RANK PROPERTIES OF FINITE CYCLIC GROUPS

Osman Kelekci

Department of Mathematics
Faculty of Arts and Sciences
Omer Halisdemir University
51240, Nigde, TURKEY

Abstract: All ranks of the finite direct product of finite cyclic groups are given which are defined as small rank, lower rank, intermediate rank, upper rank and large rank, respectively.

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1. Introduction

The concept of rank is analogous to the concept of dimension in linear algebra and important to classify algebraic structures. The dimension of a (finite-dimensional) vector space is the cardinality of a maximal linearly independent subset of the vector space, or equivalently, is the cardinality of a minimal generating set of the vector space.

Extending the notion of rank to a more general algebraic system such as a semigroup, one can find that the possible definitions of rank give different values. In this sense, Howie and Ribeiro have considered five different ranks for a semigroup which are defined below.

In previous studies, these five ranks were examined for certain semigroups like rectangular band, aperiodic Brandt semigroup, full transformation semigroup, chain, zero semigroup, free semilattice in [1], [2], [4] and [5].

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The main goal of this study is to calculate all ranks of the direct product of finite cyclic groups. For unexplained terms see [3].

2. Preliminaries

Let S be a finite semigroup. A subset U of S is called *independent* if, for every u in U , the element u does not belong to the semigroup $\langle U \setminus \{u\} \rangle$ generated by the remaining elements of U . Howie and Ribeiro introduced five different type of rank for semigroups, in [1] and [2]. These ranks are defined as follows:

- $r_1(S) = \max\{k : \text{every subset } U \text{ of } S \text{ of cardinality } k \text{ is independent}\}$
- $r_2(S) = \min\{k : \text{there exists a subset } U \text{ of } S \text{ of cardinality } k \text{ such that } U \text{ generates } S\}$
- $r_3(S) = \max\{k : \text{there exists a subset } U \text{ of } S \text{ of cardinality } k \text{ which is independent and generates } S\}$
- $r_4(S) = \max\{k : \text{there exists a subset } U \text{ of } S \text{ of cardinality } k \text{ which is independent}\}$
- $r_5(S) = \min\{k : \text{every subset } U \text{ of } S \text{ of cardinality } k \text{ generates } S\}$

It is easily seen that $r_1(S) \leq r_2(S) \leq r_3(S) \leq r_4(S) \leq r_5(S)$, and for convenience, in [2], the terminology has been used as $r_1(S)$ is small rank, $r_2(S)$ is lower rank, $r_3(S)$ is intermediate rank, $r_4(S)$ is upper rank and $r_5(S)$ is large rank. Here, the lower rank is what is normally called the *rank*, which has been extensively studied. Also $r_1(S)$ small rank and $r_5(S)$ large rank are formulated due to the word ‘every’ in their definitions as follows in [2].

Theorem 1 (small rank). *Let S be a finite semigroup, with $|S| \geq 2$.*

- (i) *If S is not a band, then $r_1(S) = 1$.*
- (ii) *If S is a band other than a royal semigroup, then $r_1(S) = 2$.*
- (iii) *If S is a royal semigroup, then $r_1(S) = |S|$. □*

Theorem 2 (large rank). *Let S be a finite semigroup and V the largest proper subsemigroup of S . Then $r_5(S) = |V| + 1$. □*

Remark 3. *A band is a semigroup which is all elements are idempotent. If lower rank $r_2(S) = |S|$, then S is called a royal semigroup.*

3. Main Results

Let C_m be the cyclic group of order m , ($m \geq 2$). It is clear that $r_2(C_m) = 1$. Then, in [1], it is proved that $r_3(C_m) = r_4(C_m) = \pi(m)$ as a special case of monogenic semigroups. Also, from the Theorem 1, $r_1(C_m) = 1$, since C_m is not a band and it is shown that in [2], $r_5(C_m) = (m/p) + 1$, here p is the smallest prime dividing m . So all ranks of C_m as stated follows.

$$r_1(C_m) = 1 \leq r_2(C_m) = 1 \leq r_3(C_m) = \pi(m) \leq r_4(C_m) = \pi(m) \leq r_5(C_m) = (m/p) + 1.$$

Now, let's start to calculate all ranks of $C_m \times C_n$.

Theorem 4. *Let C_m and C_n be the cyclic group of order m and n , ($m, n \geq 2$) respectively. Then, $r_1(C_m \times C_n) = 1$ and*

$$r_2(C_m \times C_n) = \begin{cases} 1, & \text{if } (m, n) = 1 \\ 2, & \text{if } (m, n) = d \end{cases}$$

Proof. Firstly, $C_m \times C_n$ is not a band, so from the Theorem 1, $r_1(C_m \times C_n) = 1$. Also it is clear $r_2(C_m \times C_n) = 1$ if $(m, n) = 1$. Because the set $\{(1, 1)\}$ is the minimal generating set of $C_m \times C_n$. If $(m, n) = d$, the set $\{(0, 1), (1, 0)\}$ will always generate the group $C_m \times C_n$, so $r_2(C_m \times C_n) = 2$. $\square$

Theorem 5. *Let C_m and C_n be the cyclic group of order m and n , ($m, n \geq 2$) respectively. Then,*

$$r_3(C_m \times C_n) = r_4(C_m \times C_n) = \begin{cases} \pi(mn), & \text{if } (m, n) = 1 \\ \pi(m) + \pi(n), & \text{if } (m, n) = d \end{cases}$$

$\pi(n)$ denotes the number of distinct prime factors of n .

Proof. If $(m, n) = 1$, $C_m \times C_n \cong C_{mn}$ and C_{mn} is a cyclic group of order $|mn|$. So from [1] $r_3(C_{mn}) = r_4(C_{mn}) = \pi(mn)$. Now let $(m, n) = d$. In this case $C_m \times C_n$ is not a cyclic group but it is an abelian group of order $|mn|$. Let $m = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ and $n = q_1^{b_1} q_2^{b_2} \dots q_t^{b_t}$ where $k, t \geq 1$, $p_1, p_2, \dots, p_k$ and $q_1, q_2, \dots, q_t$ are distinct primes and also $a_i, b_j \geq 1$ for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, t$. Let $\alpha_i = \frac{m}{p_i^{a_i}}$ ($i = 1, 2, \dots, k$). Then $C_m = H_1 \times H_2 \times \dots \times H_k$, where $H_i = \langle x^{\alpha_i} \rangle$ is a cyclic group of order $p_i^{a_i}$. By the equations from [1]

$$r_3(C_m \times C_n) = r_3(C_m) + r_3(C_n) \tag{1}$$

and

$$r_4(C_m \times C_n) = r_4(C_m) + r_4(C_n) \tag{2}$$

it follows that

$$\begin{aligned} r_3(C_m) &= r_3(H_1) + r_3(H_2) + \dots + r_3(H_k) = \\ &= r_2(H_1) + r_2(H_2) + \dots + r_2(H_k) = \\ &= 1 + 1 + \dots + 1 = k = \pi(m) \end{aligned}$$

and similarly

$$\begin{aligned} r_4(C_m) &= r_4(H_1) + r_4(H_2) + \dots + r_4(H_k) = \\ &= r_2(H_1) + r_2(H_2) + \dots + r_2(H_k) = \\ &= 1 + 1 + \dots + 1 = k = \pi(m) \end{aligned}$$

If we use the same argument for C_n , we find $r_3(C_n) = r_4(C_n) = t = \pi(n)$. By equations (1) and (2) it follows that $r_3(C_m \times C_n) = r_4(C_m \times C_n) = \pi(m) + \pi(n)$. Thus the proof is complete. $\square$

Theorem 6. *Let C_m and C_n be the cyclic groups of order $m \geq 2$ and $n \geq 2$. Then*

$$r_5(C_m \times C_n) = \left(m \frac{n}{p}\right) + 1$$

where p is the smallest prime dividing n .

Proof. The direct product of C_m and C_n is given by

$$C_m \times C_n \cong \{(i, j) : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}.$$

If $(m, n) = 1$, then $C_m \times C_n$ is a cyclic group and it is trivial that $r_5(C_m \times C_n) = \left(\frac{mn}{t}\right) + 1$, where t is the smallest prime dividing mn . Now suppose that $m \leq n$ and also $(m, n) = d$. $C_m \times C_n$ has a subgroup such that $C_n \cong C_n^i = \{(0, j) : 0 \leq j \leq n-1\}$. Since C_n^i is a cyclic group, there is a largest proper subsemigroup of C_n^i with $|V_n^i| = \frac{n}{p}$, where p is the smallest prime dividing n . From Theorem 2, $r_5(C_n^i) = |V_n^i| + 1$. V_n^i has the following form $V_n^i = \left\{(0, p), (0, 2p), \dots, \left(0, \frac{n}{p}p\right)\right\}$. Now let's find the largest proper subsemigroup of $C_m \times C_n$, adding all elements of C_m to V_n^i . $V_{m \times n} =$

$$\left\{ \begin{array}{l} (0, p), (0, 2p), \dots, \left(0, \frac{n}{p}p\right), (1, p), (1, 2p), \dots, \left(1, \frac{n}{p}p\right), \dots, \\ (m-1, p), (m-1, 2p), \dots, \left(m-1, \frac{n}{p}p\right) \end{array} \right\}$$

This group is the largest proper subsemigroup of $C_m \times C_n$ which does not generate $C_m \times C_n$ and its cardinality is $|V_{m \times n}| = m \frac{n}{p}$. Then if we add any

element to $V_{m \times n}$, it will generate $C_m \times C_n$ and so $r_5(C_m \times C_n) = \left(m \frac{n}{p}\right) + 1$. $\square$

For $n < m$, by the same argument it can be shown that

$$r_5(C_m \times C_n) = \left(n \frac{m}{q}\right) + 1$$

where q is the smallest prime dividing m . $\square$

We can generalize this idea to the finite product. Lets consider the following group S .

$$S = C_{m_1} \times C_{m_2} \times \dots \times C_{m_t}$$

Here C_{m_i} is the cyclic group of order $|m_i|$, ($m_i \geq 2$) and $1 \leq i \leq t$. Since S is not a band, so small rank of S is $r_1(S) = 1$. For lower rank, it is obvious that $r_2(S) = 1$, if $(m_1, m_2, \dots, m_t) = 1$. In the other case, if $(m_1, m_2, \dots, m_t) = d$, the subset

$$\{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}$$

with t elements of S will always generate S , thus $r_2(S) = t$.

If $(m_1, m_2, \dots, m_t) = 1$ then

$$S = C_{m_1} \times C_{m_2} \times \dots \times C_{m_t} \cong C_{m_1 m_2 \dots m_t}$$

and $C_{m_1 m_2 \dots m_t}$ is a cyclic group of order $|m_1 m_2 \dots m_t|$. So from [1], $r_3(S) = r_4(S) = \pi(m_1 m_2 \dots m_t)$. Now, let $(m_1, m_2, \dots, m_t) = d$. Here S is not a cyclic group but it is an abelian group of order $|m_1 m_2 \dots m_t|$. With the same method in the proof of Theorem 5, since every factor of S is a cyclic group of order $|m_i|$ and their intermediate and upper rank are $\pi(m_i)$, so from equals (1) and (2), $r_3(S) = r_4(S) = \pi(m_1) + \pi(m_2) + \dots + \pi(m_t)$.

For large rank of S , we have to find the largest proper subsemigroup of S . That is, the subsemigroup of S which does not generate S with maximum cardinality. Suppose that $m_1 \leq m_2 \leq \dots \leq m_t$. The purpose of this sequencing to find the biggest cyclic group from factors of S . Since C_{m_t} is a cyclic group, there is a largest proper subsemigroup of C_{m_t} with $|V_{m_t}| = \frac{m_t}{p}$, here p is the smallest prime dividing m_t . V_{m_t} has the following form $V_n = \left\{ (p), (2p), \dots, \left(\frac{m_t}{p} p\right) \right\}$. We can start to state the form of largest proper subsemigroup $V_{m_1 x m_2 x \dots x m_t}$ of S . Now lets add all elements of $C_{m_{t-1}}$ to V_{m_t} to find largest proper subsemigroup of last two groups $C_{m_{t-1}} \times C_{m_t}$.

$$V_{m_{t-1}xm_t} = \left\{ \begin{array}{l} (0, p), (0, 2p), \dots, \left(0, \frac{m_t}{p}p\right), \\ (1, p), (1, 2p), \dots, \left(1, \frac{m_t}{p}p\right), \dots, \\ (m_{t-1}, p), (m_{t-1}, 2p), \dots, \left(m_{t-1}, \frac{m_t}{p}p\right) \end{array} \right\}$$

and $|V_{m_{t-1}xm_t}| = (m_{t-1})\frac{m_t}{p}$. Next if we add all elements of $C_{m_{t-2}}$ to $V_{m_{t-1}xm_t}$, we obtain the largest proper subsemigroup of last three groups $C_{m_{t-2}} \times C_{m_{t-1}} \times C_{m_t}$ as follows. $V_{m_{t-2}xm_{t-1}xm_t} =$

$$= \left\{ \begin{array}{l} (0, 0, p), (0, 0, 2p), \dots, \left(0, 0, \frac{m_t}{p}p\right), \\ (0, 1, p), (0, 1, 2p), \dots, \left(0, 1, \frac{m_t}{p}p\right), \dots, \\ (0, m_{t-1}, p), (0, m_{t-1}, 2p), \dots, \left(0, m_{t-1}, \frac{m_t}{p}p\right), \\ (1, 0, p), (1, 0, 2p), \dots, \left(1, 0, \frac{m_t}{p}p\right), \dots, \\ (m_{t-2}, m_{t-1}, p), (m_{t-2}, m_{t-1}, 2p), \dots, \left(m_{t-2}, m_{t-1}, \frac{m_t}{p}p\right) \end{array} \right\}$$

and $|V_{m_{t-2}xm_{t-1}xm_t}| = (m_{t-2})(m_{t-1})\frac{m_t}{p}$. If we continue this process in this way, finally we find the largest proper subsemigroup $V_{m_1xm_2x\dots xm_t}$ of S , adding all elements of C_{m_1} to $V_{m_2x\dots xm_{t-1}xm_t}$. $V_{m_1xm_2x\dots xm_t} =$

$$= \left\{ \begin{array}{l} (0, \dots, 0, p), (0, \dots, 0, 2p), \dots, \left(0, \dots, 0, \frac{m_t}{p}p\right), \\ (0, \dots, 1, p), (0, \dots, 1, 2p), \dots, \left(0, \dots, 1, \frac{m_t}{p}p\right), \dots, \\ (0, \dots, m_{t-1}, p), (0, \dots, m_{t-1}, 2p), \dots, \left(0, \dots, m_{t-1}, \frac{m_t}{p}p\right), \\ (1, \dots, 0, p), (1, \dots, 0, 2p), \dots, \left(1, \dots, 0, \frac{m_t}{p}p\right), \dots, \\ (m_1, \dots, m_{t-1}, p), (m_1, \dots, m_{t-1}, 2p), \dots, \left(m_1, \dots, m_{t-1}, \frac{m_t}{p}p\right) \end{array} \right\}$$

This is the largest proper subsemigroup S with cardinality

$$|V_{m_2x\dots xm_{t-1}xm_t}| = m_1m_2\dots(m_{t-1})\frac{m_t}{p}$$

This subsemigroup does not generate S , but if we add only one element out of these elements of S to this subsemigroup $V_{m_1xm_2x\dots xm_t}$, it generates S . So from Theorem 2,

$$r_5(S) = \left(m_1m_2\dots(m_{t-1})\frac{m_t}{p}\right) + 1$$

□

Hence, we proved that the following theorem.

Theorem 7. Let $S = C_{m_1} \times C_{m_2} \times \dots \times C_{m_t}$ be the group of finite direct product of C_{m_i} . C_{m_i} is the cyclic group of order m_i , ($m_i \geq 2$) and $1 \leq i \leq t$. Then

$$(i) \ r_1(S) = 1$$

$$(ii) \ r_2(S) = \begin{cases} 1, & \text{if } (m_1, m_2, \dots, m_t) = 1 \\ t, & \text{if } (m_1 m_2 \dots m_t) = d \end{cases}$$

$$(iii) \ r_3(S) = r_4(S) =$$

$$= \begin{cases} \pi(m_1 m_2 \dots m_t), & \text{if } (m_1, m_2, \dots, m_t) = 1 \\ \pi(m_1) + \pi(m_2) + \dots + \pi(m_t), & \text{if } (m_1, m_2, \dots, m_t) = d \end{cases}$$

$$(iv) \ r_5(S) = \begin{cases} \left(\frac{m_1 m_2 \dots m_t}{q} \right) + 1, & \text{if } (m_1, m_2, \dots, m_t) = 1 \\ \left(m_1 m_2 \dots (m_{t-1}) \cdot \frac{m_t}{p} \right) + 1, & \text{if } (m_1, m_2, \dots, m_t) = d \end{cases}$$

Here, $\pi(m)$ denotes the number of distinct prime factors of m , q is the smallest prime dividing $m_1 m_2 \dots m_t$ and p is the smallest prime dividing m_t .

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