \delta^{U^\delta} = \delta \), that is, \( U^\delta \in \prod(\delta) \),
(2) \( U^\delta \) is the coarsest member of \( \prod(\delta) \), i.e., \( U^\delta \subseteq U \).

**Proof.** (1) is easily proved from Theorem 3.2(4).
(2) Let \( U \) be an arbitrary member of \( \prod(\delta) \). We will show that \( U_k^\delta(u) \leq U_k(u) \), for all \( u \in (L^{X \times X})^E \).

Suppose that there exists \( u \in (L^{X \times X})^E \) such that
\[
U_k^\delta(u) \not\leq U_k(u).
\]
By the definition of \( U_k^\delta \), there exists a family \( \{u_{f_i,g_i} \mid \circ i=1 \wedge u_{f_i,g_i} \subseteq u\} \) such that
\[
\circ i=1 \delta_k(f_i,g_i) \not\subseteq U_k(u).
\]
Since \( U^\delta \in \prod(\delta) \) from (1), that is, \( \delta_k(f_i,g_i) = \delta_k^U(f_i,g_i) \) for each \( i = 1,2,\ldots,m \), by the definition of \( \delta_k^U \), there exists \( v_i \in (L^{X \times X})^E \) with \( v_i[f_i] \subseteq g_i^* \) such that
\[
\circ i=1 U_k^\delta(v_i) \not\subseteq U_k(u).
\]
On the other hand, put \( v = \circ i=1 v_i \). Since \( v_i[f_i] \subseteq g_i^* \), by the definition of \( u_{f_i,g_i^*} \), we have \( v_i \subseteq u_{f_i,g_i^*} \). It follows that
\[
v = \circ i=1 v_i \subseteq \circ i=1 u_{f_i,g_i^*} \subseteq u.
\]
Hence, \( U_k(u) \geq U_k^\delta(\circ i=1 u_{f_i,g_i}) \geq U_k^\delta(v) \geq \circ i=1 U_k^\delta(v_i) \). It is a contradiction.

**Theorem 3.5.** Let \((X,\delta^X)\) and \((Y,\delta^Y)\) be \(L\)-fuzzy \((K_1,E_1)\) and \((K_2,E_2)\) soft pre-proximity spaces. Let \( \phi : X \rightarrow Y, \psi : E_1 \rightarrow E_2 \) and \( \eta : K_1 \rightarrow K_2 \) are functions. If \( \phi_{\psi,\eta} : (X,\delta^X) \rightarrow (X,\delta^X) \) is \(L\)-fuzzy soft proximally continuous, then \( \phi_{\psi,\eta} : (X,\mathcal{U}^\delta_X) \rightarrow (Y,\mathcal{U}^\delta_Y) \) is \(L\)-fuzzy soft uniformly continuous.

**Proof.**
\[
((\phi \times \phi)^{-1}(u_{f_i,g_i}),\circ i=1 u_{f_i,g_i}) \in \mathcal{U}_\psi(x,y) = (u_{f_i,g_i},\circ i=1 u_{f_i,g_i}) \in \mathcal{U}_\psi(x,y)
\]
we have
\[
\mathcal{U}^\delta_Y(\eta(\psi))(u) = \cup\{\circ i=1 (\delta(\psi))^*(f_i,g_i) \mid \circ i=1 u_{f_i,g_i} \subseteq u\}
\]
\[
\leq \cup\{\circ i=1 (\delta(\psi))^*(f_i,g_i) \mid (\phi \times \phi)^{-1}(\circ i=1 u_{f_i,g_i}) \subseteq (\phi \times \phi)^{-1}(u)\}
\]
\[
\leq \cup\{\circ i=1 (\delta(\psi))^*(f_i,g_i) \mid \circ i=1 (\phi \times \phi)^{-1}(u_{f_i,g_i}) \subseteq \phi \times \phi)^{-1}(u)\}
\]
\[
\leq \cup\{\circ i=1 (\phi^{-1}(f_i),\phi^{-1}(g_i)) \mid \circ i=1 (u_{f_i,g_i}) \subseteq (\phi \times \phi)^{-1}(u)\}
\]
\[
= \mathcal{U}_\psi^\delta((\phi \times \phi)^{-1}(u)).
\]
Hence, \( \phi_{\psi,\eta} : (X,\mathcal{U}^\delta_X) \rightarrow (Y,\mathcal{U}^\delta_Y) \) is \(L\)-fuzzy soft uniformly continuous.
Example 3.6. Let $X = \{h_i \mid i = \{1, 2, 3\}\}$ with $h_i=$house and $E = \{e, b\}$ with $e=$expensive, $b=$ beautiful. Define a binary operation $\odot$ on $[0, 1]$ by

$$x \odot y = \max\{0, x + y - 1\}, \ x \rightarrow y = \min\{1 - x + y, 1\}$$

Then $(0, 1], \wedge, \rightarrow, 0, 1)$ is a stsc-quantle.

(1) Put $l, m \in (L^X)^E$ such that

$$l_e(h_1) = 0.3, l_e(h_2) = 0.5, l_e(h_3) = 0.6$$
$$l_b(h_1) = 0.6, l_b(h_2) = 0.3, l_b(h_3) = 0.6$$
$$m_e(h_1) = 0.4, m_e(h_2) = 0.3, m_e(h_3) = 0.5$$
$$m_b(h_1) = 0.2, m_b(h_2) = 0.6, m_b(h_3) = 0.6$$

We define a $[0, 1]$-fuzzy $(E, E)$-soft preproximity $\delta : E \rightarrow [0, 1]^{([0,1]^X \times ([0,1]^X)^E}$ as follows

$$\delta_e(f, g) = \begin{cases} 
0, & \text{if } f = 0_X \text{ or } g = 0_X, \\
0.4, & \text{if } f \sqsubseteq l \sqsubseteq g^*, \\
0.7, & \text{if } 0_X \neq f \sqsubseteq l \odot l \sqsubseteq g^*, l \not\sqsubseteq g^* \\
1, & \text{otherwise}, 
\end{cases}$$

$$\delta_b(f, g) = \begin{cases} 
0, & \text{if } f = 0_X \text{ or } g = 0_X, \\
0.5, & \text{if } 0_X \neq f \sqsubseteq l \odot l \sqsubseteq g^* \neq 1_X, \\
1, & \text{otherwise}, 
\end{cases}$$

We obtain $u_{l,l^*}, u_{l\odot l, (l\odot l)^*}, u_{m,m^*} \in ([0, 1]^{X} \times X)^E$ as follows

$$u_{l,l^*}(e) = \begin{pmatrix} 1 & 1 & 1 \\
0.8 & 1 & 1 \\
0.7 & 0.9 & 1 \end{pmatrix}, \ u_{l,l^*}(b) = \begin{pmatrix} 1 & 0.7 & 1 \\
1 & 1 & 1 \\
1 & 0.7 & 1 \end{pmatrix}$$

$$u_{l\odot l, (l\odot l)^*}(e) = \begin{pmatrix} 1 & 1 & 1 \\
1 & 1 & 1 \\
0.8 & 0.8 & 1 \end{pmatrix}, \ u_{l\odot l, (l\odot l)^*}(b) = \begin{pmatrix} 1 & 0.8 & 1 \\
1 & 1 & 1 \\
1 & 0.8 & 1 \end{pmatrix}$$

$$u_{m,m^*}(e) = \begin{pmatrix} 1 & 0.9 & 1 \\
1 & 1 & 1 \\
0.9 & 0.8 & 1 \end{pmatrix}, \ u_{m,m^*}(b) = \begin{pmatrix} 1 & 1 & 1 \\
0.6 & 1 & 1 \\
0.6 & 1 & 1 \end{pmatrix}$$
By Theorem 3.2, we obtain a $[0,1]$-fuzzy soft $(E,E)$-pre-uniformity $U^\delta : E \rightarrow [0,1]^{([0,1]^{X\times X})^E}$ as follows

\[
U^\delta_e(u) = \begin{cases} 
1, & \text{if } u = 1_{X\times X}, \\
0.6, & \text{if } u_{l^*} \subseteq u \neq 1_{X\times X}, \\
0.3, & \text{if } u_{l \circ l} \leq u \not\supseteq u_{l^*}, \\
0.2, & \text{if } u_{l^*} \circ u_{l^*} \supseteq u \not\supseteq u_{l \circ l \circ l^*}, \\
0, & \text{otherwise}. 
\end{cases}
\]

\[
U^\delta_b(u) = \begin{cases} 
1, & \text{if } u = 1_{X\times X}, \\
0.5, & \text{if } u_{m,m^*} \subseteq u \neq 1_{X\times X}, \\
0, & \text{otherwise}. 
\end{cases}
\]

Since $u_{l^*}[l] = l^*$ and $u_{l \circ l} \circ l^*[l \circ l] = l^* \circ l^*$ from Lemma 3.1 (12), by Theorems 2.9 and 3.2(4), we have $\delta^\Pi = \delta$, that is, $U^\delta \in \Pi(\delta)$.

(2) Put $v, v \circ v, w \in ([0,1]^{X\times X})^E$ as

\[
v_e = \begin{pmatrix} 1 & 0.6 & 0.5 \\ 0.3 & 1 & 0.5 \\ 0.4 & 0.6 & 1 \end{pmatrix}, \quad v_b = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.7 & 1 & 0.5 \\ 0.6 & 0.6 & 1 \end{pmatrix}
\]

\[
(v \circ v)_e = \begin{pmatrix} 1 & 0.2 & 0 \\ 0 & 1 & 0 \\ 0 & 0.2 & 1 \end{pmatrix}, \quad (v \circ v)_b = \begin{pmatrix} 1 & 0 & 0 \\ 0.4 & 1 & 0 \\ 0.2 & 0.2 & 1 \end{pmatrix}
\]

\[
w_e = \begin{pmatrix} 1 & 0.4 & 0.5 \\ 0.4 & 1 & 0.5 \\ 0.4 & 0.6 & 1 \end{pmatrix}, \quad v_b = \begin{pmatrix} 1 & 0.5 & 0.3 \\ 0.3 & 1 & 0.5 \\ 0.2 & 0.3 & 1 \end{pmatrix}
\]

We define $U : E \rightarrow [0,1]^{([0,1]^{X\times X})^E}$ as follows:

\[
U_e(u) = \begin{cases} 
1, & \text{if } u = 1_{Y\times Y} \\
0.6, & \text{if } v \supseteq u \neq 1_{Y\times Y}, \\
0.3, & \text{if } v \circ v \subseteq u \not\supseteq v, \\
0, & \text{otherwise}. 
\end{cases}
\]

\[
U_b(u) = \begin{cases} 
1, & \text{if } u = 1_{Y\times Y} \\
0.5, & \text{if } w \supseteq u \neq 1_{Y\times Y}, \\
0, & \text{otherwise}. 
\end{cases}
\]

Then $U$ is a $[0,1]$-fuzzy $(E,E)$-soft pre-uniformity on $X$ (ref. [13]) . From Theorem 2.9, we obtain a $[0,1]$-fuzzy $(E,E)$-soft pre-proximity $\delta^U : E \rightarrow$...
\[ \delta^U_e(f, g) = \begin{cases} 0, & \text{if } [1_X \times X](f) \subseteq g^* \not\supseteq v[f], \\ 0.4, & \text{if } v[f] \subseteq g^* \not\supseteq (v \circ v)[f], \\ 0.7, & \text{if } (v \circ v)[f] \subseteq g^* \\ 1, & \text{otherwise}, \end{cases} \]

\[ \delta^U_b(f, g) = \begin{cases} 0, & \text{if } [1_X \times X](f) \subseteq g^* \not\supseteq w[f], \\ 0.5, & \text{if } w[f] \subseteq g^* \\ 1, & \text{otherwise}, \end{cases} \]

By Theorem 3.2, we obtain \([0,1]-\text{fuzzy } (E,E) \text{ soft pre-uniformity } U^\delta : E \to [0,1]^{([0,1]^E \times [0,1]^E)} \) as follows:

\[ U^\delta_e(u) = \begin{cases} 1, & \text{if } u = 1_X \times X \\ 0.6, & \text{if } u_{v[f],(v[f])^*} \subseteq u \neq 1_Y \times Y, \\ 0.3, & \text{if } u_{(v \circ v)[f],((v \circ v)[f])^*} \subseteq u \not\supseteq u_{v[f],(v[f])^*}, \\ 0, & \text{otherwise}. \end{cases} \]

\[ U^\delta_b(u) = \begin{cases} 1, & \text{if } u = 1_X \times X \\ 0.5, & \text{if } u_{w[f],(w[f])^*} \subseteq u \neq 1_Y \times Y, \\ 0, & \text{otherwise}. \end{cases} \]

Since \( v \leq u_{v[f],(v[f])^*} \) for \( v \in \{v, v \circ v, w\} \), we have \( U^\delta \leq U \).

References


