LINEAR CANONICAL TRANSFORM FOR INTEGRABLE BOEHMIANS

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Abstract: In this paper we extend the linear canonical transform to class of integrable Boehmians by using new definition of convolution defined in [3]. Further, we prove that the extended linear canonical transform have expected properties like linear, one-to-one, onto and continuous from one Bohmian space to another Bohmian space and obtained some basic properties.

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1. Introduction

Boehmian is a new class of generalized functions introduced by Boehme [1] which open the new door to area of research in mathematics [4, 5, 7, 8]. The construction of Boehmians is given by Mikusinski and Mikusinski. Mikusinski has studied Fourier transform of integrable Boehmians in [6]. Zayed and Abhishek singh extend the fractional Fourier transform to class of integrable Boehmians in [11, 9] respectively. Bhosale and Chaudhary [2] extended the fractional Fourier transform to distributions of compact support. The Fourier transform and fractional Fourier transform are the special cases of linear canonical transform (LCT)[3]. Let $\mathcal{L}^1(\mathbb{R})$ be the space of all complex valued absolutely integrable functions on $\mathbb{R}$ with norm $\|f\|_1 = \int_{\mathbb{R}} |f(t)|dt \leq M$, for some $M > 0$.

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The linear canonical transform of $f \in \mathcal{L}^1(\mathbb{R})$, is define as

$$F_A = \mathcal{L}_A[f(t)](u) = \begin{cases} \sqrt{\frac{1}{2\pi ib}} \int_{-\infty}^{\infty} e^{\frac{i}{2}[\frac{a}{b}t^2 - \frac{2}{b}ut + (\frac{d}{b})u^2]} f(t) dt, & \text{for } b \neq 0, \\ \sqrt{d} e^{\frac{i}{2}cd u^2} f(du), & \text{for } b = 0, \end{cases} \quad (1)$$

where $\mathcal{L}_A$ is the unitary linear canonical operator and $A = (a, b, c, d)$, a, b, c, d are real number satisfying $ad - bc = 1$. The inverse transform for linear canonical transform is given by a linear canonical transform having parameter $A^{-1} = (d, -b, -c, a)$

$$f(t) = \begin{cases} \sqrt{\frac{1}{-2\pi ib}} \int_{-\infty}^{\infty} e^{\frac{i}{2}[(\frac{-d}{b})u^2 + (\frac{2}{b})ut - (\frac{a}{b})t^2]} F_A(u) du, & \text{for } b \neq 0, \\ \sqrt{a} e^{\frac{i}{2}ca u^2} f(au), & \text{for } b = 0. \end{cases} \quad (2)$$

Definition 1.1. [3] Let $W(t, \tau) = e^{i\tau(\tau-t)\frac{a}{b}}$ be the weight function. For any two functions f and g the convolution operation $*^A$ is defined as,

$$h(t) = (f *^A g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) W(t, \tau) d\tau \quad (3)$$

Theorem 1.1. [3] (New Convolution Theorem) Let $h(t) = (f *^A g)(t)$ and $H_A(u), F_A(u), G_A(u)$ denote the linear canonical transform of $h(t), f(t)$ and $g(t)$ respectively, then

$$H_A(u) = \sqrt{2i\pi b} e^{-i(\frac{du^2}{2b})} F_A(u) G_A(u). \quad (4)$$

2. Preliminary Results

In this section we obtain some results which are required to construct space of integrable Boehmians.

Lemma 2.1. Let $f, g \in \mathcal{L}^1(\mathbb{R})$, then $(f *^A g)$ is also in $\mathcal{L}^1(\mathbb{R})$.

Lemma 2.2. The space $(\mathcal{L}^1(\mathbb{R}), *^A)$ is a commutative semi group.

Definition 2.1. Let Δ denote the collection of all sequences of continuous real functions $\{\delta_n\}$ from $\mathcal{L}^1(\mathbb{R})$ having compact support, with the following conditions:

$$(i) \quad \int_{\mathbb{R}} e^{i\frac{at^2}{2b}} \delta_n(t) dt = 1, \forall n \in \mathbb{N},$$

- (ii) $\|\delta_n\|_1 \leq M, \forall n \in \mathbb{N}$ for some $M > 0$,
- (iii) $\lim_{n \rightarrow \infty} \int_{|t| > \epsilon} |\delta_n(t)| dt \rightarrow 0$, for each $\epsilon > 0$. The members of Δ are called delta sequences. For example, let $a, b \in \mathbb{R}; b \neq 0$, consider the sequence

$$\delta_n(t) = \begin{cases} e^{-i\frac{at^2}{2b}} t, & \text{for } 0 \leq t \leq \frac{1}{n}, \\ e^{-i\frac{at^2}{2b}} n^2 (\frac{2}{n} - t), & \text{for } \frac{1}{n} \leq t \leq \frac{2}{n}, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.3. Let $\{\varphi_n\}, \{\psi_n\} \in \Delta$ then $\{\varphi_n *^A \psi_n\} \in \Delta$.

Lemma 2.4. Let $f \in \mathcal{L}^1(\mathbb{R})$ and $\{\psi_n\} \in \Delta$ then $f *^A \psi_n \rightarrow f$ as $n \rightarrow \infty$ in $\mathcal{L}^1(\mathbb{R})$.

3. LCT For Integrable Boehmians

A pair of sequences (f_n, φ_n) is called a quotient of the sequences, denoted by f_n/φ_n , where $f_n \in \mathcal{L}^1(\mathbb{R}) (n \in \mathbb{N})$, $\{\varphi_n\}$ is a delta sequence and $f_m *^A \varphi_n = f_n *^A \varphi_m$ holds $\forall m, n \in \mathbb{N}$. Two quotient of sequences f_n/φ_n and g_n/ψ_n are equivalent if $f_n *^A \psi_n = g_n *^A \varphi_n \forall n \in \mathbb{N}$. The equivalence class of quotient of sequences is called an *integrable Boehmian*. The space of all integrable Boehmians is denoted by $\mathcal{B}_{\mathcal{L}^1} = \mathcal{B}_{\mathcal{L}^1}(\mathcal{L}^1(\mathbb{R}), \Delta, *^A)$ and the members of $\mathcal{B}_{\mathcal{L}^1}$ are denoted by $F = [f_n/\varphi_n]$. The function $f \in \mathcal{L}^1(\mathbb{R})$ can be identified with the Boehmian $[(f *^A \delta_n)/\delta_n]$, where $\{\delta_n\}$ is the delta sequence. If $F = [f_n/\varphi_n]$, then $f *^A \delta_n = f_n \in \mathcal{L}^1(\mathbb{R}) \forall n \in \mathbb{N}$.

Definition 3.1. A sequence of Boehmians F_n is called Δ -convergent to a Boehmian F ($\Delta - \lim F_n = F$) if there exist a delta sequence $\{\delta_n\}$ such that $(F_n - F) *^A \delta_n \in \mathcal{L}^1(\mathbb{R})$, for every $n \in \mathbb{N}$ and that $\|(F_n - F) *^A \delta_n\|_1 \rightarrow 0$ as $n \rightarrow \infty$.

Definition 3.2. A sequence of Boehmians F_n is called δ -convergent to a Boehmian F ($\delta - \lim F_n = F$) if there exist a delta sequence $\{\delta_n\}$ such that $F_n *^A \delta_k \in \mathcal{L}^1(\mathbb{R})$ and $F *^A \delta_k \in \mathcal{L}^1(\mathbb{R})$ for every $n, k \in \mathbb{N}$ and that $\|(F_n - F) *^A \delta_k\|_1 \rightarrow 0$ as $n \rightarrow \infty$ for each $k \in \mathbb{N}$.

Let $\{\delta_n\}$ is a delta sequence, then δ_n/δ_n represents an integrable Boehmian. Since the Boehmian $[\delta_n/\delta_n]$ corresponds to Dirac delta distribution δ , all the derivative of δ are also integrable Boehmian. If $\{\delta_n\}$ is infinitely differentiable and bounded, then the k^{th} derivative of δ is define by $\delta^{(k)} = [\delta_n^{(k)}/\delta_n] \in \mathcal{B}_{\mathcal{L}^1}$, for each $k \in \mathbb{N}$. The k^{th} derivative of Boehmian $F \in \mathcal{B}_{\mathcal{L}^1}$ is define by $F^{(k)} = F *^A \delta^{(k)}$.

Lemma 3.1. Let $\Delta - \lim F_n = F$ in $\mathcal{B}_{\mathcal{L}^1}$ then $\Delta - \lim F_n^{(k)} = F^{(k)}$ for $\forall k \in \mathbb{N}$ in $\mathcal{B}_{\mathcal{L}^1}$.

Lemma 3.2. Let $\Delta - \lim_{n \rightarrow \infty} F_n = F$ and $\Delta - \lim_{n \rightarrow \infty} G_n = G$ in $\mathcal{B}_{\mathcal{L}^1}$ then $\Delta - \lim_{n \rightarrow \infty} F_n *^A G_n = F *^A G$ in $\mathcal{B}_{\mathcal{L}^1}$.

The scalar multiplication, addition and convolution in $\mathcal{B}_{\mathcal{L}^1}$ are define as,

$$\begin{aligned}\lambda[f_n/\varphi_n] &= [\lambda f_n/\varphi_n] \\ [f_n/\varphi_n] + [g_n/\psi_n] &= [(f_n *^A \psi_n + g_n *^A \varphi_n)/\varphi_n *^A \psi_n] \\ [f_n/\varphi_n] *^A [g_n/\psi_n] &= [(f_n *^A g_n)/(\varphi_n *^A \psi_n)]\end{aligned}$$

Lemma 3.3. Let $\{\delta_n\} \in \Delta$ then $\mathcal{L}_A(\delta_n)$ converges uniformly on each compact set to a constant function 1.

Let $\Delta_0 = \{\mathcal{L}_A(\delta_n); \{\delta_n\} \in \Delta\}$ be the space of complex valued functions on $\mathbb{R}$, the operation $\cdot$ is point wise multiplication and $C_0(\mathbb{R})$ be the space of all continuous functions vanishing at infinity on $\mathbb{R}$ then we construct the another space of Boehmians, denoted by $\mathcal{B} = \mathcal{B}(\mathcal{L}^1(\mathbb{R}), C_0(\mathbb{R}) \cap \mathcal{L}^1(\mathbb{R}), \cdot, \Delta_0)$. This is the range of linear canonical transform on $\mathcal{B}_{\mathcal{L}^1}$ and each element of $\mathcal{B}$ is denoted by $\mathcal{L}_A(f_n)/\mathcal{L}_A(\delta_n)$ for all $n \in \mathbb{N}$, where $\{f_n\} \in \mathcal{L}^1(\mathbb{R})$ for $n \in \mathbb{N}$.

Lemma 3.4. Let $f, g \in \mathcal{L}^1(\mathbb{R}); \varphi, \psi \in C_0(\mathbb{R})$ and $\lambda \in \mathbb{C}$ then (i) $f \cdot \varphi \in \mathcal{L}^1(\mathbb{R})$ (ii) $(f+g) \cdot \varphi = f \cdot \varphi + g \cdot \varphi$ (iii) $(\lambda f) \cdot \varphi = \lambda(f \cdot \varphi)$ (iv) $f \cdot (\varphi \cdot \psi) = (f \cdot \varphi) \cdot \psi$.

Lemma 3.5. Let $f_n \rightarrow f$ as $n \rightarrow \infty$ in $\mathcal{L}^1(\mathbb{R})$ and $\varphi \in C_0(\mathbb{R})$ then $f_n \cdot \varphi \rightarrow f \cdot \varphi$ in $\mathcal{L}^1(\mathbb{R})$.

Lemma 3.6. Let $f_n \rightarrow f$ as $n \rightarrow \infty$ in $\mathcal{L}^1(\mathbb{R})$ and $\mathcal{L}_A(\varphi_n) \in \Delta_0$ then $f_n \cdot \mathcal{L}_A(\varphi_n) \rightarrow f \cdot \mathcal{L}_A(\varphi)$ in $\mathcal{L}^1(\mathbb{R})$.

Lemma 3.7. If $\mathcal{L}_A(\varphi_n), \mathcal{L}_A(\psi_n) \in \Delta_0$ then $\mathcal{L}_A(\varphi_n) \cdot \mathcal{L}_A(\psi_n) \in \Delta_0$.

Definition 3.3. Let $\{f_n\} \in \mathcal{L}^1(\mathbb{R})$ and $\{\delta_n\} \in \Delta$ for $n \in \mathbb{N}$, we define the linear canonical transform $\mathcal{L}_A : \mathcal{B}_{\mathcal{L}^1} \rightarrow \mathcal{B}$ as

$$\mathcal{L}_A[f_n/\delta_n] = \mathcal{L}_A(f_n)/\mathcal{L}_A(\delta_n) \quad \text{for } [f_n/\delta_n] \in \mathcal{B}_{\mathcal{L}^1}. \quad (5)$$

The linear canonical transform on $\mathcal{B}_{\mathcal{L}^1}$ is well defined. Indeed if $[f_n/\delta_n] \in \mathcal{B}_{\mathcal{L}^1}$, then $f_n *^A \delta_m = f_m *^A \delta_n$ for all $m, n \in \mathbb{N}$. Applying the linear canonical transform on both sides, we get $\mathcal{L}_A(f_n)\mathcal{L}_A(\delta_m) = \mathcal{L}_A(f_m)\mathcal{L}_A(\delta_n)$ for all $m, n \in \mathbb{N}$ and hence $\mathcal{L}_A(f_n)/\mathcal{L}_A(\delta_n) \in \mathcal{B}$. Further if $[f_n/\psi_n] = [g_n/\delta_n] \in \mathcal{B}_{\mathcal{L}^1}$ then we have $f_n *^A \delta_n = g_n *^A \psi_n$ for all $n \in \mathbb{N}$. Again applying the linear canonical transform on both sides, we get $\mathcal{L}_A(f_n)\mathcal{L}_A(\delta_n) = \mathcal{L}_A(g_n)\mathcal{L}_A(\psi_n)$ for all $n \in \mathbb{N}$. i.e. $\mathcal{L}_A(f_n)/\mathcal{L}_A(\psi_n) = \mathcal{L}_A(g_n)/\mathcal{L}_A(\delta_n)$ in $\mathcal{B}$.

Lemma 3.8. *Let $[f_n/\varphi_n] \in \mathcal{B}_{\mathcal{L}^1}$ then the linear canonical transform of the sequence*

$$\mathcal{L}_A[f_n](u) = \sqrt{\frac{1}{2\pi ib}} e^{\frac{i}{2}(\frac{d}{b})u^2} \int_{-\infty}^{\infty} e^{\frac{-i}{b}ut} e^{\frac{i}{2}\frac{a}{b}t^2} f_n(t) dt \quad (6)$$

converges uniformly on each compact set in $\mathbb{R}$.

Proof. Let (δ_n) is delta sequence, then $\mathcal{L}_A(\delta_n)$ converges uniformly on each compact set to a constant function 1. Hence, for each compact set M of $\mathbb{R}$, $\mathcal{L}_A[\delta_m] > 0$ on M , for almost all $m \in \mathbb{N}$. Moreover, $\mathcal{L}_A(f_n) = \mathcal{L}_A(f_n) \frac{\mathcal{L}_A(\delta_m)}{\mathcal{L}_A(\delta_m)} = \frac{\mathcal{L}_A(f_n)}{\mathcal{L}_A(\delta_m)} \mathcal{L}_A(\delta_m)$, as $n \rightarrow \infty$ we get $\mathcal{L}_A(f_n) \rightarrow \frac{\mathcal{L}_A(f_m)}{\mathcal{L}_A(\delta_m)}$, on each compact set M . $\square$

Definition 3.4. In view of the above lemma, the linear canonical transform of an integrable Boehmian in a space of continuous functions on $\mathbb{R}$ can be defined as,

$$\mathcal{L}_A[F] = \lim_{n \rightarrow \infty} \mathcal{L}_A(f_n).$$

Now, we show that the above definition is well define. Let two Boehmians $[f_n/\varphi_n]$ and $[g_n/\psi_n]$ are the same but representation is different then $f_n *^A \psi_n = g_n *^A \varphi_n$ for all $n \in \mathbb{N}$ taking linear canonical transform on both side we have

$$\mathcal{L}_A(f_n)(\mu) \mathcal{L}_A(\psi_n)(\mu) = \mathcal{L}_A(g_n)(\mu) \mathcal{L}_A(\varphi_n)(\mu)$$

which implies that $\lim_{n \rightarrow \infty} \mathcal{L}_A(f_n) = \lim_{n \rightarrow \infty} \mathcal{L}_A(g_n)$. Hence $\mathcal{L}_A[f_n/\varphi_n] = \mathcal{L}_A[g_n/\psi_n]$.

Theorem 3.1. *The linear canonical transform $\mathcal{L}_A : \mathcal{B}_{\mathcal{L}^1} \rightarrow \mathcal{B}$ is consistent with $\mathcal{L}_A : \mathcal{L}^1(\mathbb{R}) \rightarrow \mathcal{L}^1(\mathbb{R})$.*

Theorem 3.2. *The linear canonical transform $\mathcal{L}_A : \mathcal{B}_{\mathcal{L}^1} \rightarrow \mathcal{B}$ is a bijection.*

we skip the proofs of above two theorems.

Theorem 3.3. *Let $F, G \in \mathcal{B}_{\mathcal{L}^1}$ then:*

- (a) $\mathcal{L}_A(\lambda F + G) = \lambda \mathcal{L}_A(F) + \mathcal{L}_A(G)$, for any complex λ ,
- (b) $\mathcal{L}_A(F *^A G) = \mathcal{L}_A(F) \mathcal{L}_A(G)$,
- (c) $\mathcal{L}_A(e^{ikt} F)(u) = e^{\frac{idk(2u-bk)}{2}} \mathcal{L}_A(F)(u - bk)$, for $k \in \mathbb{R}$,
- (d) $\mathcal{L}_A(F(t + \tau))(u) = e^{\frac{i u \tau}{b}} e^{\frac{i a \tau^2}{2b}} \mathcal{L}_A(e^{\frac{-i a x \tau}{b}} F(x))(u)$,
- (e) $\mathcal{L}_A(F^{(k)})(u) = \left(\frac{i u}{b}\right)^k \mathcal{L}_A(F)(u)$ for all $k \in \mathbb{N}$,

(f) If $\mathcal{L}_A(F) = 0$, then $F = 0$,

(g) Let $F_n, F \in \mathcal{B}_{\mathcal{L}^1}$, if $\Delta - \lim F_n = F$ then $\mathcal{L}_A(F_n) \rightarrow \mathcal{L}_A(F)$ uniformly on each compact set of $\mathbb{R}$.

Proof. The proofs of the above properties (a)-(f) are straightforward. Now we prove (g), let $\{\delta_m\}$ be a delta sequence such that $F_n *^A \delta_m, F *^A \delta_m \in \mathcal{L}^1(\mathbb{R})$ for all $n, m \in \mathbb{N}$ and $\|(F_n - F) *^A \delta_m\|_1 \rightarrow 0$ as $n \rightarrow \infty$ for each $m \in \mathbb{N}$. Let M be a compact set in $\mathbb{R}$ then $\mathcal{L}_A(\delta_m) > 0$ on M for all most $m \in \mathbb{N}$. Since $\mathcal{L}_A(\delta_m)$ is a continuous function and $\mathcal{L}_A(F_n) *^A \mathcal{L}_A(\delta_m) - \mathcal{L}_A(F) *^A \mathcal{L}_A(\delta_m) = ((\mathcal{L}_A(F_n) - \mathcal{L}_A(F)) *^A \mathcal{L}_A(\delta_m))$, implies $\|(\mathcal{L}_A(F_n) - \mathcal{L}_A(F)) *^A \mathcal{L}_A(\delta_m)\|_1 \rightarrow 0$ as $n \rightarrow \infty$, thus $\mathcal{L}_A(F_n) \rightarrow \mathcal{L}_A(F)$ uniformly on M . $\square$

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