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FABER POLYNOMIAL COEFFICIENT OF BI-UNIVALENT FUNCTIONS WITH RESPECT TO SYMMETRIC *q*-DERIVATIVE OPERATOR

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Abstract: In this paper, we present few applications of polynomials which will give a flavour of basic principles and behaviours. Moreover, the polynomial introduced by Faber is widely used in enormous areas of mathematical sciences including chemical engineering. Using the concepts of Faber polynomial expansions, we introduce new class of analytic bi-univalent functions in the open unit disk and obtain upper bounds for the coefficients of functions belongs to the defined class.

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1. Introduction and Definitions

Polynomials play a vital role with different equations and system of equations. Polynomials deal with real and complex solutions. Whereas, real number so-

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lutions are used widely in developing ciphers for encoding messages. When complex numbers as solutions are used to analyse the behaviour of complex systems and reactions. Polynomial equations cover various fields especially cryptography, biology, economics, coding and more important with mixing of chemical components in the correct proportion in every dosage of medication.

In advanced mathematical fields, solving imaginary polynomial equations to any systems can be analysed for its stability.

Let $\mathcal{A}$ denote the family of normalized analytic functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{U} \tag{1.1}$$

in the open disc $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are also univalent in $\mathbb{U}$ and let P be the class of functions

$$\phi(z) = 1 + \phi_1 z + \phi_2 z^2 + \ldots, \quad z \in \mathbb{D}.$$

are univalent in $\mathbb{U}$ and satisfying the condition $\mathbf{R}(\phi(z)) > 0$ in $\mathbb{U}$. While seeing Caratheodory's lemma [8] we have $|\phi_n| \leq 2$.

The well-known Koebe one-quarter theorem [8] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius 1/4. Hence every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z, (z \in \mathbb{U})$ and

$$f^{-1}(f(w)) = w, (|w| < r_0(f), r_0(f) \geq 1/4)$$

where,

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.2). For example, functions in the class Σ are given below [24]:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

In 1967, Lewin [18] introduced the class Σ of bi-univalent functions and shown that $|a_2| < 1.51$. In 1969, Netanyahu[20] showed that $max_{f\in\Sigma}|a_2| = 4/3$ and Suffridge [27] have given an example of $f \in \Sigma$ for which $|a_2| = 4/3$. Later, in 1980, Brannan and Clunie [6] improved the result as $|a_2| \leq \sqrt{2}$. In 1985, Kedzier-awski [16] proved this conjecture for a special case when the function

f and f^{-1} are starlike. In 1984, Tan [28] proved that $|a_2| \leq 1.485$ which is the best estimate for the function in the class of bi-univalent functions.

Recently, many authors have introduced and studied various subclasses of analytic and bi-univalent functions. Some of the recent analysis in this topics are [11, 12, 29, 25] for reference to the readers. Brannan and Taha [7] introduced certain subclasses of the bi-univalent function class Σ for the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$. Ali et al. [4] widen the result of Brannan and Taha using subordination.

The concept of q-analog was first introduced by Jackson [13]. Mohammed and Darus studied the geometric analog of some subclasses of analytic functions by means of the q-difference operator $D_qf(z)$ for $0 < q < 1$, [19, 13, 14]. Note that the q-difference operator plays an important role in the theory of hypergeometric series and quantum physics and engineering(see for instance [5, 9, 10, 17, 23]).

We can see some of the specialisations of $q-$calculus in the area of Geometric Function Theory for various subclasses of analytic functions. One of the most important tool to investigate subclasses of analytic function is the "$fractional\, q - calculus$". We now recall the basic definitions and formulae of fractional q-calculus.

$$(D_qf)(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad (z \neq 0) \tag{1.3}$$

$D_qf(0) = f'(0)$ provided $f'(0)$ exist and $D_q^2f(z) = D_q((D_qf)(z))$.

From (1.3), we deduce that

$$(D_qf)(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} \tag{1.4}$$

where

$$[k]_q = \frac{1-q^k}{1-q}. \tag{1.5}$$

As $k \in \mathbb{N} = \{1, 2, 3, \dots\}$; $q \to 1^-, [k]_q \to k$. For a function $h(z) = z^k$, we observe that ,

$$(D_qh)(z) = D_q(z^k) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1},$$

$$\lim_{q \to 1^-} (D_q(h(z))) = \lim_{q \to 1^-} ([k]_q z^{k-1}) = kz^{k-1} = h'(z),$$

where h' is the ordinary derivative.

As a right inverse, Jackson [13] introduced the q-integral

$$\int_0^z f(t)d_qt = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k),$$

provided that the series converges. For a function $h(z) = z^k$, we have

$$\int_0^z h(t)d_qt = \int_0^z t^k d_qt = \frac{z^{k+1}}{[k+1]_q} \quad (k \neq -1)$$

$$\lim_{q\to 1^-} \int_0^z h(t)d_qt = \lim_{q\to 1^-} \frac{z^{k+1}}{[k+1]_q} = \frac{z^{k+1}}{k+1} = \int_0^z h(t)dt,$$

where $\int_0^z h(t)dt$ is the ordinary integral. One can clearly see that $D_qf(z) \to f'(z)$ as $q \to 1^-$. This difference operator helps us to generalize the class of starlike functions S^* analytically. We can see the history of $q-$ difference operator for some subclasses of analytic functions in recent papers [15, 21, 22].

Now, the symmetric q-derivative $(\widetilde{D}_qf)(z)$ of the function $f(z)$ given by (1.1) and it is defined as

$$(\widetilde{D}_qf)(z) = \frac{f(qz) - f(q^{-1}z)}{(q-q^{-1})z}, \quad z \neq 0, \tag{1.6}$$

and $(\widetilde{D}_qf)(0) = f^{'}(0)$ provided $f^{'}(0)$ exists.

From (1.6), we deduce that

$$(\widetilde{D}_qf)(z) = 1 + \sum_{k=2}^{\infty} \widetilde{[k]}_q a_k z^{k-1}, \tag{1.7}$$

where

$$\widetilde{[k]}_q = \frac{q^k - q^{-k}}{q - q^{-1}}. \tag{1.8}$$

From equations (1.2) and (1.6), we can deduce that

$$\begin{aligned}(\widetilde{D}_qg)(w) =& \frac{g(qw) - g(q^{-1}w)}{(q-q^{-1})w} \\ =& 1 - \widetilde{[2]}_q a_2 w + \widetilde{[3]}_q (2a_2^2 - a_3)w^2 \\ & - \widetilde{[4]}_q (5a_2^3 - 5a_2a_3 + a_4)w^3 + \ldots\end{aligned} \tag{1.9}$$

Significantly, polynomial equations are considered as one of the best-equipped to model physical and real-world systems. Utility of polynomial functions remains same, even the situations vary on case by case. The properly developed polynomial modeling functions are used to solve complex biological and behavioral concepts. In this paper we use the Faber polynomial expansions to obtain upper bounds for the n-th, $(n \geq 3)$ coefficients of subclasses of analytic functions.

Motivating the result of Srivastava et.al [26], we now introduce the new function class Σ using symmetric q-difference operator. Moreover, we obtain the general coefficients $|a_n|$ of the function class Σ by means of the Faber polynomial expansions.

Definition 1. Let $\lambda \geq 0$, a function $f \in \Sigma$ is said to be in the class $\mathcal{N}_{\Sigma}^{\lambda}(\phi)$ and $\phi \in P$ if the following subordination holds

$$(\widetilde{D}_q f)(z) + \lambda z(\widetilde{D}_q(\widetilde{D}_q f)(z)) \prec \phi(z), \quad z \in \mathbb{U}, \tag{1.10}$$

and

$$(\widetilde{D}_q g)(w) + \lambda w(\widetilde{D}_q(\widetilde{D}_q g)(w)) \prec \phi(w), \quad w \in \mathbb{U}. \tag{1.11}$$

By using the Faber polynomial expansions of functions $f(z)$ belongs to the class $\mathcal{A}$ of the form (1.1), the coefficient of its inverse map $g = f^{-1}$ can be expressed as follows [3]:

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) w^n \tag{1.12}$$

where

$$\begin{aligned} K_{n-1}^{-n} = & \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ & + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ & + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2\right] \\ & + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} \left[a_6 + (-2n+5) a_3 a_4\right] + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned} \tag{1.13}$$

symbolically such expression $(-n)! \equiv \Gamma(1-n) := (-n)(-n-1)(-n-2)\ldots$ $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ $(\mathbb{N} := \{1, 2, 3, \ldots\}))$ and V_j $(7 \leq j \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$ (see for further details, [1]). In particular, the first three terms of K_{n-1}^{-n} are given by

$$K_1^{-2} = -2a_2,$$

$$K_2^{-3} = 3(2a_2^2 - a_3),$$

and

$$K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

In general, for any $p \in \mathbb{R}$, K_n^p is given by [3]

$$K_n^p = pa_n + \frac{p(p-1)}{2}D_n^2 + \frac{p!}{(p-3)!3!}D_n^3 + \cdots + \frac{p!}{(p-n)!n!}D_n^n \tag{1.14}$$

where $D_{n-1}^p = D_{n-1}^p(a_2, a_3, \dots)$ and by [2],

$$D_{n-1}^m(a_2, a_3, \dots, a_n) = \sum_{m=2}^{\infty}\left(\frac{m!}{\mu_1! \dots \mu_{n-1}!}\right)(a_2)^{\mu_1} \dots (a_n)^{\mu_{n-1}}, \quad for\ m \leq n$$

where $a_1 = 1$, and the sum is taken over all non-negative integers $\mu_1, \dots, \mu_n$ satisfies the condition

$$\mu_1 + \mu_2 + \cdots + \mu_n = m,$$

$$\mu_1 + 2\mu_2 + \cdots + n\mu_{n-1} = n - 1.$$

And it is clear that,

$$D_{n-1}^{n-1}(a_2, a_3, \dots, a_n) = a_2^{n-1}.$$

2. Main Results

In this following theorem, we are going to find the general coefficient $|a_n|$ for the subclass $\mathcal{N}_{\Sigma}^{\lambda}(\phi)$.

Theorem 1. *Let f given by (1.1) be in the class $\mathcal{N}_{\Sigma}^{\lambda}(\phi)$ and $(\lambda \geq 0)$. If $a_k = 0$ for $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{2}{\widetilde{[n]}_q(1 + \lambda\widetilde{[n-1]}_q)}, \quad (n \in \mathbb{N} \setminus \{1, 2\}). \tag{2.1}$$

Proof. Let f be given by (1.1), we have

$$(\widetilde{D}_q f)(z) + \lambda z(\widetilde{D}_q(\widetilde{D}_q f)(z)) = 1 + \sum_{n=2}^{\infty}(1 + \lambda\widetilde{[n-1]}_q)\widetilde{[n]}_q a_n z^{n-1} \tag{2.2}$$

and its inverse $g = f^{-1}$ is

$$\begin{aligned}(\widetilde{D}_q g)(w) + \lambda w(\widetilde{D}_q(\widetilde{D}_q g)(w)) &\prec \phi(w)\\ &= 1 + \sum_{n=2}^{\infty}(1+\lambda\widetilde{[n-1]}_q)K_{n-1}^{-n}(a_2, a_3, \ldots, a_n)w^{n-1}\\ &= 1 + \sum_{n=2}^{\infty}(1+\lambda\widetilde{[n-1]}_q)\widetilde{[n]}_q b_n w^{n-1}\end{aligned} \tag{2.3}$$

Furthermore, since $f \in \mathcal{N}_{\Sigma}^{\lambda}(\phi)$ and $g = f^{-1} \in \mathcal{N}_{\Sigma}^{\lambda}(\phi)$ then there exists two Schwarz functions $u(z) = c_1 z + c_2 z^2 + \ldots$ and $v(w) = d_1 w + d_2 w^2 + \ldots$ such that

$$(\widetilde{D}_q f)(z) + \lambda z(\widetilde{D}_q(\widetilde{D}_q f)(z)) = \phi(u(z)) \tag{2.4}$$

and

$$(\widetilde{D}_q g)(w) + \lambda w(\widetilde{D}_q(\widetilde{D}_q g)(w)) = \phi(v(w)) \tag{2.5}$$

where

$$\phi(u(z)) = 1 + \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\phi_k E_{n-1}^{k}(c_1, c_2, \ldots, c_{n-1})z^{n-1} \tag{2.6}$$

and

$$\phi(v(w)) = 1 + \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\phi_k E_{n-1}^{k}(d_1, d_2, \ldots, d_{n-1})w^{n-1} \tag{2.7}$$

Now, comparing the corresponding coefficients of (2.2) and (2.6), we get

$$(1+\lambda\widetilde{[n-1]}_q)\widetilde{[n]}_q a_n = \sum_{k=1}^{n-1}\phi_k E_{n-1}^{k}(c_1, c_2, \ldots, c_{n-1}), \quad n \geq 2. \tag{2.8}$$

Similarly from (2.3) and (2.7) yields

$$1 + \sum_{n=2}^{\infty}(1+\lambda\widetilde{[n-1]}_q)K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) = \sum_{k=1}^{n-1}\phi_k E_{n-1}^{k}(d_1, d_2, \ldots, d_{n-1}), \quad n \geq 2. \tag{2.9}$$

We note that $a_k = 0$ for $0 \leq k \leq n-1$, Hence,

$$\begin{aligned}&(1+\lambda\widetilde{[n-1]}_q)\widetilde{[n]}_q a_n = \phi_1 c_{n-1}\\ &(1+\lambda\widetilde{[n-1]}_q)\widetilde{[n]}_q b_n = -(1+\lambda\widetilde{[n-1]}_q)\widetilde{[n]}_q a_n = \phi_1 d_{n-1}.\end{aligned} \tag{2.10}$$

since by definition of K_n^p, we have

$$b_n = -a_n. \tag{2.11}$$

By simplification,

$$a_n = \frac{\phi_1 c_{n-1}}{(1+\lambda\widetilde{[n-1]_q})\widetilde{[n]_q}}, \tag{2.12}$$

and

$$a_n = -\frac{\phi_1 d_{n-1}}{(1+\lambda\widetilde{[n-1]_q})\widetilde{[n]_q}} \tag{2.13}$$

Taking the absolute values of (2.12) or (2.13) using the fact that $|\phi_1| \leq 2$, $|c_{n-1}| \leq 1$ and $|d_{n-1}| \leq 1$, we get

$$|a_n| = \frac{|\phi_1 c_{n-1}|}{\widetilde{[n]_q}(1+\lambda\widetilde{[n-1]_q})} = \frac{|\phi_1 d_{n-1}|}{\widetilde{[n]_q}(1+\lambda\widetilde{[n-1]_q})} \leq \frac{2}{\widetilde{[n]_q}(1+\lambda\widetilde{[n-1]_q})}, \quad n \geq 3. \qquad \square$$

Our next theorem clearly demonstrates the behavior of the early coefficients of classes of bi-univalent functions f in $\mathcal{N}_\Sigma^\lambda(\phi)$.

Theorem 2. *If $f \in \mathcal{N}_\Sigma^\lambda(\phi)$, then*

$$(i) \quad |a_2| \leq \min\left\{\frac{2q}{q^2+1}, \frac{2q\sqrt{(q^2+1)}}{\sqrt{(1+q+q^2)(1-q+q^2)(1+q\lambda+q^2)}}\right\} = \frac{2q}{q^2+1},$$

$$(ii) \quad |a_3| \leq \frac{4q^2(q^2+1)}{(1+q+q^2)(1-q+q^2)(1+q\lambda+q^2)}.$$

Proof. Put $n = 2$ and $n = 3$ in (2.10) we obtain,

$$\widetilde{[2]_q}(1+\lambda)a_2 = \phi_1 c_1. \tag{2.14}$$

$$\widetilde{[3]_q}(1+\lambda\widetilde{[2]_q})a_3 = \phi_1 c_2 + \phi_2 c_1^2. \tag{2.15}$$

and

$$\widetilde{[2]_q}(1+\lambda)b_2 = -\widetilde{[2]_q}(1+\lambda)a_2 = \phi_1 d_1. \tag{2.16}$$

$$\widetilde{[3]_q}(1+\lambda\widetilde{[2]_q})b_3 = \widetilde{[3]_q}(1+\lambda\widetilde{[2]_q})(2a_2^2 - a_3) = \phi_1 d_2 + \phi_2 d_1^2. \tag{2.17}$$

From (2.14) and (2.16), we obtain

$$\begin{aligned} |a_2| &= \frac{|\phi_1||c_1|}{\widetilde{[2]}_q(1+\lambda)} \leq \frac{2}{\widetilde{[2]}_q(1+\lambda)} \\ &= \frac{2q}{q^2+1} \end{aligned} \tag{2.18}$$

Adding (2.15) and (2.17) gives

$$2\widetilde{[3]}_q(1+\widetilde{[2]}_q\lambda)a_2^2 = \phi_1(c_2+d_2) + \phi_2(c_1^2+d_1^2)$$

or equivalently,

$$|a_2| \leq \frac{2q\sqrt{(q^2+1)}}{\sqrt{(1+q+q^2)(1-q+q^2)(1+q\lambda+q^2)}}. \tag{2.19}$$

From (2.15),

$$|a_3| = \frac{|\phi_1 c_2 + \phi_2 c_1^2|}{\widetilde{[3]}_q(1+\widetilde{[2]}_q\lambda)} \leq \frac{4q^2(q^2+1)}{(1+q+q^2)(1-q+q^2)(1+q\lambda+q^2)}. \tag{2.20}$$

Next to find the bounds of $|a_3|$, subtract (2.17) from (2.15). Hence

$$2\widetilde{[3]}_q(1+\widetilde{[2]}_q\lambda)(a_3-a_2^2) = \phi_1(c_2-d_2) + \phi_2(c_1^2-d_1^2) = \phi_1(c_2-d_2) \tag{2.21}$$

or

$$\begin{aligned} |a_3| &\leq |a_2|^2 + \frac{|\phi_1(c_2-d_2)|}{2\widetilde{[3]}_q(1+\widetilde{[2]}_q\lambda)} \\ &\leq |a_2|^2 + \frac{2}{\widetilde{[3]}_q(1+\widetilde{[2]}_q\lambda)}. \end{aligned} \tag{2.22}$$

By substituting the value of a_2^2 from (2.18) and (2.19),

$$|a_3| \leq \frac{4q^2}{(q^2+1)^2} + \frac{2q^2(q^2+1)}{(1+q+q^2)(1-q+q^2)(1+q\lambda+q^2)},$$

and

$$|a_3| \leq \frac{6q^2(q^2+1)}{(1+q+q^2)(1-q+q^2)(1+q\lambda+q^2)}.$$

Finally, we have obtain $(a_3 - a_2^2)$ from (2.21)

$$|a_3 - a_2^2| \leq \frac{2q^2(q^2+1)}{(1+q+q^2)(1-q+q^2)(1+q\lambda+q^2)}.$$

□

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